Some words on the loose end: collisions Δ ***and ..***

yesterday's theorem re connecting two config pts by an action minimizer

…ON BOARD:

Today

Oscillating about the degeneration locus

$$\Sigma = \text{degeneration locus} \supset \Delta := \bigcup \Delta_{ab} = \text{collision locus}$$

methods:

Riemannian geometry and quotient spaces
by Lie group actions

3-body problem
(in the plane)

motivating case

M-; 2002,``Infinitely many syzygies''

THM: Every zero angular momentum *, bounded* solution defined on an unbounded time interval suffers infinitely many collinearities (:= `syzygies').

collinear locus = degeneration locus

* on board: recall def of `bounded', `angular momentum'

BOUNDED: there exists a $\delta > 0$ such that $r_{ab}(t) \leq \delta$ for all a, b, t

picture of theorems in shape space,

J= 0 dynamics on shape space.
the U in shape space
Mark Levi's intuition

ON BOARD

How might this theorem generalize to more bodies (N>3)?
Or to the spatial problem? (d > 2)?

key insight after 13 years

Well, you know Rich, the ***shape space*** for the 4 body problem in 3 space is $\mathbb{R}^6$

Robert Littlejohn, Physics , Berkeley,
question period after his talk at his 2018 retirement conference

4-body problem (in 3-space)

within a month of that:

M-2019 ``Oscillating about coplanarity''

THM: Every zero angular momentum, bounded solution to the 4-body problem defined on an unbounded time interval suffers infinitely many *coplanar* instants.

coplanar locus = degeneration locus

I knew Robert's remark would be the key to finding a d = 3, N = 4 version of ``infinitely many syzygies''

I did not understand why his remark was true

Once I saw the ``why'' I could see that what I had done in `infinitely many syzygies' was the d =2 versions of a theorem that must work for the d+1 body problem in d dimensions..

Generically, d+1 points span (`determine') an affine d-plane.
Degenerate = nongeneric =configs (or d+1 -gons)
lying in a subspace of dimension d-1 or less.
Set of degenerate configurations = Degeneration locus

of ``Oscillating about the degeneration locus''

Strategy for proof

Step 1. Push Newton's Eqns down to

Shape space= **Configuration space/ Symmetries**

Observe that degeneration locus sits as a hypersurface in shape space.
Deg. locus = Collinear plane for planar 3 body problem,
Deg locus = Coplanar configurations for spatial 4 body problem.

Step 2. Let S be the signed distance of a shape from the degeneration locus.
Derive a `nice' differential equation of harmonic oscillator type for
S's evolution:

$$\frac{d^2}{dt^2}S = -Sg, g > 0$$

Here S = S(q(t)) = S evaluated along a sol'n to Newton's eqns.

Show bdd implies g > const. > 0.
End by a Sturm comparision to a harmonic osc

``Sturm comparison" with

$$\ddot{S} = -S\omega^2$$

S has a zero in any interval of time of size

$$\pi/\omega$$

implying theorem. For all d, N, with N =d+1

ON to BOARD

how to understand shape space and the dynamics on it.
answer:

Riemannian submersions and reduction
..
quotient of a manifold by a compact group G

Natural mechanical systems with symmetry (G)
and their quotients…

Onward to Step 1.

Shape space= **Configuration space/ Symmetries**

Config. sp for N-body problem in d-space: = $(\mathbb{R}^d)^N$ = d x N matrices.

elements: **q**= $[q_1, q_2, \ldots, q_N]$

Symmetry group = Isometries of d-space = translations + rotations.
b g

acts by: $[q_1, q_2, \ldots, q_N] \mapsto [g(q_1 + b), g(q_2 + b), \ldots g(q_N + b)]$

/translations $\cong \mathbb{R}^{dN}/\mathbb{R}^d = \mathbb{R}^{d(N-1)} =$ d x N-1 matrices

= M(d, N-1)

/rotations ??

Rotations act by q -> g q

action preserves deg. locus, and potential.

Shape space:= M(d, N-1)/G

two versions of shape space! depending on
if g in SO(d)
or g in O(d)

``oriented' and `unoriented' shape space

The magic of N = d+1

Configuration space/ Translations =M(d, N-1)

=M(d, d)

square matrices if N-1 = d

$\Sigma =$ degeneration locus = q's whose vertices lie in an affine d-1-space
= simplices with zero volume
= square matrices with determinant zero

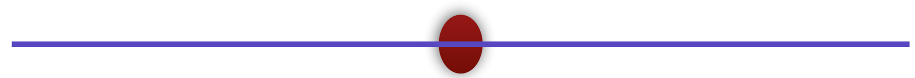
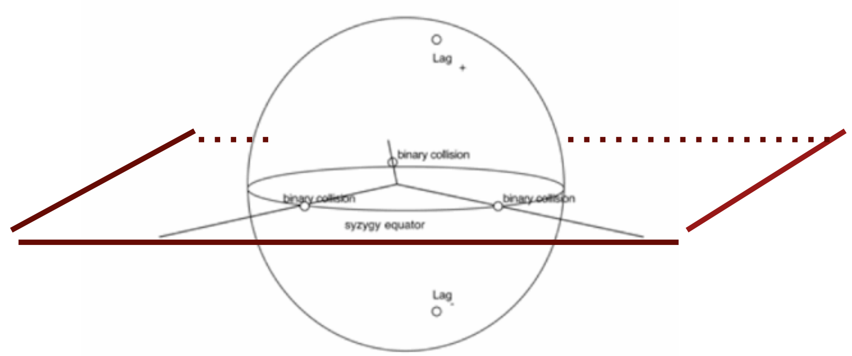
Shape space:= M(d, d)/G

action preserves degeneration locus, potential,
we denote their projections to Shape space by same symbol…

two versions again of shape space .depending on if g in SO(d)
or g in O(d)

they are…

Call the two versions the `oriented’ and `unoriented’ shape spaces

	oriented		unoriented
(d, N)			
(1, 2)		$\to \mathbb{Z}_2$	
(2, 3)		$\to \mathbb{Z}_2$	
(3, 4)	$\Sigma \subset \mathbb{R}^6$	$\to \mathbb{Z}_2$	**Cone of pos. semi-definite symmetric 3 x 3 matrices;**
(d, N=d+1)	$\Sigma \subset \mathbb{R}^{\binom{d}{2}}$	$\to \mathbb{Z}_2$	**Cone of pos. semi-definite symmetric d x d matrices;**

$$O(d)/SO(d) = \mathbb{Z}_2$$

map of forgetting orientation is a 2:1 branched cover, branched over degeneracy locus which is a hyperplane

Intuition **behind proof [M. Levi; N=3].**

Shape space is a Euclidean space endowed with a somewhat strange metric (`shape metric' induced by mass metric on config. space)

The *reduced eqns* are Newton's eqns AGAIN on this space, provided J =0.

The potential is due to a `gravitational attraction' to the binary collision locus.

This locus lies within the degeneration locus.

reduced eqns:

$$\nabla_{\dot{\sigma}}\dot{\sigma} = -\nabla \bar{V}(\sigma)$$

I told this to Mark Levi,
for the case N=3,d= 2, in 2002.

Mark: ``then the particle [=shape] must oscillate back and forth across that plane [=deg. locus]. ''

Proof now consists of implementing Mark's intuition.

$\mathbb{Z}_2$ - - Important to intuition and implementation: reflection about the degeneration locus, leaves the strange metric and the potential invariant.

Step 2. Derive a `nice' differential equation of harmonic oscillator type

$$\frac{d^2}{dt^2} S = -Sg, g > 0$$

for the ``distance'' S from the degeneration locus Σ

Here S = S(q(t)) = S evaluated along a sol'n to Newton's eqns.

M-; 2002, d=2, N=3. S = oriented area of triangle

guess: generalization is S = signed volume of simplex = det(q)

I spent a month trying to differentiate this S and derive
such a differential inequality. NEVER COULD…

Instead! $S(q) = d_{Sh}(q, \Sigma) =$ **signed distance between q and the degeneration locus**

`Distance' measured via `mass metric' (kinetic energy)
on configuration space

Fact: |S(q)| = smallest principal value of principal value decomp. of q

important: S is SO(d)-invariant so descends to a fn on Shape space.

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Deriving the needed eqn. for S.

$$\sigma(t) = \pi(q(t)); \pi : M(d,d) \to Sh(d,d+1)$$

shape curve

sol'n to Newton's eqns
having zero ang. mom. (J=0)

$$\nabla_{\dot\sigma}\dot\sigma = -\nabla\bar{V}(\sigma)$$

simple form of eq requires
J = 0 along q(t)

$$\dot{S} = \langle \nabla S, \dot\sigma \rangle$$

$$\ddot{S} = \langle \nabla S, \ddot\sigma \rangle + \langle \nabla_v \nabla S, v \rangle$$

standard computation
in Riem. geom.

$$\ddot{S} = \langle \nabla S(q), -\nabla V(q) \rangle + II_S(v,v)$$

$$= I + II$$

q solves Newt.

2nd f.f. of level sets of
S = equidistants from
deg. locus

PROP. I = -S g, g > 0 , and

$$g > \omega^2, \omega = GM/(\delta^3), M = \Sigma m_a, \text{ assuming bound } r_{ab}(t) \leq \delta$$

PROP. II = -S h, h > 0 .

Pf I: Hamilton-Jacobi or `weak KAM' + $\|\nabla S\| = 1$

+ property of potential f(r) = -1/r, where $V = G\Sigma m_a m_b f(r_{ab})$
$(f' > 0, f'' < 0, f'(r)/r \to 0)$

Pf II: curv. shape space ≥ 0, + Σ is tot. good. + `Sign & The Meaning of Curvature.'

REST ON THE BOARD …

…odds & ends of talk in two slides to follow:

``Sturm comparison" with

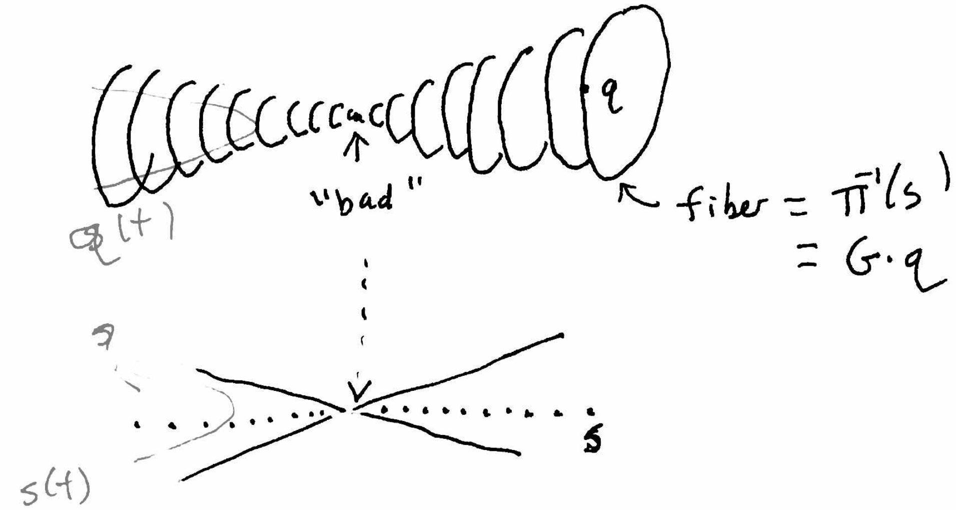
$$\ddot{S} = -S\omega^2$$

S has a zero in any interval of time of size

$$\pi/\omega$$

implying theorem. For all d, N, with N =d+1

$$\begin{array}{c} M \\ G \big\downarrow \pi \\ Sh \end{array}$$



A metric submersion:

$$\mathrm{dist}_{Sh}(s_1, s_2) = \mathrm{dist}_M(\pi^{-1}(s_1), \pi^{-1}(s_2))$$

uses that G acts by isometries

<u>Prop</u> For $s(t) = \pi(q(t))$, missing 'bad points'

$$\ddot{q} = -\nabla V(q) \quad \& \quad J(q, \dot{q}) = 0$$

$$\Leftrightarrow \nabla^{S}_{\dot{s}} \dot{s} = -\nabla_S V_S(s))$$

induced Riem metric on Sh.

<u>Prop</u>: [O'neill fmla] π is curvature non decreasing:

$$K(v \wedge w) = K(\pi v \wedge \pi w) + \text{pos term.}$$

if v, w are "horizontal" in M.

↪