Geometric Phases for the Free Rigid Body with Variable Inertia Tensor

Patrick Tantalo
Department of Mathematical Sciences
301 Science-Engineering
University of Arkansas
Fayetteville, AR 72701
email: ptantalo@comp.uark.edu

September 16, 1999

Abstract

The Hannay-Berry connection[8,14,16] provides a geometric setting for the classical Hannay angles associated to an integrable Hamiltonian which depends on a slowly varying (adiabatic) parameter. In this paper we compute the connection one-form and curvature associated to an asymmetrical rigid body whose inertia tensor varies slowly with time. The connection in this case is defined on a bundle whose base is the space of inertia tensors and whose fiber is the union of those regions in phase space which admit local action-angle charts. Each element in the base induces dynamics in the corresponding fiber which are completely integrable. The connection is defined by averaging the trivial connection over the $\mathbb{T}^3$ action induced by this local set of parameter dependent action-angle variables. As a corollary we compute explicit formulae for the holonomy of special classes of loops in the space of inertia tensors. For example it is proved that any loop consisting of inertia tensors which are simultaneously diagonalizable has trivial holonomy, and hence the Hannay angles are zero. We also compute the holonomy of certain loops for which the moments of inertia are constant, while the principal axes undergo a rotation.

Contents

1 Introduction

1.1 Background

In this paper we will study geometric aspects of the Euler-Poinsot system (or Free Rigid body) where the inertia tensor is made to vary with time. In particular we will consider loops in the space of inertia tensors and will restrict our attention to the adiabatic setting in which the moments and axes of inertia move slowly relative to the overall rotational motion of the system.

More generally we may consider a family of integrable Hamiltonians H_m, depending on a parameter m, which lies in a smooth manifold M. In our case M is the set of inertia tensors. When the parameter undergoes an adiabatic circuit $t \mapsto m(\epsilon t)$ for ϵ small, $0 \leq t \leq 1/\epsilon$, we obtain a non-autonomous Hamiltonian system. It is known[3,8,10] that in such systems the action variables remain (almost) adiabatically constant, while the shift in the angle variables splits into a *dynamic phase*

$$\Delta\boldsymbol{\theta}_{\mathrm{dyn}} = \int_0^{1/\epsilon} \boldsymbol{\omega}(\mathbf{I}(t), \epsilon t)dt,$$

which depends on the parameterization; and a *geometric phase*

$$\Delta\boldsymbol{\theta}_{\mathrm{geom}} = \int_\gamma \langle d_M \boldsymbol{\theta} \rangle,$$

depending only on the image $\gamma \subset M$ of the circuit. Here $\epsilon \geq 0$ is called the slowness parameter, $(\mathbf{I}, \boldsymbol{\theta})$ denote parameter dependent action-angle variables, $\boldsymbol{\omega} = \partial H_m / \partial \mathbf{I}$ the frequency vector, d_M the exterior derivative with respect to the parameters, and $\langle \cdot \rangle$ the operation of averaging over invariant phase space tori. $\Delta\boldsymbol{\theta}_{\mathrm{geom}}$ is also called the *classical Hannay angles.* Hannay[10] and Berry[4] explain the geometric phase by noting that when one writes Hamilton's equations in coordinates $(\mathbf{I}, \boldsymbol{\theta})$ there is an extra term in the equation giving the angle rate of change, owing to the dependence on parameters. Hannay[10] averages this equation to get, in the adiabatic limit ($\epsilon \to 0$), the total angle shift:

$$\Delta\boldsymbol{\theta} = \Delta\boldsymbol{\theta}_{\mathrm{dyn}} + \Delta\boldsymbol{\theta}_{\mathrm{geom}}. \tag{1}$$

$\Delta\boldsymbol{\theta}_{\mathrm{geom}}$ is the classical analog of a phase occurring in quantum mechanics discovered by Berry[3]. Simon[17] gave a geometric interpretation as the holonomy of a natural connection on a Hermitian line bundle. Montgomery[16], Golin, Knauf, and Marmi[8], and Marsden, Montgomery, and Ratiu[14] have given an analogous geometric interpretation for the classical Hannay angles, which extends to Hamiltonians that are invariant under *non*-Abelian parameter dependent phase space symmetries. The connection in this case is called the Hannay-Berry (HB) connection. For integrable Hamiltonians, it is defined by averaging the trivial connection on $M \times P$ (where P is the phase space) over the $\mathbb{T}^n$ action induced by the M dependent local action angle coordinates[14].

The Hannay angles (or equivalently the holonomy of the HB connection) have been computed in a number of simple examples such as: families of harmonic oscillators [3,16], the Foucault pendulum

[10,11,16], and the ball in the rotating hoop [11,14]. The goal of this paper is to compute curvature and holonomy of the HB connection in an example which exhibits more complexity: the asymmetrical free rigid body. In this case the parameter is a positive definite symmetric matrix, the inertia tensor, determined by the mass distribution of the body. Each such matrix induces a completely integrable system on the phase space $T^*SO(3) \cong SO(3) \times \mathbb{R}^3 = \{(\text{orientations}, \text{body angular momenta})\}$, via the corresponding kinetic energy function. By allowing the inertia tensor to vary with time, we obtain the non-autonomous Hamiltonian system which is the subject of the present work.

(FIX THIS!) This system is a good approximation to the dynamics of a slowly deforming free space structure. The equations for a slowly deforming body contain a higher order term in the slowness parameter ϵ, arising from the kinetic energy of the moving parts. If the masses undergo an adiabatic cycle the new state of the system can be approximated by calculating the dynamic and geometric phases and using (1). The dynamic phase is (typically) computed numerically and is $\mathcal{O}(1/\epsilon)$, the same order as the time scale of the cycle. The geometric phase is $\mathcal{O}(1)$ and is intrinsic to the physical paths of the moving parts. The reader is referred to [18] for further details of this approximation.

1.2 Results

The main result of this paper is the calculation of the HB connection one-form coefficients for the rigid body (Propositions 4.1 and 4.2). We find explicit formulas for all but one of the connection form coefficients. The remaining coefficient is determined by a P.D.E. whose coefficients and Right hand side are elliptic integrals (Equation (16)). As a corollary we obtain Theorem 4.1 giving the curvature form, excluding the three components which involve the non-explicit connection coefficient. Since the parameter space is six dimensional, there are fifteen terms in the curvature form. Among the twelve known terms ten are zero, indicating the presence of many loops with trivial holonomy.

In general given an explicit connection form it is a non-trivial (often impossible) problem to find analytic expressions for the holonomy. (We do not consider path ordered exponentials to be analytic expressions since they are in essence *defined* as the solutions to the parallel transport equations.) Analytic expressions usually are available only when the situation "Abelianizes." In our case this happens for certain loops lying in special submanifolds of the parameter space. With this in mind we are able to obtain explicit formulae for the holonomy (geometric phases) for two particular classes of loops, namely (I) those in which the principal axes undergo a rotation while the eigenvalues are fixed; and (II) those in which the axes are fixed while the eigenvalues vary. The holonomy of other loops could be determined numerically using the above expressions for the connection form.

In §4.5 we prove the following results concerning parallel transport and holonomy.

Theorem 1.1 *Consider a path $m(t)$ in which the inertia tensor is rotated about a fixed vector $\xi \in \mathbb{R}^3$. The time t parallel transport map of the HB connection acts on the body angular momentum vector $z_0 \in \mathbb{R}^3$ by a composition of two motions: first let z_0 evolve along the frozen rigid body dynamics (i.e. $m = m(0)$ in equation (3)) for time $u(z_0)t$ (where the function u is defined in §4.5 equation (29)); then rotate the resulting vector about ξ by t radians.*

Corollary 1.1 *Consider the loop given by rotation of the inertia tensor about ξ by 2π radians. The holonomy operator acts on z_0 by evolving along the rigid body trajectory (corresponding to $m(0)$) for time $2\pi u(z_0)$.*

Corollary 1.2 *Consider the loop given by rotation of the inertia tensor about ξ by $\pm\pi$ radians, where ξ is parallel to one of the principal axes of $m(0)$. If ξ is parallel to either of two principal axes then the holonomy operator rotates z_0 by π radians about that axis. If ξ is parallel to the remaining axis, z_0 is transported for time $\pi u(z_0)$ along the rigid body dynamics, then rotated by π radians* *about the given axis. Which principal axis yields the distinguished holonomy is determined by the action-angle coordinate domain from which z_0 is selected.*

We remark that Corollary 1.2 is analogous to results obtained for the harmonic oscillator[3,16]. For rotations of the quadratic form defining the harmonic oscillator the Hannay angle is a rotation of the phase plane by π radians. As a consequence of Theorem 4.1 giving the curvature form, we obtain

Theorem 1.2 *For loops consisting of inertia tensors which are simultaneously diagonalizable, the Hannay angles are zero.*

This is again analogous to the harmonic oscillator in that dialations of the quadratic form yield zero Hannay angle. Theorem 1.2 has the following physical interpretation regarding the free space structure discussed earlier. If the adjustable masses undergo an adiabatic cycle in such a way that the principal axes of the inertia tensor are fixed (e.g. if the masses move along the principal axes) then the change in state of the system is given by the dynamic phase only.

The set of inertia tensors with double eigenvalues has two components in the full space of inertia tensors; namely those where the double eigenvalue is the smaller, or the larger of the two eigenvalues, respectively. We show in §4.6 that the HB connection can be extended to one of these components, but not both simultaneously. The reason for this is that as two eigenvalues collide, some of the action-angle coordinate domains disappear. When two eigenvalues are equal the system is an axially symmetric body. The parallel transport equations are particularly simple in this case and the phases are computed in §4.6.

The free rigid body, while completely integrable is also completely integrable in the non-Abelian sense[7]. Equivalently, the system admits more than the required three integrals, which necessarily then do not Poisson commute. The energy and the three components of angular momentum are independent (but non-commuting) first integrals, and hence the generic trajectories lie on a two dimensional submanifold of the six dimensional phase space $SO(3) \times \mathbb{R}^3$. This submanifold is a 2-torus[1,2]. Three commuting integrals are given by: an arbitrarily chosen component of angular momentum, the length of the angular momentum, and the energy. The freedom in choosing the first integral implies that trajectories lie in many different invariant 3-tori, and hence on their intersection, the invariant 2-torus mentioned above. After passing to action-angle variables this torus is parametrized by two of the angles[18,19]. Thus $\mathbb{T}^2$ acts in a Hamiltonian manner on the phase space and therefore defines an HB connection via the averaging procedure. We show in §4.3 that this connection is in fact identical to the one defined by the $\mathbb{T}^3$ action.

Acknowledgments

I wish to express my sincere thanks to Tudor Ratiu and Richard Montgomery for many discussions regarding this problem. Also I thank Jerry Marsden for a number of helpful comments. Special thanks are also due to Debra Lewis for her detailed reading of several early drafts, and for much good advice on the exposition.

2 The Hannay-Berry Connection

We review here basic definitions and results concerning the Hannay-Berry connection which generalized the Hannay angles to non-integrable Hamiltonians. Although our interest is in the rigid body, which *is* completely integrable, the results here provide a global setting for the problem as well as facilitate the computations to follow. The proofs are omitted and can be found in [8,14,16]. Our exposition closely follows that in [16].

2.1 Families of Hamiltonian Group Actions

Let (P, ω) be a symplectic manifold, G a Lie group with Lie algebra $\mathfrak{g}$, and M a manifold which we call the parameter space. Let π_M and π_P denote projections of $M \times P$ onto M and P respectively, and suppose $2n = \dim(P)$, and $k = \dim(M)$. Let $E \subset M \times P$ be a smooth submanifold for which $\pi_M|E : E \to M$ is a smooth (not necessarily trivial) subbundle, and each fiber $E_m = \pi_M^{-1}(m) \cap E$ is an open submanifold of $\pi_M^{-1}(m) = \{m\} \times P$. Thus E_m is a symplectic manifold with symplectic form $\pi_P^* \omega | E_m$.

Definition 2.1 *A smooth action of G on E is called a family of Hamiltonian G actions if the following are satisfied:*

1. *G preserves the fibers E_m of $\pi_M|E$.*
2. *The action restricted to each fiber is symplectic.*
3. *The action admits a parameterized momentum map $I : E \to \mathfrak{g}^*$.*

We elaborate briefly on (3). Let d_P and d_M denote the exterior derivatives in the P and M directions respectively. That is, for $f \in C^\infty(E)$

$$d_P f = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q^i} dq^i + \frac{\partial f}{\partial p_i} dp_i \right) \qquad \text{and} \qquad d_M f = \sum_{j=1}^{k} \frac{\partial f}{\partial m_j} dm_j,$$

where $\{q^i, p_i\}_{i=1}^n$ and $\{m_j\}_{j=1}^k$ are local coordinates on P and M respectively. The Hamiltonian vector field of $f \in C^\infty(E)$ is defined as the unique vector field $X_f \in \mathfrak{X}(E)$ which is vertical with respect to $\pi_M|E$ and satisfies

$$\pi_P^* \omega(X_f, \cdot) = d_P f.$$

For $\xi \in \mathfrak{g}$, let I^ξ denote the function on E given by $I^\xi(m, x) = \langle I(m, x), \xi \rangle$, where $\langle \cdot, \cdot \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ is the natural paring. Then condition (3) means that for each $\xi \in \mathfrak{g}$,

$$X_{I^\xi} = \xi_E,$$

where $\xi_E(m, x) = \frac{d}{dt}\Big|_{t=0} \exp t\xi \cdot (m, x)$ is the infinitesimal generator of the G action on E corresponding to ξ. (See [1,15] for the basic theory regarding momentum maps.)

A trivial example is given by a Hamiltonian action of G on P with momentum map $\mathbf{J} : P \to \mathfrak{g}^*$. Let G act trivially on M and take the diagonal action on $E := M \times P$. In this case $I(m, x) = \mathbf{J}(x)$ and there is no dependence on the "parameters" M. We will be concerned primarily with Hamiltonian actions which depend nontrivially on a parameter.

It will be helpful to keep the following example in mind while reading §3.2 and during the calculations in Section §4.

Example. Let $H \in C^\infty(M \times P)$. Since X_H is vertical, the fibers of π_M are invariant under the flow. We may consider X_H to be a Hamiltonian system on $P \cong \pi_M^{-1}(m)$ which depends on the parameter $m \in M$. We say that H defines a *parameter dependent integrable system* if for each $m \in M$, $H(m, \cdot) \in C^\infty(P)$ is integrable in the usual sense. This means that there are functions $f_1, \ldots, f_n = H \in C^\infty(M \times P)$ such that for each $m \in M$, $\{f_i(m, \cdot), f_j(m, \cdot)\} = 0$ (using the induced Poisson structure on $\pi_M^{-1}(m)$), and $d_P f_1(m, x) \wedge \ldots \wedge d_P f_n(m, x) \neq 0$ for almost every $x \in P$. By the Arnold-Liouville Theorem[2] the regular compact connected level sets of $f(m, \cdot) = (f_1(m, \cdot), \ldots, f_n(m, \cdot))$ are diffeomorphic to $\mathbb{T}^n$ and about such a torus there exist local action-angle coordinates (I, Θ). That is we have independent functions $I = (I_1, \ldots, I_n)$ (actions) locally defined in (m, x), whose

Hamiltonian flows are 2π-periodic. The angle variables are constructed by selecting a Lagrangian submanifold transversal to the above tori and transporting it by the flow of I_i. The angle coordinate Θ_i of a point is then the time at which the transported submanifold reaches the given point. Also we have a local diffeomorphism ϕ, on $\mathbb{R}^n$, depending on $m \in M$, such that $f(m,x) = \phi(m, I(m,x))$. In particular H is a function of m and I.

Assume for the moment that I is globally defined. Then by transporting points along the trajectories of $X_{I_1}, \ldots, X_{I_n}$ (or equivalently by advancing the angle coordinates conjugate to $I_1, \ldots, I_n$) we have an action of $\mathbb{T}^n$ on $M \times P$. We see readily that this is a family of Hamiltonian $\mathbb{T}^n$ actions. Indeed, (1) in the definition is satisfied since X_{I_i} $(1 \leq i \leq n)$ is vertical, and (2) holds since Hamiltonian flows are necessarily symplectic. By definition of the $\mathbb{T}^n$ action we have for $\xi \in \mathbb{R}^n = \mathfrak{Lie}(\mathbb{T}^n)$,

$$\xi_{M\times P} = \xi_1 X_{I_1} + \cdots + \xi_n X_{I_n} = X_{\xi_1 I_1 + \cdots + \xi_n I_n} = X_{I\xi},$$

showing that (3) holds.

Now suppose that the action-angle coordinates are *not* globally defined. The Arnold-Liouville Theorem implies that for each parameter value $m \in M$, there is an open submanifold $P(m) \subset P$ on which $f(m, \cdot) = (f_1(m,\cdot), \ldots, f_n(m,\cdot))$ is a proper submersion. Each level set of $f(m,\cdot) \mid P(m)$ is a disjoint union of Liouville tori, and each connected component of $P(m)$ is a union of action angle chart domains. Typically $P(m)$ is obtained from P by removing certain submanifolds of codimension at least one, so that $P(m)$ is dense in P. As $m \in M$ is allowed to vary, the connected components of $P(m)$ may move, change topology, or even disappear. Let us assume that the number of components remains constant for all $m \in M$; say

$$P(m) = \bigcup_{i=1}^{k} P_i(m),$$

where the $P_i(m)$ are connected, open, and pairwise disjoint. If we follow the i^{th} component as m executes a loop in M, it may be that it does not return to itself. In other words the labeling map from $\mathcal{P}_m :=$ {components of $P(m)$} to $\{1, \ldots, k\}$ may be defined only locally on M. Thus we have an action of the loop group of M (based at m) on $\mathcal{P}_m$. In fact this action depends only on the homotopy class of the loop and hence $\pi_1(M,m)$ acts on $\mathcal{P}_m$. When considering the rigid body we will sometimes restrict our attention to a single orbit of this action. Now define

$$E = \{(m,x) \in M \times P \mid x \in P(m)\}$$

and consider the subbundle $\pi_M|E : E \to M$, with fibers $E_m = \{m\} \times P(m)$. The previous discussion indicates that $\pi_M|E$ may not be trivial. Assume that each component $P_i(m)$ admits a single action-angle chart (which is what happens for the rigid body.) Just as before, $\mathbb{T}^n$ acts in a Hamiltonian manner on $P_i(m)$, $1 \leq i \leq k$, and hence also on the fiber E_m, thus forming a family of Hamiltonian $\mathbb{T}^n$ actions on E.

2.2 Definition and Geometric Properties of the HB Connection

Let $E \subset M \times P$ be equipped with a family of Hamiltonian G actions, and assume G is compact and connected. We will denote the action by $\Phi_g(m,x)$ for $g \in G$, $(m,x) \in E$. Let dg denote normalized Haar measure on G, and let σ be a tensor field defined along (not necessarily on) a G invariant submanifold of E. The average of σ over G is the tensor field of the same type defined by $\langle\sigma\rangle = \int_G \Phi_g^* \sigma \, dg$. Observe that if σ is G-invariant, which means that $\Phi_g^*\sigma = \sigma$ for $g \in G$, then $\langle\sigma\rangle = \sigma$. Conversely $\langle\sigma\rangle$ is itself G-invariant by the translation invariance of Haar measure, so that $\langle\sigma\rangle = \sigma$ implies $\Phi_g^*\sigma = \sigma$. Note that the map $\sigma \mapsto \langle\sigma\rangle$ is $\mathbb{R}$ linear, and in fact linear over the ring of G-invariant functions.

Let $v \in T_m M$ and let $v \oplus 0$ denote the vector field along E_m whose value at (m,x) is $(v,0) \in T_{(m,x)}E$. (i.e. $v \oplus 0$ is the horizontal lift of v by the trivial connection on $\pi_M : M \times P \to M$.)

Definition 2.2 *The Hannay-Berry (HB) connection is the Ehresman connection on $\pi_M|E : E \to M$ whose horizontal lift is given by*

$$\mathrm{Hor}_{(m,x)}(v) = \langle v \oplus 0 \rangle (m,x).$$

Note that if the G action is independent of m, then $\Phi_g^*(v \oplus 0) = v \oplus 0$ whence $\langle v \oplus 0 \rangle = v \oplus 0$, and the HB connection is trivial in this case.

The motivation for this definition comes from the example of the previous section. Consider a parameter dependent integrable system with local action angle variables $(\mathbf{I}, \boldsymbol{\theta})$. The following is proved in [14,16].

Theorem 2.1 *If $\langle d_M \mathbf{I} \rangle = 0$, then for sufficiently small loops, the holonomy of the HB connection is the Hannay angles.*

Remark. Sufficiently small in this case means that the loop must lie in a region in M over which the m dependent action-angle coordinates can be consistently defined. Thus the HB connection serves to generalize the Hannay angles to non-integrable systems which admit a family of Hamiltonian G actions.

For $m \in M$ define $P(m) := \{x \in P \mid (m,x) \in E_m\} = \pi_P(E_m)$.

Definition 2.3 *Let the bundle $\pi_M : E \subset M \times P \to M$ be equipped with a family of Hamiltonian G actions, with parametrized momentum map $\mathbf{I} : E \to \mathfrak{g}^*$. A Hamiltonian connection for this family is an Ehresman connection on $\pi_M|E$ satisfying:*

1. *$D\mathbf{I} = 0$, where D denotes the covariant differentiation operator.*
2. *For each $v \in T_m M$, there is a function $\mathbf{K} \cdot v \in C^\infty(P(m))$ such that the horizontal lift is given by*
$$\mathrm{Hor}_{(m,x)}(v) = v \oplus X_{\mathbf{K}\cdot v}(x),$$
for $x \in P(m)$.
3. *$\langle \mathbf{K} \cdot v \rangle$ is a constant, which can be taken to be zero.*

Property (1) says that $\mathbf{I}$ is constant along the horizontal lift $c(t) = (m(t), x(t))$ of a curve $m(t) \in M$. By (2), the parallel transport equations are

$$c'(t) = \mathrm{Hor}_{c(t)} \cdot m'(t) = m'(t) \oplus X_{\mathbf{K}\cdot m'(t)}(x(t)),$$

so that

$$x'(t) = X_{\mathbf{K}\cdot m'(t)}(x(t)).$$

Thus $x(t)$ is the flow of a time dependent Hamiltonian vector field on $P(m(t))$. One says that *parallel transport is Hamiltonian.* Regarding (3), note that by (2) $(\mathbf{K} \cdot v)(m,x)$ need only be defined up to an additive constant. Thus we may replace $\mathbf{K} \cdot v$ by $\mathbf{K} \cdot v - \langle \mathbf{K} \cdot v \rangle$ which has average zero.

If $Z \in \mathfrak{X}(M)$ we can regard $\mathbf{K} \cdot Z$ as a function on E which is defined up to addition of a smooth function on M. The map $Z \mapsto \mathbf{K} \cdot Z$ can clearly be taken to be linear. The operator $\mathbf{K}$ is thus a one-form on M taking values in the ring $C^\infty(E)/C^\infty(M)$, and determines the connection. $\mathbf{K}$ is called the *Hamiltonian one-form* for the connection. The following is proved in [14,16].

Theorem 2.2 *A family of Hamiltonian G actions admits a Hamiltonian connection if and only if the adiabatic condition $\langle d_M \mathbf{I} \rangle = 0$ holds. Furthermore, if such a connection exists it is unique and equals the Hannay-Berry connection. In particular the HB connection is Hamiltonian.*

Let $v \in T_m M$ and $\xi \in \mathfrak{g}$. Using (2) of the definition, we have $D\mathbf{I}^\xi \cdot v = d\mathbf{I}^\xi \cdot \mathrm{Hor}(v) = d\mathbf{I}^\xi \cdot (v \oplus X_{\mathbf{K}\cdot v}) = d_M \mathbf{I}^\xi \cdot v + d_P \mathbf{I}^\xi \cdot X_{\mathbf{K}\cdot v} = d_M \mathbf{I}^\xi \cdot v + \{\mathbf{I}^\xi, \mathbf{K}\cdot v\}$, so by (1), $\mathbf{K}\cdot v$ necessarily satisfies $d_M \mathbf{I}^\xi \cdot v + \{\mathbf{I}^\xi, \mathbf{K}\cdot v\} = 0$ for each $\xi \in \mathfrak{g}$. This PDE is not sufficient however, to determine the function $\mathbf{K}\cdot v$. Instead we have

Proposition 2.1 *Suppose one can find a function $\widetilde{\mathbf{K}}\cdot v \in C^\infty(P(m))$ such that for all $\xi \in \mathfrak{g}$*

$$d_M \mathbf{I}^\xi \cdot v + \{\mathbf{I}^\xi, \widetilde{\mathbf{K}}\cdot v\} = 0. \tag{2}$$

Then $\mathbf{K}\cdot v = \widetilde{\mathbf{K}}\cdot v \;-\langle \widetilde{\mathbf{K}}\cdot v\;\rangle$ is the Hamiltonian 1-form for the HB connection.

We say that $\widetilde{\mathbf{K}}\cdot v$ *almost generates parallel translation.* This proposition provides a procedure for calculating the horizontal lift operator, which is sometimes more feasible than computing $\mathrm{Hor}\cdot v$ directly from its definition. This is because it is easier to average a function than a vector field. (For the latter, one must differentiate the group action $\Phi_g : E \to E,\ g \in G$ with respect to $x \in P(m)$.) Of course we have the added step of solving (2), but this PDE is sometimes quite simple.

If the family of Hamiltonian G actions admits additional symmetries the above procedure simplifies considerably.

Proposition 2.2 *Suppose another Lie group H, with Lie algebra $\mathfrak{h}$, acts on P in a Hamiltonian manner with equivariant momentum map $\mathbf{J} : P \to \mathfrak{h}^*$. Suppose also that H acts on M in such a way that the corresponding diagonal action $h\cdot(m,x) = (h\cdot m, h\cdot x),\ h \in H$, preserves $\mathbf{I} : E \to \mathfrak{g}^*$. Then the action of $\mathbf{K}$ on vectors tangent to the H orbits in M is given by*

$$\mathbf{K}\cdot \eta_M = \mathbf{J}^\eta - \langle \mathbf{J}^\eta \rangle,$$

where $\eta \in \mathfrak{h}$, and η_M denotes the infinitesimal generator of the H action on M.

Propositions 2.1 and 2.2 will be our main tools for computing the Hamiltonian one-form $\mathbf{K}$ for the rigid body. Their proofs can be found in [16].

2.3 Curvature of the HB Connection

Recall that the curvature of an Ehresman connection is the vertical bundle valued two form given by the covariant derivative of the connection one form. (See [14] for a concise treatment of Ehresman connections.) The curvature induces a two form on the base (also called the curvature) by composition with the horizontal lift operator. Maintaining the notation of previous sections, let $\mathrm{curv}(V_1, V_2)(m,x)$ denote the curvature of an Ehresman connection on $\pi_M : E \subset M \times P \to M$ applied to $V_1, V_2 \in T_{(m,x)}E,\ (m,x) \in E$. The induced form is then

$$\overline{\Omega}(v_1, v_2)(m,x) = \mathrm{curv}(\mathrm{Hor}\cdot v_1, \mathrm{Hor}\cdot v_2)(m,x),$$

where $v_1, v_2 \in T_m M$.

For the HB connection the form $\overline{\Omega}$ is Hamiltonian in the following sense.

Theorem 2.3 *Let $m \in M$ and $v_1, v_2 \in T_m M$. Then there is a smooth function $\Omega(v_1, v_2) : E_m \to \mathbb{R}$ such that*

$$\overline{\Omega}(v_1, v_2)(m,x) = X_{\Omega(v_1,v_2)}(m,x).$$

Ω is given by

$$\Omega(v_1, v_2) = \langle \{\mathbf{K}\cdot v_1, \mathbf{K}\cdot v_2\} \rangle.$$

The proof can be found in [14,16]. We will abuse the terminology slightly and call Ω the curvature of the HB connection.

Remark. $\Omega(v_1, v_2)$ is defined only up to addition of a constant, so for $Z_1, Z_2 \in \mathfrak{X}(M)$, $\Omega(Z_1, Z_2)$ is a smooth function on E defined up to addition of a smooth function on M. Thus Ω is a two-form on M taking values in the ring $C^\infty(E)/C^\infty(M)$.

3 Rigid Body Dynamics

In section 3.1 we review the necessary facts concerning the dynamics of rigid body motion, including the solutions to the Euler equations which will be needed for later calculations. The reader familiar with the basic material may safely skip §3.1 or just skim to establish notation. §3.2 presents a complete set of action integrals for the rigid body, including one which does not seem to appear in the standard literature[1,2,12,13]. In §3.3 we study the space of inertia tensors.

3.1 The Euler Equations and Left Trivialization

The configuration space of the rigid body is the Lie group $SO(3)$ consisting of 3×3 orientation preserving orthogonal matrices. Each such matrix represents a rotation of the body in space from a fixed reference configuration. Let $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$ denote a coordinate frame in $\mathbb{R}^3$ attached to the body, and let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be an inertial frame. The origin of both frames coincides with the center of mass of the body. The phase space of the rigid body is the cotangent bundle $T^*SO(3)$ which is naturally diffeomorphic to $SO(3) \times \mathbb{R}^3$ via the body coordinate system, or left trivialization of $T^*SO(3)$. For $(g, z) \in SO(3) \times \mathbb{R}^3$, g represents a configuration of the body and z represents the angular momentum as seen from the body fixed frame $\{\mathbf{E}_i\}$. We also identify the Lie algebras $(so(3), [\ ,\ ])$ and $(\mathbb{R}^3, \times)$ via the map $a \in \mathbb{R}^3 \mapsto \hat{a} \in so(3)$, where $\hat{a}b = a \times b$ for any $b \in \mathbb{R}^3$. One checks that $\widehat{ga} = g\hat{a}g^{-1}$ for $a \in \mathbb{R}^3$, $g \in SO(3)$. $SO(3)$ acts on $SO(3) \times \mathbb{R}^3$ by $g \cdot (h, z) = (gh, z)$ for $g, h \in SO(3)$, $z \in \mathbb{R}^3$, which is the cotangent lift of the left action of $SO(3)$ on itself given by left multiplication. Thus a function on $SO(3) \times \mathbb{R}^3$ is left invariant exactly when it depends only on $z \in \mathbb{R}^3$. The Poisson bracket of two left invariant functions $f_1,\ f_2$ is given by

$$\{f_1, f_2\}(z) = -\langle z, \nabla f_1(z) \times \nabla f_2(z)\rangle,$$

which is the standard left Lie Poisson structure on $\mathbb{R}^3$[15]. The Hamiltonian vector field of a left invariant function is

$$X_f(g, z) = (g \cdot \widehat{\nabla f(z)}, z \times \nabla f(z)).$$

We denote by $\Phi_t^f(g, z)$ the flow of X_f with initial point (g, z). The infinitesimal generator of the lifted left action corresponding to $\xi \in \mathbb{R}^3$ is $\xi_P(g, z) = \frac{d}{dt}\Big|_{t=0} \exp t\hat{\xi} \cdot (g, z) = (\hat{\xi}g, 0)$. The equivariant momentum map $\mathbf{J} : P \to \mathbb{R}^3$ for the lifted left action is given by right translation to the identity[1,15], which is $\mathbf{J}(g, z) = gz$ in body coordinates. Its value is the angular momentum of the system as seen from the inertial frame $\{\mathbf{e}_i\}$. Thus $X_{J^\xi}(g, z) = (\hat{\xi}g, 0)$, where $\mathbf{J}^\xi(g, z) := \langle \mathbf{J}(g, z), \xi\rangle = \langle gz, \xi\rangle$, and the Hamiltonian flow of $\mathbf{J}^\xi$ is given by $\Phi_t^{J^\xi}(g, z) = (\exp t\hat{\xi} \cdot g, z)$.

The rigid body Hamiltonian is the left invariant function $H_m(z) = \frac{1}{2}\langle z, m^{-1}z\rangle$. Here m denotes the inertia tensor, a positive definite symmetric matrix. If m is diagonal with respect to $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$, say $m = diag(\lambda_1, \lambda_2, \lambda_3)$, then

$$H_m(z) = \frac{1}{2}\left(\frac{z_1^2}{\lambda_1} + \frac{z_2^2}{\lambda_2} + \frac{z_3^2}{\lambda_3}\right),$$

with coordinates z_i relative to $\{\mathbf{E}_i\}$. The λ_i are the principal moments of inertia, and the vectors $\mathbf{E}_i$ are the principal axes. One finds that $X_{H_m}(g,z) = (g \cdot \widehat{m^{-1}z}, z \times m^{-1}z)$. Thus the flow $\Phi_t^{H_m}(g_0, z_0)$ of X_{H_m} is obtained by solving:

(3) $$\dot{z} = z \times m^{-1}z$$

and

(4) $$\dot{g} = g \cdot \widehat{m^{-1}z},$$

with initial conditions $g(0) = g_0$ and $z(0) = z_0$. Equations (3) are the Euler equations for a rigid body, and their solution, $z(t)$, is called a reduced trajectory. The solution of (4), is called the reconstructed trajectory, and gives the attitude of the body with respect to the inertial coordinate frame. (See [1,15] for a discussion of Lie-Poisson reduction and reconstruction.) A fixed point of (3) corresponds to steady rotation of the body about its spatial angular momentum vector.

In the calculations that are to follow we will need the explicit solutions to the Euler equations (3). Assume that $\lambda_1 \geq \lambda_2 \geq \lambda_3$. The functions H_m and $\mathbf{J}$ are conserved so $r = \|\mathbf{J}(g,z)\|$ and $c = H_m(z)$ are constant along the solutions of (3). Thus $z(t)$ lies in the intersection of the sphere $S_r^2 = \{z \in \mathbb{R}^3 \mid \|z\| = r\}$ and the ellipsoid $H_m^{-1}(c)$ for all t. If the body is spherical ($\lambda_1 = \lambda_2 = \lambda_3$) then $S_r^2 = H_m^{-1}(c)$ and each point on S_r^2 is a fixed point of (3). If $\lambda_1 = \lambda_2 > \lambda_3$ then $H_m^{-1}(c)$ is an ellipsoid of revolution and the equator $S_r^2 \cap \{z \in \mathbb{R}^3 \mid z_3 = 0\}$ consists of unstable fixed points, while the poles $\{\pm r\mathbf{E}_3\}$ are stable fixed points and the remaining points of S_r^2 lie in closed orbits. This case (as well as the case $\lambda_1 > \lambda_2 = \lambda_3$) is known as the axially symmetric body.

Now assume that the body is asymmetrical (i.e. $\lambda_1 > \lambda_2 > \lambda_3$.) One checks that the condition $S_r^2 \cap H_m^{-1}(c) \neq \emptyset$ implies $\lambda_1 \geq r^2/2c \geq \lambda_3$. If $r^2/2c = \lambda_1$ or λ_3 then $S_r^2 \cap H_m^{-1}(c) = \{\pm r\mathbf{E}_1\}$ or $\{\pm r\mathbf{E}_3\}$ which are the stable fixed points of (3). If $r^2/2c = \lambda_2$ then $z(t)$ lies in one of the planes $z_3 = \sqrt{\eta}\, z_1$ or $z_3 = -\sqrt{\eta}\, z_1$, where $\eta = \lambda_3(\lambda_1 - \lambda_2)/\lambda_1(\lambda_2 - \lambda_3)$. We call these the *separatrix planes.* They intersect the sphere S_r^2 in the two unstable fixed points $\{\pm r\mathbf{E}_2\}$, and in the four heteroclinic orbits, which connect the two unstable equilibria. If $\lambda_1 > \lambda_2 > r^2/2c > \lambda_3$ or $\lambda_1 > r^2/2c > \lambda_2 > \lambda_3$ then $S_r^2 \cap H_m^{-1}(c)$ consists of a pair of closed orbits on the sphere.

In the case $\lambda_1 > \lambda_2 > r^2/2c > \lambda_3$ the solution is given by

(5) $$\begin{aligned} z_1(t) &= P\,\mathrm{cn}(s(t-t_0),k) \\ z_2(t) &= -Q\,\mathrm{sn}(s(t-t_0),k) \\ z_3(t) &= \pm R\,\mathrm{dn}(s(t-t_0),k), \end{aligned}$$

where P, Q, R, s, k are positive and satisfy

(6) $$P^2 = \frac{\lambda_1(r^2 - 2c\lambda_3)}{(\lambda_1 - \lambda_3)} \qquad Q^2 = \frac{\lambda_2(r^2 - 2c\lambda_3)}{(\lambda_2 - \lambda_3)} \qquad R^2 = \frac{\lambda_3(2c\lambda_1 - r^2)}{(\lambda_1 - \lambda_3)}$$

$$s^2 = \frac{(\lambda_2 - \lambda_3)(2c\lambda_1 - r^2)}{\lambda_1\lambda_2\lambda_3} \qquad k^2 = \frac{(\lambda_1 - \lambda_2)(r^2 - 2c\lambda_3)}{(\lambda_2 - \lambda_3)(2c\lambda_1 - r^2)}.$$

Here $\mathrm{cn}(\cdot,k)$, $\mathrm{sn}(\cdot,k)$, $\mathrm{dn}(\cdot,k)$ denote the Jacobi elliptic functions of modulus k. The constants r, c, t_0 and the sign in z_3 are chosen in accordance with initial conditions. The solutions (5) are periodic with period $T = 4s^{-1}K(k)$ where $K(k) = \int_0^{\pi/2}(1 - k^2\sin^2\phi)^{-1/2}d\phi$ is the complete elliptic integral of the first kind. For the solution in case $\lambda_1 > r^2/2c > \lambda_2 > \lambda_3$ the reader is referred to Lawden[13] or [18]. The group equations (4) can also be solved explicitly by transforming to Euler angles on $SO(3)$. Details can be found in Landau and Lifshitz[12] or Lawden[13].

3.2 Action-Angle Coordinates

Here we obtain a complete set of independent action integrals: $\mathbf{I} = (I_1, I_2, I_3)$. We refer the reader to [18,19] for the proofs of the following facts. Let $I_1(g,z) := \langle \mathbf{J}(g,z), \mathbf{e}_3 \rangle = \langle gz, \mathbf{e}_3 \rangle$ and $I_2(g,z) := \|\mathbf{J}(g,z)\| = \|z\|$. One verifies that I_1, I_2, and H_m Poisson commute. Note the definition of I_1 is arbitrary in that we could have taken $I_1 = \langle \mathbf{J}, \mathbf{u} \rangle$ with $\mathbf{u}$ any unit vector in $\mathbb{R}^3$. We choose $\mathbf{u} = \mathbf{e}_3$ merely for definiteness. This ambiguity implies that a typical trajectory lies on many *different* invariant 3-tori in the phase space $SO(3) \times \mathbb{R}^3$, and hence on their intersection, an invariant 2-torus. Therefore the system has a resonance which is independent of initial conditions. Define $f := (I_1, I_2, H_m) : SO(3) \times \mathbb{R}^3 \to \mathbb{R}^3$ and let

$$W := \{(g,z) \in SO(3) \times \mathbb{R}^3 \mid gz \times \mathbf{e}_3 \neq 0, \text{ and } z \times m^{-1} z \neq 0\}.$$

Observe that W is the complement of the set of points at which the spatial angular momentum vector is parallel to $\mathbf{e}_3$ or the body angular momentum is parallel to one of the principal axes. W is open and dense in $SO(3) \times \mathbb{R}^3$. It can be shown that $f|_W$ is a submersion so that f defines a completely integrable system.

We now remove those points $(g,z) \in W$ for which $f^{-1}(f(g,z)) \cap W$ is non-compact. Recall from §3.1 that the separatrix planes in $\mathbb{R}^3$ are given by $z_3^2 = \eta z_1^2$. Define

$$\begin{aligned} U_1(m) &:= \{z \in \mathbb{R}^3 \mid z_3^2 < \eta z_1^2\} - \{z_1 \text{ axis}\} = \{z \in \mathbb{R}^3 \mid \lambda_1 > r^2/2c > \lambda_2\} \\ U_3(m) &:= \{z \in \mathbb{R}^3 \mid z_3^2 > \eta z_1^2\} - \{z_3 \text{ axis}\} = \{z \in \mathbb{R}^3 \mid \lambda_2 > r^2/2c > \lambda_3\}, \end{aligned}$$

where $r = \|z\|$ and $c = H_m(z)$ as in §3.1. Thus $U_i(m) \subset \mathbb{R}^3$ is open and consists of all closed orbits which encircle the z_i-axis ($i = 1,\ 3$). The connected components of $U_i(m)$ are

$$U_i^+(m) = \{z \in U_i \mid z_i > 0\} \qquad \text{and} \qquad U_i^-(m) = \{z \in U_i \mid z_i < 0\}.$$

Define the projection $\pi : SO(3) \times \mathbb{R}^3 \to \mathbb{R}^3$, $(g,z) \mapsto z$, and set $P_i^\pm(m) = \pi^{-1}(U_i^\pm(m)) \cap W$, for $i = 1, 3$. Each $P_i^\pm(m) \subset SO(3) \times \mathbb{R}^3$ is connected and open. Let

$$P(m) = P_1^+(m) \cup P_1^-(m) \cup P_3^+(m) \cup P_3^-(m).$$

It can be shown that $f|P(m)$ is a proper submersion and in fact each level set of f in $P(m)$ is a disjoint union of two Liouville tori (one in $P_i^+(m)$ the other in $P_i^-(m)$, $i = 1$ or 3.) Also each $P_i^\pm(m)$ admits a single action-angle chart [18].

The Hamiltonian vector fields for I_1 and I_2 are

$$X_{I_1}(g,z) = (\hat{\mathbf{e}}_1 g, 0), \qquad \text{and} \qquad X_{I_2}(g,z) = \left(g \frac{\hat{z}}{\|z\|}, 0\right),$$

with flows $\Phi_t^{I_1}(g,z) = ((\exp t\hat{\mathbf{e}}_1)g, z)$ and $\Phi_t^{I_2}(g,z) = \left(g \exp t\frac{\hat{z}}{\|z\|}, z\right) = \left((\exp t\frac{\widehat{gz}}{\|z\|})g, z\right)$, which are 2π periodic. In [18,19] it is shown that a third action integral is given by

$$I_3(g,z) := \frac{A(z)}{2\pi\|z\|} \qquad (z \in U_i(m)),$$

where $A(z)$ denotes the oriented surface area enclosed by the periodic trajectory of (3) passing through $z \in S_r^2$ ($r = \|z\|$.) Note that the curve $S_r^2 \cap H_m^{-1}(c)$ actually encloses two regions on the sphere. We take the region contained in one of the sets $U_i^\pm(m)$, i.e. the enclosed area does not contain any separatricies. The orientation is given by the direction of the orbit.

It is known that for one degree of freedom systems, an action integral is given by $(2\pi)^{-1}$ times the symplectic area enclosed by a periodic orbit through a given point. In fact I_3 above is an example of this. Applying Marsden-Weinstein reduction[1,15] to $\mathbf{J}^{-1}(\mu) \subset SO(3) \times \mathbb{R}^3$ for μ a regular value of $\mathbf{J}$, one obtains the reduced manifold S^2_r with reduced symplectic form r^{-1} times the standard oriented area form on S^2_r. Thus I_3 is an action integral for the reduced system.

Observe I_1 and I_2 do not depend on $m \in M$ so that $d_M I_1 = d_M I_2 = 0$. One can show directly that $\langle d_M I_3 \rangle = 0$ so that $\mathbf{I} = (I_1, I_2, I_3)$ satisfies the hypothesis of Theorem 2.2.

3.3 The Space of Inertia Tensors

Let M_1 be the set of all possible rigid body inertia tensors, i.e. real positive definite symmetric matrices whose eigenvalues satisfy $\lambda_i + \lambda_j > \lambda_k$ (i, j, k cyclic.) M_1 is an open subset of the six dimensional vector space of 3×3 symmetric matrices. Observe that the elements of M_1 are in one to one correspondence with the ellipsoids in $\mathbb{R}^3$ which are the level sets of the corresponding quadratic forms. It will be very helpful to keep this identification in mind when considering loops in the space of inertia tensors. Let $\Sigma \subset M_1$ be the set of matrices with exactly two distinct eigenvalues and $\Sigma' \subset M_1$ those with just one eigenvalue. Σ and Σ' are (regular) submanifolds of M_1 of dimensions four and one respectively (see Arnold[2] Appendix 10) and are therefore closed. The set $M = M_1 - (\Sigma \cup \Sigma')$ of matrices with three distinct eigenvalues is thus an open submanifold of M_1. M will be the parameter space in our problem.

There is a natural (trivial) fibration of M which will figure heavily in our calculation of the HB connection for the rigid body. The fibers are orbits of the conjugation action of $SO(3)$ on M: $m \mapsto gmg^{-1}$. In other words each fiber consists of all $m \in M$ with a fixed set of (distinct) eigenvalues. The base of the fibration (or rather any section parallel to the base) consists of all $m \in M$ which are diagonal with respect to a fixed set of principal axes. Thus we may write

$$M \cong B \times F,$$

where

$$B = \{\lambda \in \mathbb{R}^3 \mid \lambda_1 > \lambda_2 > \lambda_3 > 0 \text{ and } \lambda_i + \lambda_j > \lambda_k \ (i, j, k \text{ cyclic})\},$$

and

$$F = SO(3)/\{\text{diagonal matricies}\}.$$

It can be shown [18] that the homotopy group of F is the quaternion group $\{\pm 1, \pm i, \pm j, \pm k\}$ where i, j, k satisfy $i^2 = j^2 = k^2 = ijk = -1$.

****

In general gmg^{-1} is diagonal for some g, and $SO(3)_{gmg^{-1}} = g \cdot SO(3)_m \cdot g^{-1}$. Thus *all isotropy groups are conjugate.* It follows from a standard result of the theory of compact group actions (Corollary 2.5 p.309 of Bredon[5]) that

Lemma 3.1 *$M/SO(3)$ is a smooth manifold and the canonical projection $\pi : M \to M/SO(3)$ is a locally trivial fiber bundle with typical fiber $SO(3)/G_0$.*

In fact, as we'll see, $M/SO(3)$ is contractible and hence π is globally trivial. Define $\pi_B : M \to B$ by $m \mapsto \lambda = (\lambda_1, \lambda_2 \lambda_3)$ where λ_i $(1 \leq i \leq 3)$ are the distinct eigenvalues of m taken in descending order. Note that π_B is clearly onto and smooth since the eigenvalues are smooth functions of m. Since the action preserves eigenvalues, π_B drops to a smooth function on the quotient. That is, we have a smooth function $\phi : M/SO(3) \to B$ satisfying $\phi \circ \pi = \pi_B$. We claim that ϕ is a diffeomorphism. Observe that π_B admits a smooth section $\sigma : B \to M$, $\lambda = (\lambda_1, \lambda_2, \lambda_3) \mapsto \text{diag}(\lambda_1, \lambda_2, \lambda_3)$. Write $[m] = \pi(m)$ for the equivalence class of $m \in M$, so that $\phi([m]) = \pi_B(m)$. Then $(\pi \circ \sigma) \circ \phi([m]) = \pi(\sigma \circ \pi_B(m)) = \pi(m) = [m]$. Also $\phi \circ (\pi \circ \sigma) = \pi_B \circ \sigma = \text{identity}_B$, whence $\pi \circ \sigma : B \to M/SO(3)$ is a smooth inverse to ϕ. This shows that $M/SO(3) \cong B$, and in fact that π and π_B are isomorphic as fiber bundles.

Theorem 3.1 *$\pi : M \to M/SO(3)$ is a trivial fiber bundle isomorphic to $\pi_B : B \times F \to B$, with $F := SO(3)/G_0$. The fiber has homotopy group $\pi_1(F) \cong Q$, where Q is the quaternion group.*

Proof: The first statement follows from Lemma 3.1 and the preceding discussion. For the second statement, let $\mathbb{H}$ denote the quaternions and identify $S^3 \subset \mathbb{R}^4$ with the quaternions of unit length. The quaternion group is $Q = \{\pm 1, \pm i, \pm j, \pm k\}$. If $q = q_0 + q_1 i + q_2 j + q_3 k \in \mathbb{H}$, its conjugate is $\overline{q} = q_0 - q_1 i - q_2 j - q_3 k$, and its squared length is $|q|^2 = q\overline{q}$. The covering projection $\rho : S^3 \to SO(3)$ is defined as follows. The map $\mathbb{H} \to \mathbb{H}$, $x \mapsto qx\overline{q}$, $x \in \mathbb{H}$, $q \in S^3$ is orthogonal since it preserves lengths. It also preserves the purely real quaternions, and so also their orthogonal complement, $\text{span}\{i, j, k\} \cong \mathbb{R}^3$. Let $\rho(q) \in O(3)$ be the restriction of this map to $\mathbb{R}^3$: $\rho(q) \cdot v = qv\overline{q}$, for $v = v_1 i + v_2 j + v_3 k \in \mathbb{R}^3$. The matrix of $\rho(q)$ is computed to be

$$\rho(q) = \begin{pmatrix} 2(q_0^2 + q_1^2) - 1 & 2(q_1 q_2 - q_0 q_3) & 2(q_1 q_3 + q_0 q_2) \\ 2(q_1 q_2 + q_0 q_3) & 2(q_0^2 + q_2^2) - 1 & 2(q_2 q_3 - q_1 q_0) \\ 2(q_1 q_3 - q_0 q_2) & 2(q_2 q_3 + q_0 q_1) & 2(q_0^2 + q_3^2) - 1 \end{pmatrix}. \tag{7}$$

We see that $\det(\rho(q)) = 1$, whence $\rho(q) \in SO(3)$, and ρ is clearly smooth. Since $\dim(S^3) = \dim(SO(3)) = 3$, invariance of domain implies ρ is onto. For q, $p \in S^3$ and $v \in \mathbb{R}^3$ we have $\rho(qp) \cdot v = qpv\overline{qp} = q(pv\overline{p})\overline{q} = \rho(q) \circ \rho(p) \cdot v$, showing that ρ is a homomorphism. From (7) we obtain $\ker(\rho) = \{1, -1\} \subset Q$, and $\rho(Q) = G_0$. Thus the surjective homomorphism $\rho : S^3 \to SO(3)$ induces a diffeomorphism $\bar{\rho} : S^3/Q \to SO(3)/G_0$ of quotient manifolds, showing that $F \cong S^3/Q$. Now the natural projection $S^3 \to S^3/Q$ is a bundle with discrete fiber Q. The long exact sequence of homotopy groups arising from this fibering (see Gray[9]) is

$$\cdots \to \pi_{n+1}(S^3/Q) \to \pi_n(Q) \to \pi_n(S^3) \to \pi_n(S^3/Q) \to \pi_{n-1}(Q) \to \cdots,$$

which leads to $0 \to \pi_1(S^3/Q) \to \pi_0(Q) \to 0$, since $\pi_1(S^3) = \pi_0(S^3) = 0$. Thus $\pi_1(S^3/Q) \cong \pi_0(Q)$ and since Q is a discrete group, $\pi_0(Q) \cong Q$. Therefore $\pi_1(F) = Q$ and the proof is complete. ///

Under the identification $M \cong B \times F$, the orbits $F_\lambda := SO(3) \cdot \sigma(\lambda)$, $\lambda \in B$, correspond to the fibers $\{\lambda\} \times F \cong S^3/Q$ which are compact. Note that since B is contractible, M contracts onto a single fiber F_λ, and hence $\pi_1(M) = \pi_1(F_\lambda) = Q$. We can realize the isomorphism $Q \to \pi_1(F_\lambda)$ explicitly as follows. Define γ_1 to be the constant curve at $\sigma(\lambda) \in F_\lambda$ and

$$\begin{aligned} \gamma_{-1}(t) &= \exp(t\widehat{\xi}) \cdot \sigma(\lambda) && (0 \le t \le 2\pi), \\ \gamma_{\pm i}(t) &= \exp(\pm t\widehat{\mathbf{E}_1}) \cdot \sigma(\lambda) && (0 \le t \le \pi), \\ \gamma_{\pm j}(t) &= \exp(\pm t\widehat{\mathbf{E}_2}) \cdot \sigma(\lambda) && (0 \le t \le \pi), \\ \gamma_{\pm k}(t) &= \exp(\pm t\widehat{\mathbf{E}_3}) \cdot \sigma(\lambda) && (0 \le t \le \pi). \end{aligned}$$

In the definition of γ_{-1} we may take ξ to be any unit vector in $\mathbb{R}^3$. (If we choose some other unit vector $\eta \in \mathbb{R}^3$, then since $t \mapsto \exp t\hat{\xi}$ and $t \mapsto \exp t\hat{\eta}$ are homotopic in $SO(3)$, the loops $t \mapsto \exp t\hat{\xi} \cdot \sigma(\lambda)$ and $t \mapsto \exp t\hat{\eta} \cdot \sigma(\lambda)$ are homotopic in F_λ.) The isomorphism $Q \cong \pi_1(F_\lambda)$ is then given by $a \in Q \mapsto [\gamma_a]$, where $[\gamma_a]$ denotes the homotopy class of γ_a. We shall be interested in calculating the holonomy of the HB connection on loops belonging to these classes.

4 The Hannay-Berry Connection for the Rigid Body

As before $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$ denote inertial and body frames in $\mathbb{R}^3$ respectively. However, we will not always assume that $\{\mathbf{E}_i\}$ diagonalizes m. It will be necessary to consider a third frame, denoted by $\{\mathcal{E}_1(m), \mathcal{E}_2(m), \mathcal{E}_3(m)\}$, which diagonalizes the inertia tensor. Note that this basis cannot be defined in a consistent manner along certain non-contractible loops in M; namely those corresponding to the quaternions i, j, k under the isomorphism of the last section. When considering loops $m(t)$, we will take $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$ and $\{\mathcal{E}_1(m(t)), \mathcal{E}_2(m(t)), \mathcal{E}_3(m(t))\}$ to coincide initially. We define the sets $U_i^{\pm}(m) \subset \mathbb{R}^3$ and $P_i^{\pm}(m),\ P(m) \subset S(3) \times \mathbb{R}^3$ $(i = 1, 3)$ just as in §3.2 except that all coordinates are taken relative to the diagonalizing basis $\{\mathcal{E}_i(m)\}$. If for instance, the inertia tensor is rotated by π about $\mathbf{E}_1$ (which is a loop corresponding to $i \in Q$) then $U_3^+(m)$ and $U_3^-(m)$ change places while $U_1^+(m)$ and $U_1^-(m)$ return to their original positions. Similarly such a loop causes $P_3^+(m)$ and $P_3^-(m)$ to interchange while $P_1^+(m)$ and $P_1^-(m)$ return. Examination of other examples reveals that the orbits of the connected components $\{P_1^+(m),\ P_1^-(m),\ P_3^+(m),\ P_3^-(m)\}$ of $P(m)$ under the action of $\pi_1(M) \cong Q$ mentioned in the example of §2.1 are precisely

$$\{P_1^+(m), P_1^-(m)\} \qquad \text{and} \qquad \{P_3^+(m), P_3^-(m)\}.$$

In some of the subsequent calculations we restrict our attention to the orbit $\{P_3^{\pm}(m)\}$, i.e. we consider initial conditions $z \in \mathbb{R}^3$ satisfying $\lambda_1 > \lambda_2 > r^2/2c > \lambda_3$. The reader is referred to [18] for the other cases. Define $E := \{(m, g, z) \mid (g, z) \in P(m)\} \subset M \times (SO(3) \times \mathbb{R}^3)$. Note that the $\mathbb{T}^3$ action on $\pi_M|E : E \to M$ given by the parameter dependent integrable system H_m forms a family of Hamiltonian group actions by the example of §2.1. We consider the HB connection on $\pi_M|E : E \to M$.

4.1 The Hamiltonian One-Form, Case I: $v \in TF_\lambda$

In this section we compute the Hamiltonian one-form in the direction of vectors tangent to the orbits F_λ, of the $SO(3)$ action on M from §3.3. To do this we appeal to Proposition 2.2 which gives $\mathbf{K} \cdot \xi_M(m)$ in terms of a momentum map for an $SO(3)$ action on $P(m)$.

Consider the left $SO(3)$ action on $SO(3) \times \mathbb{R}^3$ given in body coordinates by

$$h \cdot (g, z) = (gh^{-1}, hz), \tag{8}$$

for $h \in SO(3)$. This is the cotangent lift of the left action on $SO(3)$ given by *right* multiplication by h^{-1}. We remark that this action commutes with the lifted left action. For $\xi \in \mathbb{R}^3$, the infinitesimal generator is $\xi_P(g, z) = \frac{d}{dt}\Big|_{t=0} \exp t\hat{\xi} \cdot (g, z) = -(g\hat{\xi}, z \times \xi)$. Define $\mathbf{L} : SO(3) \times \mathbb{R}^3 \to \mathbb{R}^3$, by $\mathbf{L}(g, z) = -z$. Then $\mathbf{L}^\xi(g, z) = -\langle \xi, z \rangle$ so that $\nabla \mathbf{L}^\xi = -\xi$, and hence by §3.1 $X_{\mathbf{L}^\xi}(g, z) = (g \cdot \widehat{\nabla \mathbf{L}^\xi}, z \times \nabla \mathbf{L}^\xi) = \xi_P(g, z)$. Thus $\mathbf{L}$ is a momentum map for the action (8). One checks that $\mathbf{L}$ is equivariant with respect to (8) and the usual action of $SO(3)$ on $\mathbb{R}^3$.

To utilize Proposition 2.2 we must first check that the actions $\mathbf{I} = (I_1, I_2, I_3)$ for the rigid body are invariant under the corresponding diagonal action on $E \subset M \times (SO(3) \times \mathbb{R}^3)$:

$$h \cdot (m, g, z) = (hmh^{-1}, gh^{-1}, hz), \tag{9}$$

for $h \in SO(3),\ m \in M,\ (g, z) \in P(m)$. Recall that $I_1 = \langle \mathbf{J}, \mathbf{e}_1 \rangle$, and $I_2 = \|\mathbf{J}\|$ do not depend on the inertia tensor m. One verifies that $\mathbf{J}$ is invariant under (8) whence $I_1,\ I_2$ are invariant under (9).

To check the invariance of I_3 we first note that the energy $H(m, z) = \frac{1}{2}\langle z, m^{-1}z \rangle$ is invariant:

$$H(hmh^{-1}, hz) = \frac{1}{2}\langle hz, (hmh^{-1})^{-1}hz \rangle = \frac{1}{2}\langle hz, hm^{-1}z \rangle = H(m, z).$$

Thus the original integrals in involution $f = (I_1, I_2, H_m)$, are invariant under the diagonal action (9). We noted in §3.2 that the value of I_3 is $1/2\pi r$ times the oriented area of the spherical cap enclosed by the curve $S^2_r \cap H_m^{-1}(c)$. Observe that replacing m by $hmh^{-1},\ h \in SO(3)$ has the effect of rigidly rotating the level sets of H about the origin, i.e.

$$H^{-1}_{hmh^{-1}}(c) = h \cdot H_m^{-1}(c).$$

This rotation does not alter the area of the spherical cap, and therefore I_3 is invariant as required. The invariance of $\mathbf{I} = (I_1, I_2, I_3)$ under the diagonal action together with Proposition 2.2 yields

Lemma 4.1 *The Hamiltonian* 1*-form acting on vectors tangent to the* $SO(3)$ *orbits in* M *is given by* $\mathbf{K} \cdot \xi_M(m) = \mathbf{L}^\xi - \langle \mathbf{L}^\xi \rangle$, *for* $m \in M,\ \xi \in \mathbb{R}^3$.

We remark that $\mathbf{K} \cdot \xi_M(m)$ is a left invariant function on $P(m)$ since $\mathbf{L}^\xi$ is. The following lemma will facilitate the computation of the average $\langle \mathbf{L}^\xi \rangle$. We shall see that for left invariant functions the averaging operation (over the $\mathbb{T}^3$ action) can be replaced by the time average over the rigid body dynamics.

Lemma 4.2 *Fix* $m \in M$ *and suppose* $F : P(m) \to \mathbb{R}$ *is continuous and left invariant. Let* $\Phi_t^{H_m}$ *denote the flow of the Euler equations (3) in* $\mathbb{R}^3$ *with period* T*, and define*

$$\langle F \rangle_{H_m}(z) := \frac{1}{T} \int_0^T F(\Phi_t^{H_m}(z))dt.$$

Then $\langle F \rangle = \langle F \rangle_{H_m}$.

Proof: Let $\langle \cdot \rangle_{\mathbb{T}^2}$ denote averaging over the $\mathbb{T}^2$ action generated by the flows of I_2 and I_3. Since F is left invariant, it is invariant under the Hamiltonian flow of I_1:

$$\Phi_t^{I_1}(g, z) = ((\exp t\hat{\mathbf{e}}_1)g, z)\,.$$

By Fubini's Theorem we have immediately that $\langle F \rangle = \langle F \rangle_{\mathbb{T}^2}$, so we must show that $\langle F \rangle_{\mathbb{T}^2} = \langle F \rangle_{H_m}$.

Let ω_i denote the frequency of rigid body motion with respect to the angles θ_i conjugate to I_i $(i = 1, 2, 3.)$ If the initial point z is such that ω_2/ω_3 is irrational then the flow of H_m is a dense winding on the 2-torus parametrized by θ_2, θ_3. Since in this case we know that the time average equals the space average (see Arnold[2]) we have $\langle F \rangle_{\mathbb{T}^2}(z) = \langle F \rangle_{H_m}(z)$. Its clear that $\langle F \rangle_{\mathbb{T}^2}$ and $\langle F \rangle_{H_m}$ are continuous functions so to show they are equal its sufficient to show they coincide on a dense subset of $P(m)$. Thus we must check that ω_2/ω_3 is irrational for a dense set of initial conditions. In [18,19] it is shown that ω_2/ω_3 is real analytic being given by a combination of algebraic operations and complete elliptic integrals. Thus the critical points of ω_2/ω_3 are isolated and therefore any neighborhood of $(g, z) \in P(m)$ contains a regular point of ω_2/ω_3, and hence also a point at which ω_2/ω_3 is irrational. ///

Remark. In general the time average of a function is defined as

$$\lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau F(\Phi_t^{H_m}(g, z)dt,$$

which is not continuous. The key point in the last lemma is that by left invariance of F one need only consider a reduced trajectory, which is periodic. The result of averaging such a function over a period *is* continuous.

The next result shows that the operator $\mathbf{K}$ is invariant under a certain $SO(3)$ action.

Lemma 4.3 *Let* $\xi \in \mathbb{R}^3,\ m \in M,\ z \in U(m),$ *and* $h \in SO(3)$*. Then*

$$\left(\mathbf{K} \cdot (h\xi)_M(hmh^{-1})\right)(hz) = (\mathbf{K} \cdot \xi_M(m))\,(z).$$

Proof: Since $\mathbf{L}^{h\xi}(hz) = -\langle h\xi, hz\rangle = -\langle h, z\rangle = \mathbf{L}^{\xi}(z)$, we need only show $\langle\mathbf{L}^{h\xi}\rangle(hmh^{-1}, hz) = \langle\mathbf{L}^{\xi}\rangle(m, z)$. Note that even though $\mathbf{L}^{\xi}$ does not depend on m, its average does, since the $\mathbb{T}^3$ action we average over does. By Lemma 4.2 its sufficient to show $\langle\mathbf{L}^{h\xi}\rangle_{H_{hmh^{-1}}}(hz) = \langle\mathbf{L}^{\xi}\rangle_{H_m}(z)$.

With a slight abuse of notation let $\Phi_t^{H_m}(z)$ denote the flow of the Euler equations (3) in $\mathbb{R}^3$ with energy H_m and initial point $z \in U_i(m)$, $i = 1$ or 3. The flow satisfies $\Phi_t^{H_{hmh^{-1}}}(hz) = h \cdot \Phi_t^{H_m}(z)$, i.e. by rotating a given trajectory by $h \in SO(3)$, we obtain a trajectory for the system with rotated inertia tensor and rotated initial point. Thus

$$\mathbf{L}^{h\xi}(\Phi_t^{H_{hmh^{-1}}}(hz)) = -\langle h\xi, h \cdot \Phi_t^{H_m}(z)\rangle = -\langle \xi, \Phi_t^{H_m}(z)\rangle = \mathbf{L}^{\xi}(\Phi_t^{H_m}(z)),$$

and therefore

$$\langle\mathbf{L}^{h\xi}\rangle_{H_{hmh^{-1}}}(hz) = \frac{1}{T}\int_0^T \mathbf{L}^{h\xi}(\Phi_t^{H_{hmh^{-1}}}(hz))dt = \langle\mathbf{L}^{\xi}\rangle_{H_m}(z),$$

as required. ///

As a consequence of Lemma 4.3 we need only calculate $\mathbf{K} \cdot \xi_M(m)$ for m which are diagonal with respect to the fixed basis $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$. If we set $m = \sigma(\lambda) = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$, $\lambda \in B$, then

$$\left(\mathbf{K} \cdot \xi_M(h\sigma(\lambda)h^{-1})\right)(z) = \left(\mathbf{K} \cdot (h^{-1}\xi)_M(\sigma(\lambda))\right)(h^{-1}z) \tag{10}$$

for any $h \in SO(3)$. Note that $z \in U(h\sigma(\lambda)h^{-1}) = h \cdot U(\sigma(\lambda))$ implies $h^{-1}z \in U(\sigma(\lambda))$, so the right hand side is defined whenever the left is.

Proposition 4.1 *Let $m = \sigma(\lambda)$, $\lambda \in B$, $z = (z_1, z_2, z_3) \in U_i(m)$, $i = 1, 3$, and $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$, with all coordinates relative to $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$. Then*

$$(\mathbf{K} \cdot \xi_M(m))(z) = G_i(z)\xi_i - \langle\xi, z\rangle \qquad \text{for } z \in U_i(m),$$

where

$$G_1(z) = \frac{\epsilon\pi\sqrt{z_1^2 + \mu z_2^2}}{2K(k_1(z))} \qquad G_3(z) = \frac{\epsilon\pi\sqrt{\nu z_2^2 + z_3^2}}{2K(k_3(z))}$$

$$\mu = \frac{\lambda_1(\lambda_2 - \lambda_3)}{\lambda_2(\lambda_1 - \lambda_3)} \qquad \nu = \frac{\lambda_3(\lambda_1 - \lambda_2)}{\lambda_2(\lambda_1 - \lambda_3)}$$

$$k_1(z) = \sqrt{\frac{c_2 z_2^2 + c_3 z_3^2}{c_1 z_1^2 + c_2 z_2^2}} = k_3(z)^{-1}$$

and

$$\begin{aligned} c_1 &= \lambda_2\lambda_3(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3) \\ c_2 &= \lambda_1\lambda_3(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3) \\ c_3 &= \lambda_1\lambda_2(\lambda_2 - \lambda_3)(\lambda_1 - \lambda_3). \end{aligned}$$

Also $\epsilon = \pm 1$ according to whether $z \in U_i^+(m)$ or $U_i^-(m)$, and $K(k)$ denotes the complete elliptic integral of the first kind[6].

Proof: By Lemmas 4.1 and 4.2 we have $(\mathbf{K} \cdot \xi_M(m))(z) = \mathbf{L}^{\xi}(z) - \langle\mathbf{L}^{\xi}\rangle(m, z) = -\langle\xi, z\rangle - \langle\mathbf{L}^{\xi}\rangle_{H_m}(z)$, so we must show that the identity

$$\langle\mathbf{L}^{\xi}\rangle_{H_m}(z) = -\xi_i G_i(z). \tag{11}$$

holds for $z \in U_i(m)$, $i = 1, 3$. As in the previous proof let $\Phi_t^{H_m}(z)$ denote the solution to the Euler equations (3) with initial point z. For $z \in U_i(m)$, $\Phi_t^{H_m}(z)$ is periodic with period T, whence

$$\langle \mathbf{L}^\xi \rangle_{H_m}(z) = \frac{1}{T}\int_0^T \mathbf{L}^\xi(\Phi_t^{H_m}(z))\, dt = -\frac{1}{T}\int_0^T \left\langle \xi, \Phi_t^{H_m}(z) \right\rangle\, dt = -\left\langle \xi\ ,\ \frac{1}{T}\int_0^T \Phi_t^{H_m}(z)\, dt \right\rangle.$$

Hence (11) is satisfied if and only if

$$\frac{1}{T}\int_0^T \Phi_t^{H_m}(z)\, dt = G_i(z)\mathbf{E}_i \tag{12}$$

for $z \in U_i(m)$. Now $\Phi_t^{H_m}(z) = z_1(t)\mathbf{E}_1 + z_2(t)\mathbf{E}_2 + z_3(t)\mathbf{E}_3$ is given by (5) for the initial point $z(0) = z \in U_3(m)$. In this case we have

$$\int_0^T z_1(t)\, dt = \int_0^T z_2(t)\, dt = 0,$$

since $\mathrm{cn}(\cdot, k)$ and $\mathrm{sn}(\cdot, k)$ have average zero over one period. Thus

$$\frac{1}{T}\int_0^T \Phi_t^{H_m}(z)\, dt = \left(\frac{1}{T}\int_0^T z_3(t)\, dt\right)\mathbf{E}_3.$$

Let R, s, k, ϵ be as in (6), then

$$\frac{1}{T}\int_0^T z_3(t)\, dt = \frac{\epsilon R}{T}\int_0^T \mathrm{dn}(s(t-t_0), k)\, dt = \frac{\epsilon R}{4K(k)}\int_0^{4K(k)} \mathrm{dn}(u, k)\, du = \frac{\epsilon \pi R}{2K(k)}.$$

In the second equality above we have set $u = s(t - t_0)$ and used $T = 4s^{-1}K(k)$. Using (6), $r = \|z\|$, and $c = H_m(z)$, we obtain

$$R^2 = \nu z_2^2 + z_3^2 \qquad \text{and} \qquad k^2 = \frac{c_1 z_1^2 + c_2 z_2^2}{c_2 z_2^2 + c_3 z_3^2} = k_3(z)^2.$$

Therefore

$$\frac{1}{T}\int_0^T z_3(t)\, dt = \frac{\pi\sqrt{\nu z_2^2 + z_3^2}}{2K(k_3(z))} = G_3(z),$$

proving (12) and hence (11) for $i = 3$. We omit the case $i = 1$ and refer the reader to [18] for further details. ///

Proposition 4.1 and Equation (10) yield

$$\left(\mathbf{K} \cdot \xi_M(h\sigma(\lambda)h^{-1})\right)(z) = G_i(h^{-1}z)\langle \xi, h\mathbf{E}_i \rangle - \langle \xi, z \rangle \tag{13}$$

for any $h \in SO(3)$ and $z \in U_i(h\sigma(\lambda)h^{-1})$, $i = 1, 3$. Equation (13) now gives the expression for $\mathbf{K} \cdot \xi_M(m)$ for any $m \in M$.

4.2 The Hamiltonian One-Form, Case II: $v \in TB$

Our goal in this section is to compute $\mathbf{K} \cdot v$ for vectors $v \in TM$ tangent to B under the identification $M \cong B \times F$ in §3.3. These are vectors tangent to curves in M along which the principle axes of inertia remain fixed and the moments of inertia are allowed to vary. Throughout we assume m is diagonal with respect to the fixed frame $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$, i.e $m = \sigma(\lambda)$, $\lambda \in B$.

If we solve the system

$$d_M I_j \cdot v + \{I_j, \widetilde{\mathbf{K}} \cdot v\} = 0 \qquad (1 \le j \le 3) \tag{14}$$

for the unknown function $\widetilde{\mathbf{K}} \cdot v$, then by Proposition 2.1 we may take $\mathbf{K} \cdot v = \widetilde{\mathbf{K}} \cdot v - \langle \widetilde{\mathbf{K}} \cdot v \rangle$. Recall that $I_1 = \langle \mathbf{J}, \mathbf{e}_1 \rangle$, and $I_2 = \|\mathbf{J}\|$ do not depend on the parameter m so $d_M I_j \cdot v = 0$ for $j = 1, 2$. Thus (14) reads

$$\begin{cases} \{I_1, \widetilde{\mathbf{K}} \cdot v\} &= 0 \\ \{I_2, \widetilde{\mathbf{K}} \cdot v\} &= 0 \\ d_m I_3 \cdot v + \{I_3, \widetilde{\mathbf{K}} \cdot v\} &= 0 \end{cases}$$

The first two equations imply that $\widetilde{\mathbf{K}} \cdot v$ is constant along the flows $\Phi_t^{I_1}(g, z) = ((\exp t\hat{\mathbf{e}}_1)g, z)$ and $\Phi_t^{I_2}(g, z) = \left(g \exp t \frac{\hat{z}}{\|z\|}, z\right)$. If we assume that $(\widetilde{\mathbf{K}} \cdot v)(g, z)$ is left invariant then the first two equations are automatically satisfied. Therefore it is sufficient to find a smooth function $\widetilde{\mathbf{K}} \cdot v$, depending only on $z \in U_i(m)$, which satisfies the single equation

$$\{\widetilde{\mathbf{K}} \cdot v, I_3\} = d_M I_3 \cdot v. \tag{15}$$

Now recall that $I_3 = A/2\pi r$ where A is the oriented spherical area enclosed by one of the periodic trajectories of (3). If $z \in U_3(m)$ we see form the direction of the trajectories (5) that $A = -|A|$, where $|A|$ denotes the strictly positive area. A straightforward calculation yields

$$|A| = \int\int_D \frac{PQr\, dxdy}{\sqrt{r^2 - P^2x^2 - Q^2y^2}},$$

where $D \subset \mathbb{R}^2$ is the unit disc, and P^2, Q^2 are given by (6) (This area integral can be written as a combination of complete elliptic integrals[18].) Thus for $z \in U_3(m)$

$$I_3(z) = -\frac{1}{2\pi} \int\int_D \frac{PQ\, dxdy}{\sqrt{r^2 - P^2x^2 - Q^2y^2}}.$$

Using $r^2 = z_1^2 + z_2^2 + z_3^2$ and $2c = z_1^2/\lambda_1 + z_2^2/\lambda_2 + z_3^2/\lambda_3$ we have

$$P^2 = z_1^2 + \mu z_2^2, \qquad \text{and} \qquad Q^2 = \mu^{-1} z_1^2 + z_2^2,$$

where $\mu = \lambda_1(\lambda_2 - \lambda_3)/\lambda_2(\lambda_1 - \lambda_3)$ is as in Proposition 4.1. Now one checks that $\mu = 1/(1+\eta)$ where $\eta = \lambda_3(\lambda_1 - \lambda_2)/\lambda_1(\lambda_2 - \lambda_3)$ gives the slope of the separatrix planes in $\mathbb{R}^3$. Thus $I_3(z)$ depends on m only through η for $z \in U_3(m)$. Similar arguments show the same is true for $z \in U_1(m)$. Hence if we let $\lambda \in B$ vary along a level set of $\eta = \eta(\lambda)$, then $U_1(m)$ and $U_3(m)$ are unchanged and $I_3(z)$ is constant.

One checks that $\nabla_\lambda \eta \neq 0$ for all $\lambda \in B$. Since B is contractible it follows that η can serve as a coordinate function on B. If we set $\eta_1 = \eta$, then there exist smooth functions η_2, η_3 on B such that the gradients $\nabla_\lambda \eta_j$ are linearly independent at each $\lambda \in B$. The coordinate vector fields $\partial/\partial\eta_j$ are then convenient directions along which to compute $\mathbf{K} \cdot v$.

Proposition 4.2 $\mathbf{K} \cdot \partial/\partial\eta_j = 0$ *for* $j = 2, 3$.

Proof: The preceding discussion implies that $d_M I_3 \cdot \partial/\partial\eta_2 = d_M I_3 \cdot \partial/\partial\eta_3 = 0$. Thus for $v = \partial/\partial\eta_j$, $j = 2, 3$, (15) reads $\{\widetilde{\mathbf{K}} \cdot v, I_3\} = 0$, which has the simple solution $\widetilde{\mathbf{K}} \cdot v = 0$. Therefore

$$\mathbf{K} \cdot \frac{\partial}{\partial\eta_j} = \widetilde{\mathbf{K}} \cdot \frac{\partial}{\partial\eta_j} - \left\langle \widetilde{\mathbf{K}} \cdot \frac{\partial}{\partial\eta_j} \right\rangle = 0$$

for $j = 2, 3$, as required. ///

For $v = \partial/\partial\eta_1$, (15) becomes

$$\{\widetilde{\mathbf{K}} \cdot v, I_3\} = \frac{\partial I_3}{\partial \eta_1}. \tag{16}$$

Since $\widetilde{\mathbf{K}} \cdot v$ is left invariant we have by §3.1

$$\{\widetilde{\mathbf{K}} \cdot v, I_3\}(z) = -\Big\langle z\ ,\ \nabla(\widetilde{\mathbf{K}} \cdot v) \times \nabla I_3 \Big\rangle = \Big\langle z \times \nabla I_3\ ,\ \nabla(\widetilde{\mathbf{K}} \cdot v) \Big\rangle = \mathcal{D}(\widetilde{\mathbf{K}} \cdot v)(z),$$

where $\mathcal{D}$ is the linear differential operator $\mathcal{D} = \langle z \times \nabla I_3, \nabla\rangle$. Thus (16) can be written as $\mathcal{D}(\widetilde{\mathbf{K}} \cdot v) = \partial I_3/\partial\eta_1$. It would be a formidable task to solve this linear PDE since its coefficients and right hand side are elliptic integrals. Fortunately it is not necessary to know $\mathbf{K} \cdot \partial/\partial\eta_1$ explicitly in order to compute the holonomy of loops in B.

4.3 $\mathbb{T}^2$ Averaging

The basis for the averaging principle is the fact that for multifrequency systems without resonances, the time average over a dynamic trajectory can be replaced by the space average (Arnold[2] Chapter 10). It was mentioned in the proof of Lemma 4.2 that the generic trajectories of the rigid body are dense windings on the 2-torus parametrized by the angles θ_2, θ_3 conjugate to I_2, I_3. Viewing this as a three frequency system, it has a proper resonance, while as a two frequency system it is generically non-resonant. It would therefore seem more reasonable to study the HB connection associated to the family of Hamiltonian $\mathbb{T}^2$ actions induced by the flows of I_2, I_3. In fact the results would be identical to those already obtained, as we now show.

In calculating the Hamiltonian one-form in case I: $v \in TF_\lambda$, we find that we must replace the average appearing in the statement of Lemma 4.1 with the average over the $\mathbb{T}^2$ action. But we argue in the proof of Lemma 4.2 that since the momentum map $\mathbf{L}$ is left invariant, and hence doesn't see the flow of I_1, the $\mathbb{T}^2$ and $\mathbb{T}^3$ averages of $\mathbf{L}$ coincide. Thus the Hamiltonian one-forms are identical in this case. In case II: $v \in TB$, Proposition 2.1 indicates that we must simply remove the first equation from system (14). Proceeding as before we find it is sufficient to solve the single equation (15). The Hamiltonian one-forms are again identical. Since the Hamiltonian one-form uniquely determines the HB connection, it follows that the connections associated to the $\mathbb{T}^2$ and $\mathbb{T}^3$ actions are identical.

Observe however that the $\mathbb{T}^2$ action can be defined over a slightly larger region in $SO(3) \times \mathbb{R}^3$ than can the $\mathbb{T}^3$ action. In particular there is no need to assure that dI_1 be independent of dI_2 and dH_m, which means we can drop the requirement $gz \times \mathbf{e}_1 \neq 0$ in the definition of $W \subset SO(3) \times \mathbb{R}^3$ (see §3.2). For $m \in M$, set

$$W' = \{(g, z) \in SO(3) \times \mathbb{R}^3 \mid z \times m^{-1}z \neq 0\},$$

and define $P(m) \subset SO(3) \times \mathbb{R}^3$, $E \subset M \times (SO(3) \times \mathbb{R}^3)$ as in §3.2, with W replaced by W'. Our formulas for the Hamiltonian one-form, curvature, and holonomy are then valid on this slightly larger bundle.

That these two actions give the same results ultimately derives from the arbitrariness of the definition of the action I_1. Recall we could have taken $I_1 = \langle \mathbf{J}, \mathbf{u}\rangle$ where $\mathbf{u} \in \mathbb{R}^3$ is any unit vector. Then the corresponding action-angle charts do not cover points (g, z) at which $gz \times \mathbf{u} = 0$. The different $\mathbb{T}^3$ actions (one for each choice of $\mathbf{u}$) induce connections defined on bundles which exclude different codimension one submanifolds of $SO(3) \times \mathbb{R}^3$. The resulting formulas are the same in each case, so we need not leave out any such submanifold.

4.4 Curvature

We now turn our attention to calculation of the curvature form on M, described in §2.3. Recall that for $m \in M,\ v_1, v_2 \in T_m M$,

$$\Omega(v_1, v_2) = \langle\{\mathbf{K}\cdot v_1, \mathbf{K}\cdot v_2\}\rangle \tag{17}$$

gives the smooth function on E_m whose Hamiltonian vector field is the curvature applied to $\mathrm{Hor}\cdot v_1, \mathrm{Hor}\cdot v_2$. Throughout we assume $m = \sigma(\lambda) = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3),\ \lambda \in B$ and identify $M \cong B \times F_\lambda$ as in §3.3. We will calculate Ω on a conveniently chosen basis for $T_m M$. Note that since the $SO(3)$ action on M is locally free, the map $\mathbb{R}^3 \to T_m F_\lambda$ given by $\xi \mapsto \xi_M(m)$ is an isomorphism, so that the infinitesimal generators $\{(\mathbf{E}_j)_M(m) \mid 1 \le j \le 3\}$ form a basis of $T_m F_\lambda$. We will use the coordinates (η_1, η_2, η_3), described in §4.2 on B, and the coordinate vector fields $\{\partial/\partial\eta_1, \partial/\partial\eta_2, \partial/\partial\eta_3\}$ as a basis on $T_m B$.

Define

$$\begin{aligned} \mathcal{F}^i &= \mathbf{K}\cdot\frac{\partial}{\partial\eta_i} && 1 \le i \le 3 \\ \mathcal{F}_j &= \mathbf{K}\cdot(\mathbf{E}_j)_M(m) && 1 \le j \le 3, \end{aligned}$$

and

$$\begin{aligned} \Omega^{ij} &= \langle\{\mathcal{F}^i, \mathcal{F}^j\}\rangle && 1 \le i < j \le 3 \\ \Omega_{ij} &= \langle\{\mathcal{F}_i, \mathcal{F}_j\}\rangle && 1 \le i < j \le 3 \\ \Omega^i_j &= \langle\{\mathcal{F}^i, \mathcal{F}_j\}\rangle && 1 \le i, j \le 3. \end{aligned}$$

With these conventions we have

Theorem 4.1 *Let $G_i,\ i = 1, 3$ be defined as in Proposition 4.1, $z \in U_i(m)$, and $\lambda \in B$. Then*

$$\text{on } B: \quad \left\{\ \Omega^{23} = \Omega^{13} = \Omega^{12} = 0, \right. \tag{18}$$

$$\text{on } F_\lambda: \quad \begin{cases} \Omega_{23}(z) = \begin{cases} -G_1(z) & z \in U_1(m) \\ 0 & z \in U_3(m) \end{cases} \\ \Omega_{13}(z) = 0 \\ \Omega_{12}(z) = \begin{cases} 0 & z \in U_1(m) \\ -G_3(z) & z \in U_3(m), \end{cases} \end{cases} \tag{19}$$

$$\text{cross terms:} \quad \left\{\ \Omega^2_j = \Omega^3_j = 0 \quad 1 \le j \le 3. \right. \tag{20}$$

The terms $\Omega^1_1,\ \Omega^1_2,\ \Omega^1_3$ remain unknown.

Proof of (18) and (20): Proposition 4.2 yields $\mathcal{F}^2 = \mathcal{F}^3 = 0$ while $\mathcal{F}^1$ is unknown. Thus $\{\mathcal{F}^i, \mathcal{F}^j\} = 0,\ 1 \le i < j \le 3$, which proves (18). Also $\{\mathcal{F}^i, \mathcal{F}_j\} = 0$ for $i = 2,\ 3$ and $1 \le j \le 3$, proving (20). ///

The presence of so many zero terms in the curvature suggests that there are many loops in M with trivial holonomy. Since B is contractible, an immediate consequence is that the holonomy about any loop lying in B is trivial. Thus the Hannay angles corresponding to any loop in M consisting of inertia tensors which are simultaneously diagonalizable, are zero. This proves Theorem 1.2. To prove (19) we first establish two lemmas. It follows immediately from Proposition 4.1 that

Lemma 4.4 *For* $z \in U_i(m),\ i = 1, 3$

$$\begin{aligned}
\mathcal{F}_1(z) &= \begin{cases} G_1(z) - z_1 & z \in U_1(m) \\ -z_1 & z \in U_3(m) \end{cases} \\
\mathcal{F}_2(z) &= -z_2 \\
\mathcal{F}_3(z) &= \begin{cases} -z_3 & z \in U_1(m) \\ G_3(z) - z_3 & z \in U_3(m), \end{cases}
\end{aligned}$$

where G_1 *and* G_3 *are defined in Proposition 4.1.*

Lemma 4.5 *If* $z \in U_1(m)$ *then*

$$\text{(21)}\qquad \begin{cases} \{\mathcal{F}_1, \mathcal{F}_2\}(z) = -z_3 + z_1 z_3 f_1 \\ \{\mathcal{F}_1, \mathcal{F}_3\}(z) = z_2 + z_1 z_2 f_2 \\ \{\mathcal{F}_2, \mathcal{F}_3\}(z) = -z_1, \end{cases}$$

and if $z \in U_3(m)$

$$\text{(22)}\qquad \begin{cases} \{\mathcal{F}_1, \mathcal{F}_2\}(z) = -z_3 \\ \{\mathcal{F}_1, \mathcal{F}_3\}(z) = z_2 + z_2 z_3 f_3 \\ \{\mathcal{F}_2, \mathcal{F}_3\}(z) = -z_1 + z_1 z_3 f_4, \end{cases}$$

where f_1, f_2, f_3, f_4 *are certain smooth functions of* (z_1^2, z_2^2, z_3^2).

Proof: The Poisson bracket for left invariant functions is given in §3.1 as $\{\mathcal{F}_i, \mathcal{F}_j\}(z) = -\langle z, \nabla\mathcal{F}_i \times \nabla\mathcal{F}_j\rangle$. Let $z \in U_3(m)$, then by Lemma 4.4

$$\begin{aligned}
\nabla\mathcal{F}_1 &= -\mathbf{E}_1 \\
\nabla\mathcal{F}_2 &= -\mathbf{E}_2 \\
\nabla\mathcal{F}_3 &= \nabla G_3 - \mathbf{E}_3.
\end{aligned}$$

Now observe from the statement of Proposition 4.1 that $G_3(z)$ is actually a function of (z_1^2, z_2^2, z_3^2). Hence $\nabla G_3 = (z_1 g_1, z_2 g_2, z_3 g_3)$ for some smooth functions g_1, g_2, g_3 of (z_1^2, z_2^2, z_3^2). We compute:

$$\begin{aligned}
\nabla\mathcal{F}_1 \times \nabla\mathcal{F}_2 &= \mathbf{E}_3 \\
\nabla\mathcal{F}_1 \times \nabla\mathcal{F}_3 &= (z_3 g_3 - 1)\mathbf{E}_2 - z_2 g_2 \mathbf{E}_3 \\
\nabla\mathcal{F}_2 \times \nabla\mathcal{F}_3 &= (1 - z_3 g_3)\mathbf{E}_1 + z_1 g_1 \mathbf{E}_3,
\end{aligned}$$

whence

$$\begin{aligned}
\{\mathcal{F}_1, \mathcal{F}_2\}(z) &= -z_3 \\
\{\mathcal{F}_1, \mathcal{F}_3\}(z) &= z_2 + z_2 z_3 (g_2 - g_3) \\
\{\mathcal{F}_2, \mathcal{F}_3\}(z) &= -z_1 + z_1 z_3 (g_3 - g_1).
\end{aligned}$$

Observe that $f_3 := g_2 - g_3$ and $f_4 := g_3 - g_1$ are smooth functions of (z_1^2, z_2^2, z_3^2) as required. This proves (22). Equation (21) is proved for the case $z \in U_1(m)$ similarly. ///

Proof of (19): By (17) we must average the expressions in Lemma 4.5 over the $\mathbb{T}^3$ action induced by $I_1,\ I_2,\ I_3$. Since in all cases $\{\mathcal{F}_i, \mathcal{F}_j\}$ is left invariant, we may instead take the time average over the rigid body dynamics. (See Lemma 4.2 and the remark immediately following.) Thus we substitute into the expressions (21) and (22) the appropriate solution to the Euler equations (3) and average over one period of the motion.

We review a few facts concerning the Jacobi elliptic functions. (See Byrd and Friedman[6] or Lawden[13] for additional details.) The functions $\mathrm{cn}(u,k)$, $\mathrm{sn}(u,k)$, $(k^2<1)$, are periodic in u of period $4K(k)$, while $\mathrm{dn}(u,k)$ has period $2K(k)$. Now $\mathrm{cn}(u,k)$ is an even function with respect to the point $u=0$, and odd with respect to $u=K$; $\mathrm{sn}(u,k)$ is odd with respect to $u=0$, and even with respect to $u=K$; finally $\mathrm{dn}(u,k)$ is even with respect to both $u=0$ and $u=K$.

Let $z\in U_3(m)$. Then combining the above information with (5) we see that $z(t)$ (with initial point $z(0)=z$) has period $T=4s^{-1}K(k)$; $z_1(t)$ is even with respect to $t=0$, odd with respect to $t=T/4$; $z_2(t)$ is odd with respect to $t=0$, even with respect to $t=T/4$; and $z_3(t)$ is even with respect to both $t=0$ and $t=T/4$.

Now the integral of an odd periodic function over one of its periods is zero. We see from (22) that $\{\mathcal{F}_2,\mathcal{F}_3\}(z(t))$ is an odd function with respect to $t=T/4$, while $\{\mathcal{F}_1,\mathcal{F}_3\}(z(t))$ is odd with respect to $t=0$. Hence $\langle\{\mathcal{F}_2,\mathcal{F}_3\}\rangle=\langle\{\mathcal{F}_1,\mathcal{F}_3\}\rangle=0$, and

$$\langle\{\mathcal{F}_1,\mathcal{F}_2\}\rangle(z)=-\langle z_3\rangle_{H_m}=-\frac{1}{T}\int_0^T z_3(t)\,dt=-G_3(z).$$

The last calculation was carried out in the proof of Proposition 4.1. Thus for $z\in U_3(m)$

$$\begin{cases}\Omega_{23}(z) &= 0\\ \Omega_{13}(z) &= 0\\ \Omega_{12}(z) &= -G_3(z).\end{cases}$$

In a similar manner we obtain from (21) that for $z\in U_1(m)$

$$\begin{cases}\Omega_{23}(z) &= -G_1(z)\\ \Omega_{13}(z) &= 0\\ \Omega_{12}(z) &= 0.\end{cases}$$

This completes the proof. ///

4.5 Holonomy

As in the last section we identify $M\cong B\times F_\lambda$. We saw in the comments following Theorem 4.1 that if $m(t)$ is a loop lying in B, then its holonomy is trivial; equivalently the Hannay angles are zero. In this section we compute the holonomy of certain loops lying in F_λ, $\lambda\in B$. These are closed curves in M along which the principal moments of inertia remain fixed, while the principal axes rotate about a fixed vector in $\mathbb{R}^3$. Throughout this section we consider curves of the form

$$m(t)=(\exp t\hat{\xi})\cdot\sigma(\lambda)=(\exp t\hat{\xi})\sigma(\lambda)(\exp t\hat{\xi})^{-1} \tag{23}$$

where $\sigma(\lambda)=\mathrm{diag}(\lambda_1,\lambda_2,\lambda_3)$, $\lambda\in B$, and $\xi\in\mathbb{R}^3$ with $\|\xi\|=1$. These are integral curves of the infinitesimal generator ξ_M, i.e. $m'(t)=\xi_M(m(t))$. Observe from §3.3 that if $0\le t\le 2\pi$ then $m(t)$ is a loop whose homotopy class corresponds to $-1\in Q$ under the isomorphism of §3.3. If $\xi=\pm\mathbf{E}_1,\pm\mathbf{E}_2,\pm\mathbf{E}_3$ and $0\le t\le\pi$ we have loops corresponding to $\pm i,\pm j,\pm k\in Q$ respectively. We shall examine these cases in turn.

Let $z\in U_i(m(t))$, $i=1,3$. Then by (13) and (23)

$$(\mathbf{K}\cdot m'(t))(z)=(\mathbf{K}\cdot\xi_M(m(t)))(z)=G_i\left((\exp t\hat{\xi})^{-1}z\right)\langle\xi,(\exp t\hat{\xi})\mathbf{E}_i\rangle-\langle\xi,z\rangle.$$

Thus

$$(\mathbf{K}\cdot m'(t))\,(z) = G_i\left((\exp t\hat{\xi})^{-1}z\right)\xi_i - \langle \xi, z\rangle, \tag{24}$$

since $\exp t\hat{\xi} \in SO(3)$ is a rotation about ξ. Note that $U_i(m(t)) = \exp t\hat{\xi}\cdot U_i(\sigma(\lambda))$, so that $(\exp t\hat{\xi})^{-1}z \in U_i(\sigma(\lambda))$ as required by the definition of G_i in Proposition 4.1.

The Hamiltonian vector field of $\mathbf{K}\cdot m'(t)$ yields the parallel transport equations for the path $m(t)$, which we wish to solve. These equations are

$$\begin{cases} \dot{g} &=& g(\widehat{\nabla(\mathbf{K}\cdot m'(t))(z)} \\ \dot{z} &=& z\times(\nabla(\mathbf{K}\cdot m'(t))(z)), \end{cases} \tag{25}$$

where $\nabla(\mathbf{K}\cdot m'(t))\,(z) = (\exp t\hat{\xi})\nabla G_i((\exp t\hat{\xi})^{-1}z)\xi_i - \xi$ for $z\in U_i(\sigma(\lambda))$. System (25) becomes autonomous upon going to a frame which rotates about ξ. Set

$$y(t) = (\exp t\hat{\xi})^{-1}z(t) \qquad \text{and} \qquad h(t) = g(t)(\exp t\hat{\xi}), \tag{26}$$

which we observe is just $(\exp t\hat{\xi})^{-1}$ acting by (8) on $(g(t), z(t))$. Then (25) becomes

$$\begin{cases} \dot{h} &=& h\widehat{\nabla G_i(y)}\xi_i \\ \dot{y} &=& y\times\nabla G_i(y)\xi_i. \end{cases} \tag{27}$$

In the remainder of this section we take $i = 3$ for definiteness. (The case $i = 1$ is entirely similar.) First consider the case $\xi\in\mathbf{E}_3^{\perp}$. Then $\xi_3 = 0$ and the right hand side of (27) is zero. In fact (25) is autonomous and can be solved directly. We have

$$(g(t), z(t)) = (\exp t\hat{\xi})\cdot(g_0, z_0)$$

using the action (8). For loops $m(t)$, $0\le t\le 2\pi$ given by (23) the holonomy is trivial. If $\xi = \pm\mathbf{E}_j$, $j = 1,2$ and $0\le t\le\pi$ the holonomy is

$$(g_0, z_0)\mapsto(\exp\pi\hat{\mathbf{E}}_j)\cdot(g_0, z_0). \tag{28}$$

Now consider the general case $\xi\notin\mathbf{E}_3^{\perp}$. The proof of Proposition 4.1 shows that $G_3(y)$ is a function of $r^2 = \|y\|^2$ and $c = H_{m_0}(y)$ where $m_0 = \sigma(\lambda)$. Namely $G_3(y) = \widetilde{G}_3(c, r^2) = \epsilon\pi R/2K(k)$, where R, k, ϵ are as in (6) and $K(k)$ denotes the complete elliptic integral of the first kind. Define

$$u(y) := \xi_3\frac{\partial\widetilde{G}_3}{\partial c} \qquad \text{and} \qquad v(y) := 2\xi_3\frac{\partial\widetilde{G}_3}{\partial(r^2)}. \tag{29}$$

(Note that u and v depend on ξ only through ξ_3.) Thus $\xi_3\nabla G_3(y) = u(y)\nabla H_{m_0}(y) + v(y)y = (um_0^{-1} + vI)y$, and (27) becomes

$$\begin{cases} \dot{h} &=& h\widehat{(um_0^{-1} + vI)y} \\ \dot{y} &=& y\times(um_0^{-1} + vI)y. \end{cases} \tag{30}$$

Equations (30) are rigid body equations with "inertia matrix" $um_0^{-1} + vI$. A calculation shows that $u/\lambda_j + v$ is negative for $j = 1,2$ and positive for $j = 3$. Thus $um_0^{-1} + vI$ is non-degenerate but not positive definite. The second equation in (30) reduces to

$$\dot{y} = u(y)(y\times m_0^{-1}y). \tag{31}$$

Note that u and v are constant along solutions to the Euler equations (3). It follows that the solution of (31) is given by (5) with t replaced by $u(y_0)t$. Thus u and v can be treated as constants (depending on y_0) in (30).

At this point we concentrate on (31). Let $\Phi_t^{H_{m_0}}(x)$ denote the solution to (3) with initial point $x \in U_3(m_0)$. Then

$$y(t) = \Phi_{u(y_0)t}^{H_{m_0}}(y_0)$$

is the solution to (31), and (26) implies that

$$z(t) = (\exp t\hat{\xi})\Phi_{u(z_0)t}^{H_{m_0}}(z_0) \tag{32}$$

is the vector part of the solution to (25). (Note that $y_0 = z_0$.) Thus the parallel transport of the body angular momentum vector is a composition of the scaled Euler flow and a steady rotation about ξ. The (vector part of the) holonomy of $m(t)$, $0 \le t \le 2\pi$ is then

$$z_0 \mapsto \Phi_{2\pi u(z_0)}^{H_{m_0}}(z_0). \tag{33}$$

Notice that if $\xi \in \mathbf{E}_3^\perp$ so that $\xi_3 = 0$, then $u \equiv 0$, making the above holonomy trivial. This coincides with our earlier result. If $\xi = \pm\mathbf{E}_3$, $0 \le t \le \pi$ the holonomy is

$$z_0 \mapsto (\exp \pi\hat{\mathbf{E}}_3)\Phi_{\pi u(z_0)}^{H_{m_0}}(z_0). \tag{34}$$

Equations (28), (32), (33), and (34) prove Theorem 1.1 and its Corollaries.

4.6 The Axially Symmetric Body

The calculation of phases simplifies considerably when two of the principal moments of inertia are equal. Observe that $\Sigma \subset M_1$ (see §3.3) contains two connected components. One component Σ_1 consists of inertia tensors whose double eigenvalue is the smaller eigenvalue ($\lambda_1 > \lambda_2 = \lambda_3$), and the other component Σ_3 has double eigenvalues larger ($\lambda_1 = \lambda_2 > \lambda_3$). Observe that $\mathrm{cl}(\Sigma_1) \cap \mathrm{cl}(\Sigma_3) = \Sigma'$. We shall see that the HB connection can be extended to just one of these components, but not both simultaneously.

Suppose $m \in M$ approaches Σ_3. From §3.1 we see that $(\lambda_1 - \lambda_2) \to 0$ implies $\eta \to 0$, showing that the two separatrix planes become the $z_1 z_2$ plane. Thus the sets $U_1^\pm(m)$ and $P_1^\pm(m)$ are squeezed out of existence and the parallel transport of an initial point in $P_1^\pm(m)$ cannot be continued. Similarly the parallel transport of an initial point in $P_3^\pm(m)$ cannot be prolonged while m passes through Σ_1. Thus the HB connection cannot be extended to Σ_1 and Σ_3 simultaneously.

On the other hand if we consider only initial points in $P_3^\pm(m)$ the connection *can* be extended to Σ_3 as we now show. The loop $m(t) \in M$, $0 \le t \le \pi$ given by (23) with $\xi = \mathbf{E}_3$ encircles $\Sigma_3 \subset M_1$. As $(\lambda_1 - \lambda_2) \to 0$ this loop contracts to a point in Σ_3. To show that the connection can be extended to Σ_3 its sufficient to show that the holonomy of such a loop becomes trivial in the limit. From equations (6) and Proposition 4.1 we see that $(\lambda_1 - \lambda_2) \to 0$ implies $k \to 0$ and $\nu \to 0$, whence $K(k) \to \pi/2$ and $G_3(z) \to z_3$ for $z \in U_3(\sigma(\lambda))$. Therefore $(\mathbf{K} \cdot m'(t))(z) \to 0$, showing that the parallel transport (and holonomy) becomes trivial as required. A similar argument would show that the connection can be extended to Σ_1 as long as we only attempt to transport points in $P_1^\pm(\sigma(\lambda))$.

We now compute the holonomy of loops lying in Σ_3. First recall that the parallel transport along curves of type II (see §4.2) is trivial if $m = \sigma(\lambda)$ varies along a level set of $\eta = \eta(\lambda)$, $\lambda \in B$. But we saw above that $\eta = 0$ for $m \in \Sigma_3$, hence the holonomy of such loops is trivial. Now consider loops of type I (see §4.1) with initial point $(g_0, z_0) \in P_3^\pm(\sigma(\lambda))$. The above discussion shows that $G_3(z) = z_3$ so that $\nabla G_3(z) = \mathbf{E}_3$. Equation (27) becomes

$$\begin{cases} \dot{h} &= h\xi_3\widehat{\mathbf{E}}_3 \\ \dot{y} &= y \times \xi_3\mathbf{E}_3 = -\xi_3\widehat{\mathbf{E}}_3 y \end{cases}$$

with solution

$$h(t) = h_0(\exp t\xi_3\widehat{\mathbf{E}}_3) \qquad y(t) = (\exp t\xi_3\widehat{\mathbf{E}}_3)^{-1}y_0.$$

By (26) the parallel transport is

$$(g(t), z(t)) = (\exp t\hat{\xi})(\exp t\xi_3\widehat{\mathbf{E}}_3)^{-1}\cdot(g_0, z_0),$$

using the action (8). The holonomy about $m(t),\ 0 \le t \le 2\pi$ is

$$(g_0, z_0) \mapsto (\exp 2\pi\xi_3\widehat{\mathbf{E}}_3)^{-1}\cdot(g_0, z_0).$$

(Observe that this is trivial if $\xi = \mathbf{E}_3$ or $\xi \in \mathbf{E}_3^\perp$.) The holonomy of $m(t),\ 0 \le t \le \pi$ with $\xi = \pm\mathbf{E}_j,\ j = 1, 2$ is

$$(g_0, z_0) \mapsto (\exp \pi\widehat{\mathbf{E}}_j)(\exp \pi\xi_3\widehat{\mathbf{E}}_3)^{-1}\cdot(g_0, z_0).$$

One may verify that these results are consistent with those in §4.5 by taking the limit as $(\lambda_1 - \lambda_2) \to 0$ in equation (32).

5 Conclusions

References

1. Abraham, R., J.E. Marsden: *Foundations of Mechanics*. Addison-Wesley (1978).

2. Arnold, V.I.: *Mathematical Methods of Classical Mechanics (Second Edition)*. Springer-Verlag (1989).

3. Berry, M.: Quantal Phase Factors Accompanying Adiabatic Changes. *Proc. R. Soc. Lond. A*, **392**, 45-57 (1984).

4. Berry, M.: Classical Adiabatic Angles and Quantal Adiabatic Phase. *J. Phys. A: Math. Gen.*, **18**, 15-27 (1985).

5. Bredon, G.E.: *Introduction to Compact Transformation Groups*. Academic Press (1972).

6. Byrd, P.F., M.D. Friedman: *Handbook of Elliptic Integrals for Engineers and Scientists (second edition)*. Springer-Verlag (1971).

7. Fomenko, A.T., V.V. Trofimov: *Integrable Systems on Lie Algebras and Symmetric Spaces*. Gordon and Breach (1988).

8. Golin, S., A. Knauf, S. Marmi: The Hannay angles: Geometry, Adiabaticity, and an Example. *Commun. Math. Phys.*, **123**, 95-122 (1989).

9. Gray, B.: *Homotopy Theory; an Introduction to Algebraic Topology*. Academic Press (1975).

10. Hannay, J.H.: Angle Variable Holonomy in Adiabatic Excursion of an Integrable Hamiltonian. *J. Phys. A: Math. Gen.*, **18** 221-230 (1985).

11. Koiller, J.: Classical Adiabatic Angles for Slowly Moving Mechanical Systems. *Contemporary Mathematics*, **97**, 159-185 (1989).

12. Landau, L.D., E.M. Lifshitz: *Mechanics (third edition)*. Pergamon Press (1976).

13. Lawden, D.F.: *Elliptic Functions and Applications*. Springer-Verlag (1989).

14. Marsden, J.E., R. Montgomery and T. Ratiu: *Reduction, Symmetry, and Phases in Mechanics.* Memoirs AMS, **436** (1990).

15. Marsden, J.E., T.S. Ratiu: *An Introduction to Mechanics and Symmetry.* To appear.

16. Montgomery, R.: The Connection Whose Holonomy is the Classical Adiabatic Angles of Hannay and Berry and Its Generalization to the Non-Integrable Case. *Commun. Math. Phys.*, **120**, 269-294 (1988).

17. Simon, B.: Holonomy, the Quantum Adiabatic Theorem and Berry's Phase. *Phys. Rev. Lett.*, **51**, 2167-2170 (1983).

18. Tantalo, P.: *Geometric Phases for the Free Rigid Body with Variable Inertia Tensor.* Ph.D. thesis.

19. Tantalo, P.: Action Angle Coordinates for the Asymmetrical Rigid Body. Preprint.

20. Weinstein, A.: Connections of Berry and Hannay Type for Moving Lagrangian Submanifolds. *Adv. in Math.*, **82**, 133-159, (1990).