Scattering and Metric Lines

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in `Geometry, Mechanics and Dynamics'
organized by

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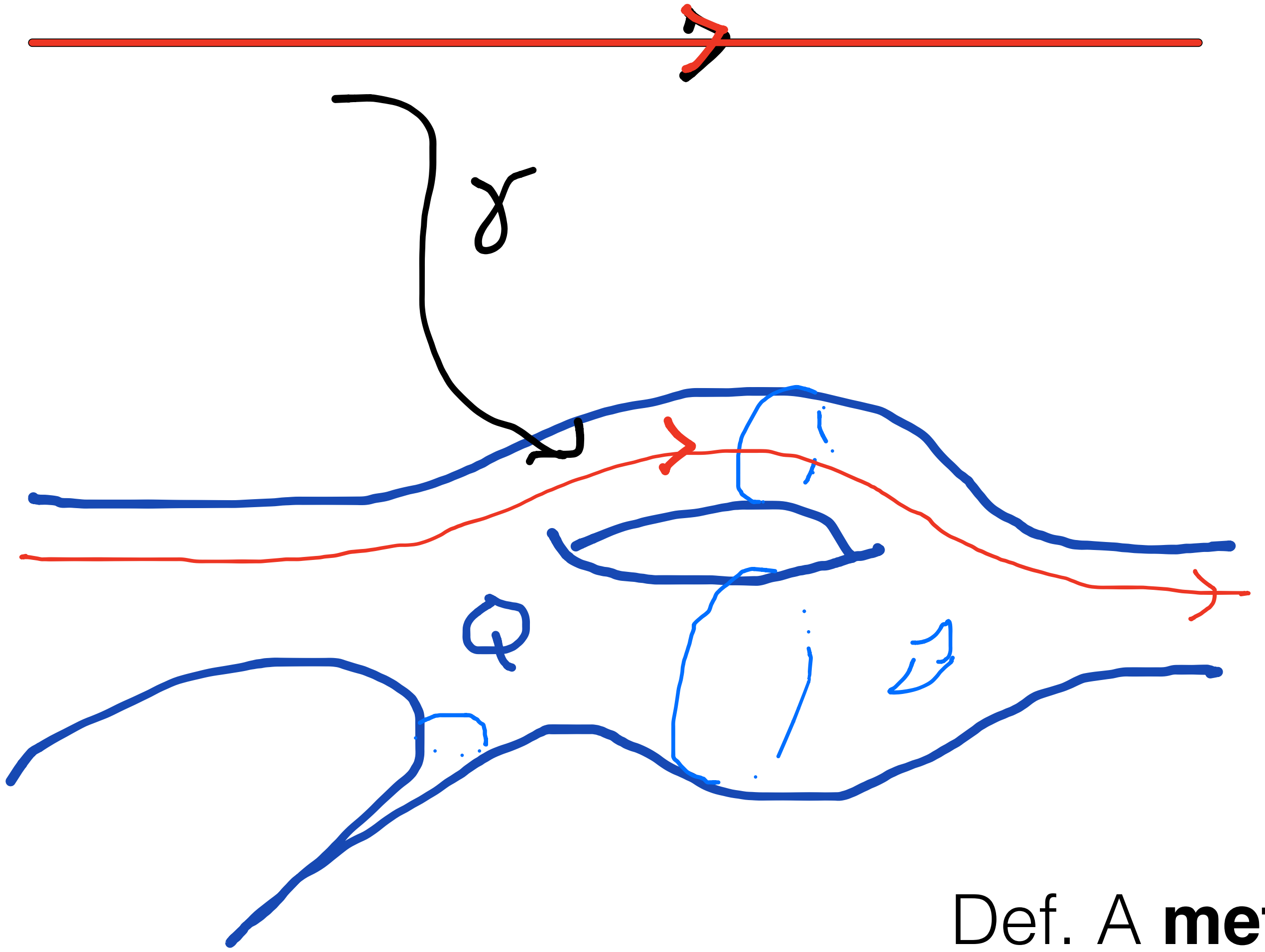
via Zoom, June 2, 2020

(*) : am retiring, July 1, 2020:
- keep me in mind for post C-virus longish term invites in 2021 or 2022



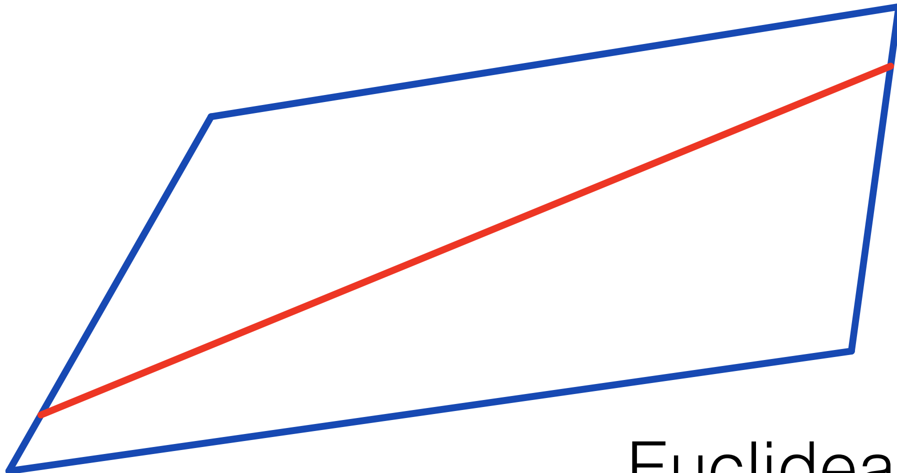
Def. A **metric line** in a metric space (Q,d) is an isometric image of the real line:

$$\gamma : \mathbb{R} \to Q, d(\gamma(t), \gamma(s)) = |t - s|, (\forall t, s \in \mathbb{R})$$

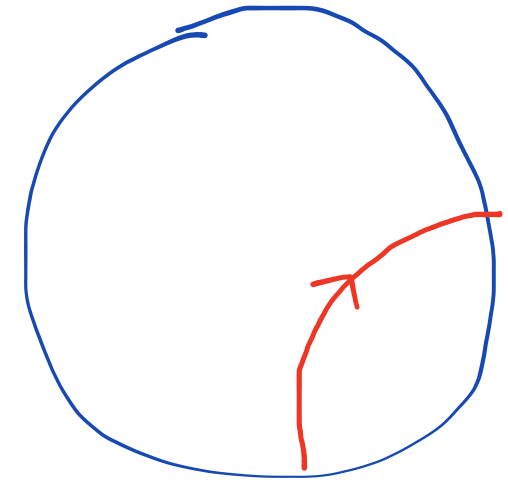
equivalently: a globally minimizing geodesic.

Def. A **metric ray :** isometric image of the closed half-line [0, \infty):

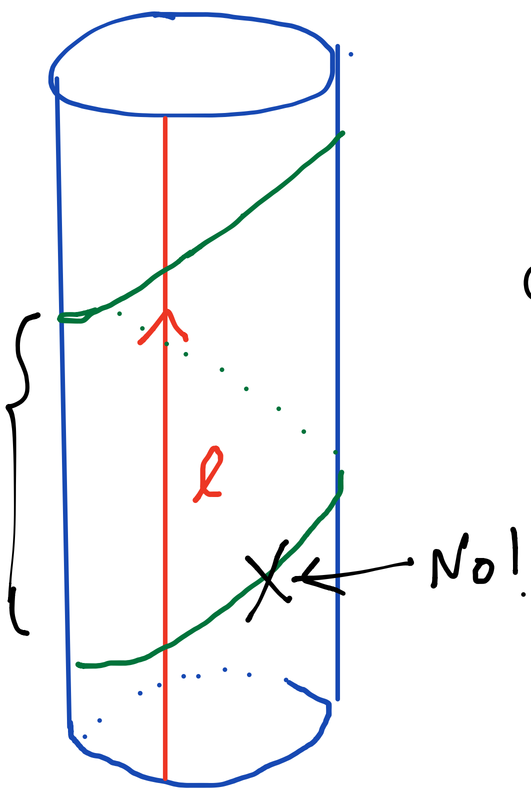
Def. A **minimizing geodesic :** isometric image of a compact interval [a,b].



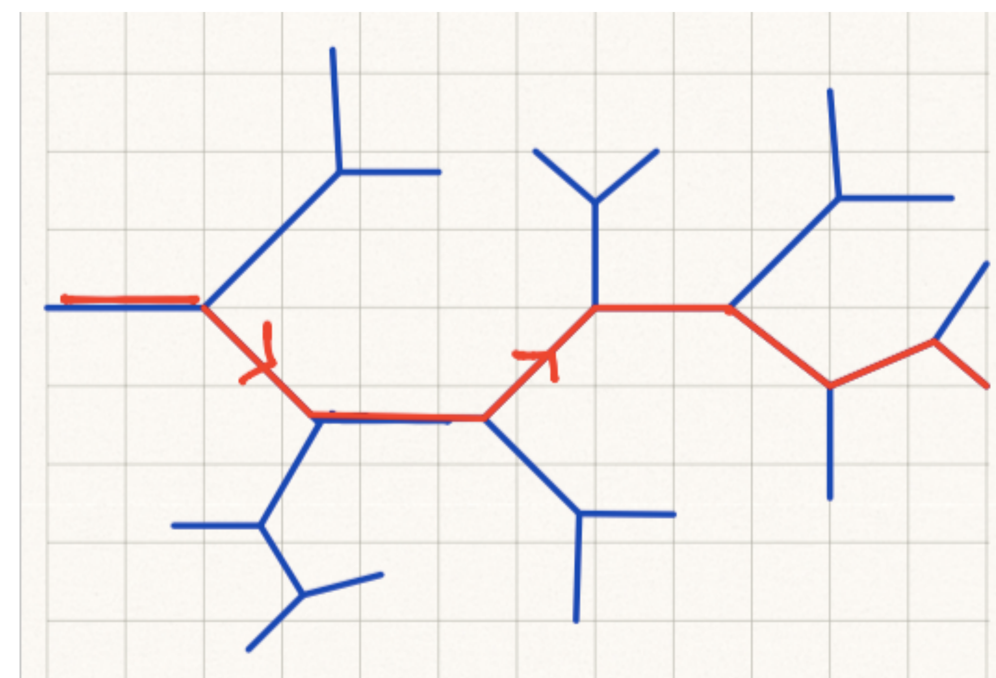
Euclidean space



Hyperbolic plane



cylinder



metric tree

1. **What are the metric lines for the N-body problem?**

using the Jacobi-Maupertuis metric formulation of dynamics to measure distances

2. **What are the metric lines for homogeneous subRiemannian geometries?**

commonality: like those of Riemannian geometries,
the geodesics of these geometries are generated by Hamiltonian flows

3. **Scattering in the N-body problem: how do asymptotic** (Euclidean) **rays at t = -\infty get mapped to asymptotic lines at t = + \infty?**

Why care? …

I. **What are the metric lines for the N-body problem?**

-> *use the Jacobi-Maupertuis metric formulation of its dynamics*

report on work of E Maderna and A Venturelli

thanks A Albouy , V Barutello, H Sanchez, Maderna, Venturelli

2. **What are the metric lines for homogeneous subRiemannian geometries?**

commonality: like those of Riemannian geometries,
the geodesics of these geometries are generated by Hamiltonian flows

with A Ardentov, G Bor, E Le Donne, Y Sachkov

report on work of A. Anzaldo-Meneses[a)] and F. Monroy-Perez; A Doddoli

3. **Scattering in the N-body problem: how do asymptotic** (Euclidean) **lines**
at t = -\infty get mapped to asymptotic lines at t = + \infty?

with Nathan Duignan, Rick Moeckel and Guowei Yu

thanks A Knauf, J Fejoz, T Seara, A Delshams, M Zworski, R Mazzeo

Warm-up: Kepler problem = 2-body problem

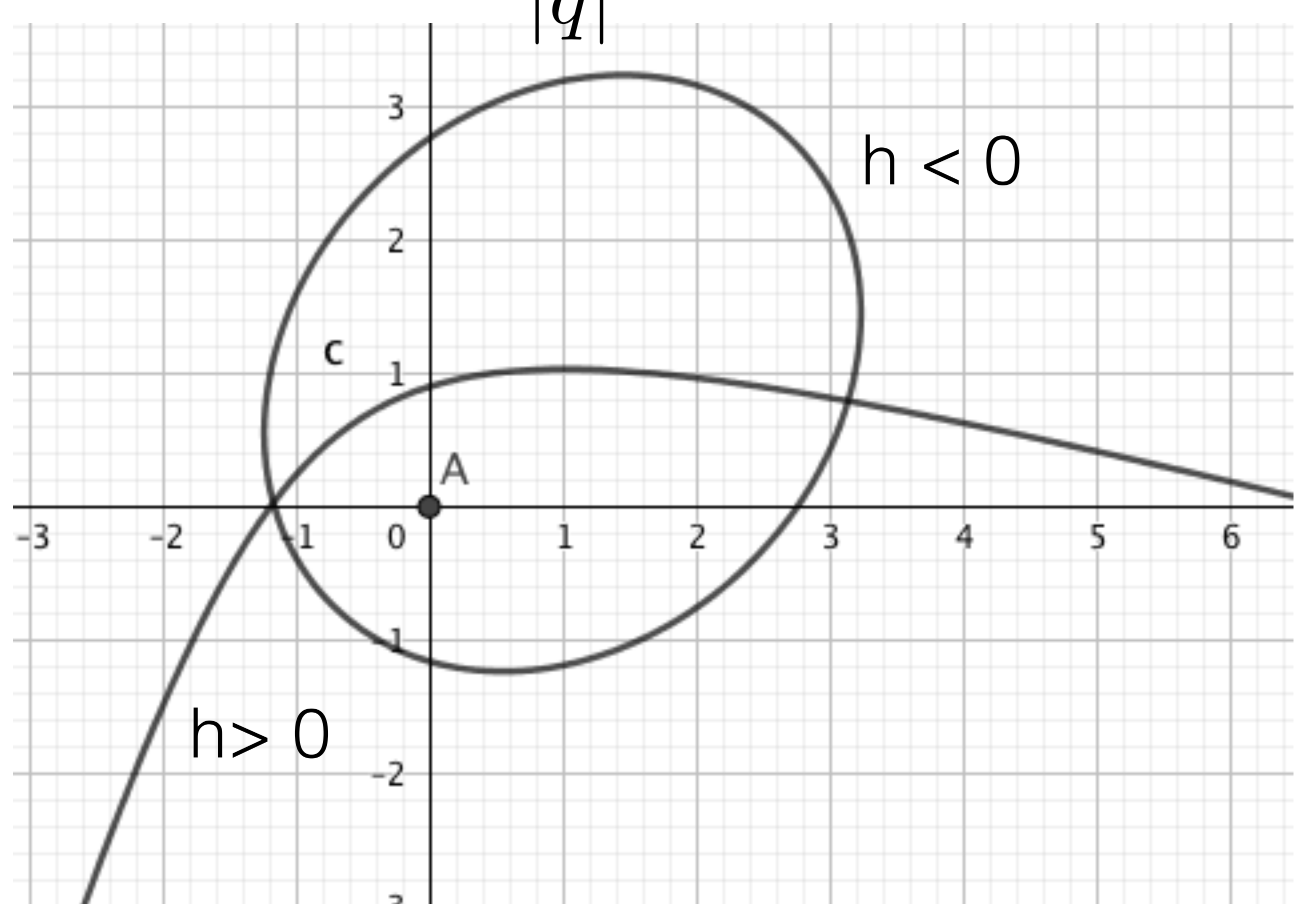
$$\ddot{q} = -\frac{q}{|q|^2}$$

$$E(q, \dot{q}) = \frac{1}{2}|\dot{q}|^2 - \frac{1}{|q|}$$
$$= h$$

Jac.-Maup. metric: $$\boxed{ds_h^2 = 2(h + \frac{1}{|q|})|dq|^2}$$

on domain $$\Omega_h = \{q \in \mathbb{R}^2 : h + \frac{1}{|q|} \geq 0\}$$ = `Hill region'

geodesics
=solutions having
energy h, up to
a reparam.
=Kepler conics



Metric properties.

Ω_h is a complete metric space.

Riem., except at the Hill boundary $\partial\Omega_h$

and collision q=0. Solutions are metric geodesics

up until they hit the Hill boundary or **collision (hit the `Sun')**

beyond which instant they cannot be continued as geodesics.

The conformal factor vanishes at the Hill boundary and is infinite at collision

$$ds_h^2 = 2(h + \frac{1}{|q|})|dq|^2$$

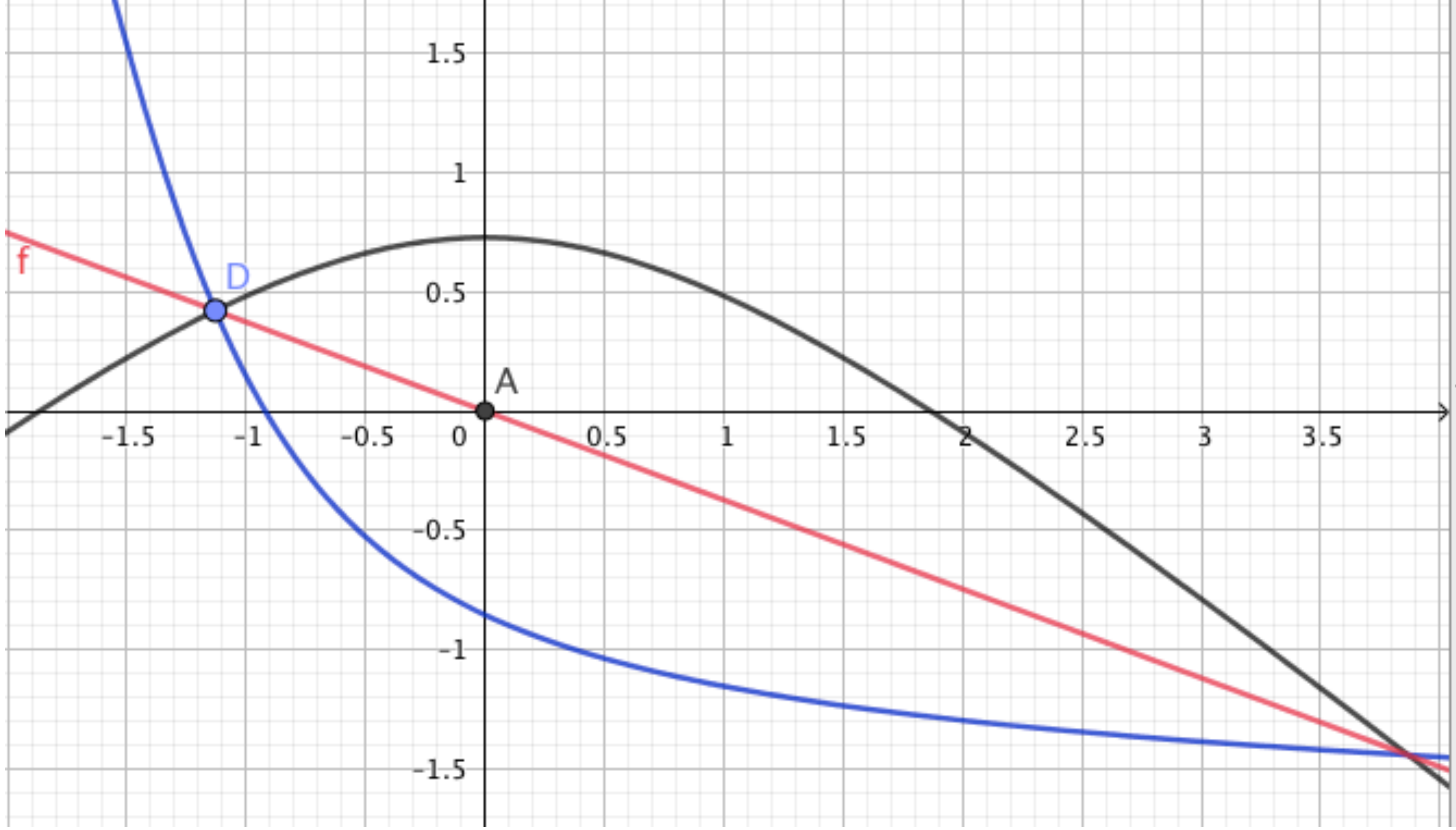
h < 0. $\Omega_h = B(2a)$

h= -1/2a. No metric lines!

h > 0, (or h =0). $\Omega_h = \mathbb{R}^2$ still **no metric lines.**

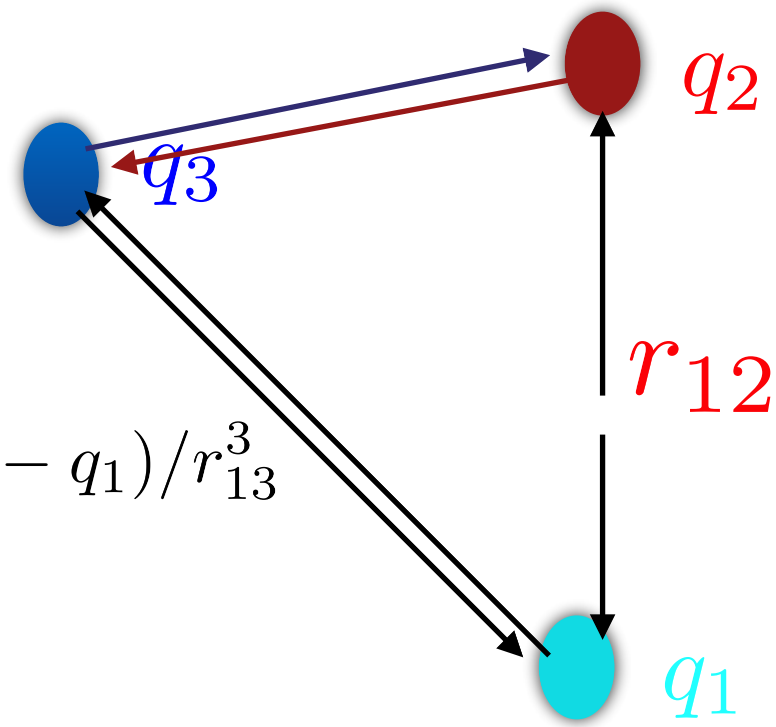
many **metric rays**: all the Kepler hyperbolas (or parabolas) up to aphelion (closest approach to `sun')

¿ Why .. ?



cut point/ reflection argument

N-bodies, i =1, 2, .. N.



$$F_{13} = Gm_1m_3(q_3 - q_1)/r_{13}^3$$

N=3 or greater: **Conjecture: there are no metric lines**

for the JM metric
(which depends on energy h).

What's known? JM metric depends on energy h :

h= 0: [da Luz-Maderna]**No metric lines.** Many **metric rays**
``On the free time minimizers of the Newtonian N-body problem''

h> 0: [Maderna-Venturelli] Many metric **rays.**

any **lines?** -open.

h < 0: N= 3, ang. mom zero: no **metric rays,** so no **metric lines** (*).

conjecture : no metric rays if h < 0

(*) proof: `Infinitely many syzygies, II' implies cut points along any sol'n)

Set-up and eqns.

$$q = (q_1, \dots, q_N) \in \mathbb{E} := \mathbb{R}^{Nd} \qquad q_a \in \mathbb{R}^d, a = 1, \dots, N$$

Conserved energy

$$E(q, \dot{q}) = \frac{1}{2}\langle \dot{q}, \dot{q} \rangle_m - G \sum \frac{m_a m_b}{r_{ab}}$$
$$= h.$$
$$= K(\dot{q}) - U(q)$$

where

$$2K(\dot{q}) = \langle \dot{q}, \dot{q} \rangle_m = \sum m_i \|\dot{q}_i\|^2 =$$

and

$$U(q) = G \sum \frac{m_a m_b}{r_{ab}}$$

Newton's eqns: $\iff \ddot{q} = \nabla_m U(q)$

where $\langle \nabla_m U(q), w \rangle_m = dU(q)(w)$

Solutions for fixed E = h are reparam's of geodesics for the JM -metric:

$ds_h^2 = 2(h + U(q))|dq|_m^2$ on $\Omega_h = \{q : h + U(q) \geq 0\}$

Ω_h is a complete metric space.
Riemannian **except** at the Hill boundary h + U(q) = 0
and at the collision locus h + U(q) = + ∞

Solutions to Newton at energy h are metric geodesics
up until they hit the Hill boundary
or **the collision locus**
beyond which instant they cannot be continued
as geodesics.

$$h \geq 0 \implies \Omega_h = \mathbb{R}^{Nd}$$

Dynamical implications of positive energy.

$$I(q) = \|q\|_m^2$$

$$h \geq 0 \implies \Omega_h = \mathbb{R}^{Nd}$$

A solution is ***bounded*** iff I(q(t)) is bounded.

$$\dot{I} = 2\langle q, \dot{q} \rangle_m$$

$$\begin{aligned} \ddot{I} &= 2\langle \dot{q}, \dot{q} \rangle_m + 2\langle q, \ddot{q} \rangle_m \\ &= 4K - 2U(q) \\ &= 4h + 2U(q) \end{aligned}$$

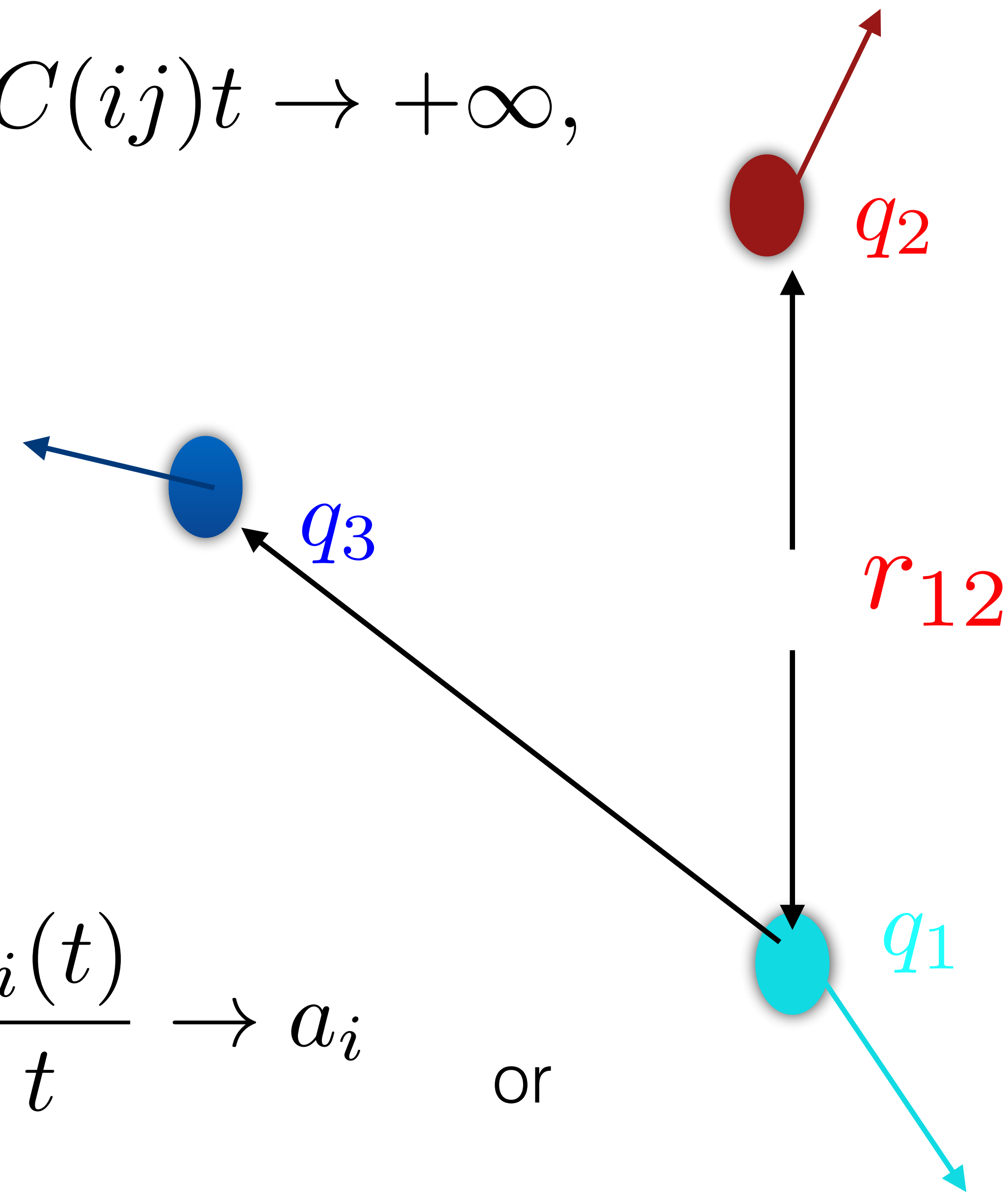
Since U > 0 : $h \geq 0 \implies \ddot{I} > 0$ along a solution.

$$h \geq 0 \text{ and defined for } t \in [0, \infty) \implies \text{ unbounded}$$

($\text{periodic} \implies \text{bounded} \implies h < 0$)

Def a solution is ***hyperbolic*** iff

$$r_{ij}(t) \sim C(ij)t \to +\infty,$$



equivalently: $\dfrac{q_i(t)}{t} \to a_i$ or $\dot{q}_i(t) \to a_i \neq 0$

$$a_i \neq a_j, i \neq j$$

Note: then h = K(a) > 0.

Thm: [Chazy, 1920s]: any hyperbolic solution q(t) satisfies

$$q(t) = at + (\nabla_m U(a)) \log t + c + f(t) \qquad \text{as t } \to \infty$$

with f(t) = O(log(t)/t), and f(t) = g(1/t, log(t)), g analytic in its two variables.

and $a \in \mathbb{R}^{Nd} \setminus \{ \text{ collisions } \}$

Think of a as an asymptotic position at infinity.

Question: Given a , q_0 in $\mathbb{R}^{Nd}$ with **a** not a collision configuration.
Does there exist a hyperbolic solution connecting q_0 at time 0 to **a** at time ∞ ?

Thm [Maderna-Venturelli; 2019]. YES. Moreover this solution is a metric **ray** for the JM metric with energy h = K(a)= (1/2) |a|^2 .

Method of proof: Weak KAM, a la Fathi
for
$H(q, dS(q)) = h$

so: calculus of variations + some PDE

Metric input: Buseman, Buseman functions as
solutions to the (weak) Hamilton-Jacobi eqns
some Gromov ideas re the boundary at infinity

change gears

subRiemannian geometry

2. **SubRiemannian geometry**

$$X = \sum X^\mu(q) \frac{\partial}{\partial q^\mu} \qquad\qquad Y = \sum Y^\mu(q) \frac{\partial}{\partial q^\mu}$$

smooth vector fields on an n-dim. manifold Q.

Def. A path q(t) in Q is ``horizontal'' if $\dot{q} = u_1(t)X(q(t)) + u_2(t)Y(q(t))$

sR Geodesic problem: find the *shortest* horizontal path q(t) joining q_0 to q_1.

where $$\ell(q(\cdot)) = \int \sqrt{u_1(t)^2 + u_2(t)^2}dt$$

Such a path, if it exists, is a ***sR geodesic***.

[Chow-Rashevskii]

If X, Y, [X,Y], [X, [X, Y]], … eventually span TQ and if Q is connected then any two points are joined by a horiz. curve and the corresponding distance function:

$$d(q_0, q_1) = \inf\{\ell(q(\cdot)) : q(t) \text{ horizontal } q \text{ joins } q_0 \text{ to } q_1\}$$

gives Q the same topology as the manifold topology.

and sR **geodesics** exist, at least locally

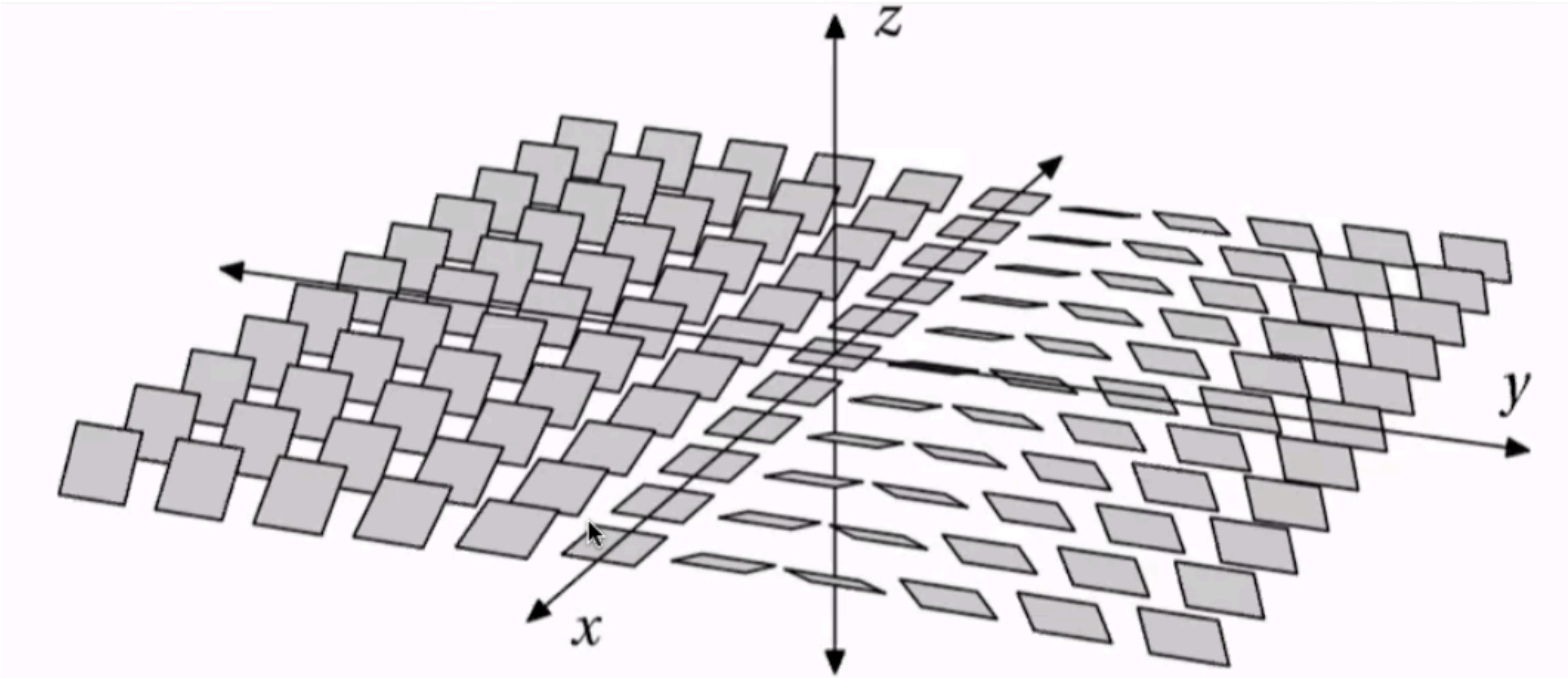
Geodesics: (most) are generated by

$$H = \frac{1}{2}(P_1^2 + P_2^2) \qquad : T^*Q \to \mathbb{R}$$

$$P_1 = P_X = \sum p_\mu X^\mu(q) \qquad\qquad P_2 = P_Y = \sum p_\mu Y^\mu(q)$$

Example: $Q = \mathbb{R}^3$

$$X = \frac{\partial}{\partial x} + A_1(x,y)\frac{\partial}{\partial z} \qquad Y = \frac{\partial}{\partial y} + A_2(x,y)\frac{\partial}{\partial z}$$



(A_1 (x,y) = 0, A_2 (x,y) = x : standard contact distribution.)

Then

$$H = \frac{1}{2}\{(p_x + A_1(x,y)p_z)^2 + (p_x + A_2(x,y)p_z)^2\}$$

no z's

so $\dot{p}_z = 0$

View the const. parameter p_z as electric **charge**

Then H is the Hamiltonian of a particle of mass 1 and this charge moving in the xy plane under the influence of the magnetic field B(x,y) where

$$B(x,y) = \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}$$

$$\{P_1, P_2\} = -B(x,y)p_z$$

Eqns of motion:

1) For plane curve part:

c(t)= (x(t), y(t)) $= \pi(x(t), y(t), z(t)); \pi : Q = \mathbb{R}^3 \to \mathbb{R}^2$

$$\boxed{\kappa(s) = \lambda B(x(s), y(s))}$$

s = arc length

$\lambda = p_z$ = ``charge''

$\kappa(s) =$ plane curvature of c(s)

2) z(t) determined from c(t) by horizontality (by being tangent to distribution D = span(X,Y):

$$z(s) = z(0) + \int_{c([0,s])} A_1(x,y)dx + A_2(x,y)dy$$

(call (x(s), y(s), z(s)) = ``horizontal lift'' of c(s) = (x(s), y(s).)

Observe:

Straight lines in the plane are solutions (with charge 0), for **any** B(x,y)

Their horizontal lifts are always metric lines

since $$\pi : Q = \mathbb{R}^3 \to \mathbb{R}^2$$

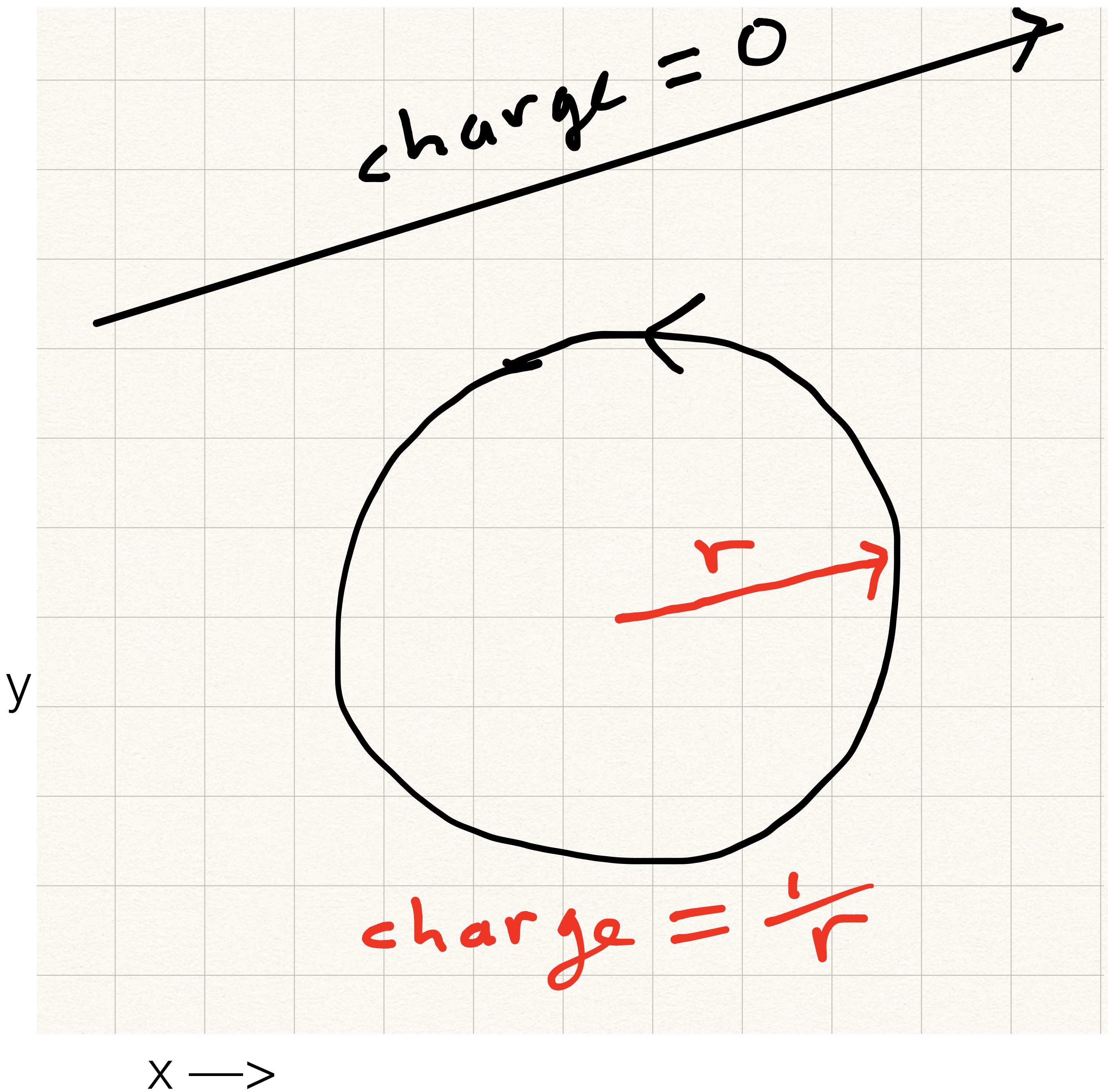
satisfies $$\ell(\gamma) = \ell_{\mathbb{R}^2}(\pi \circ \gamma)$$

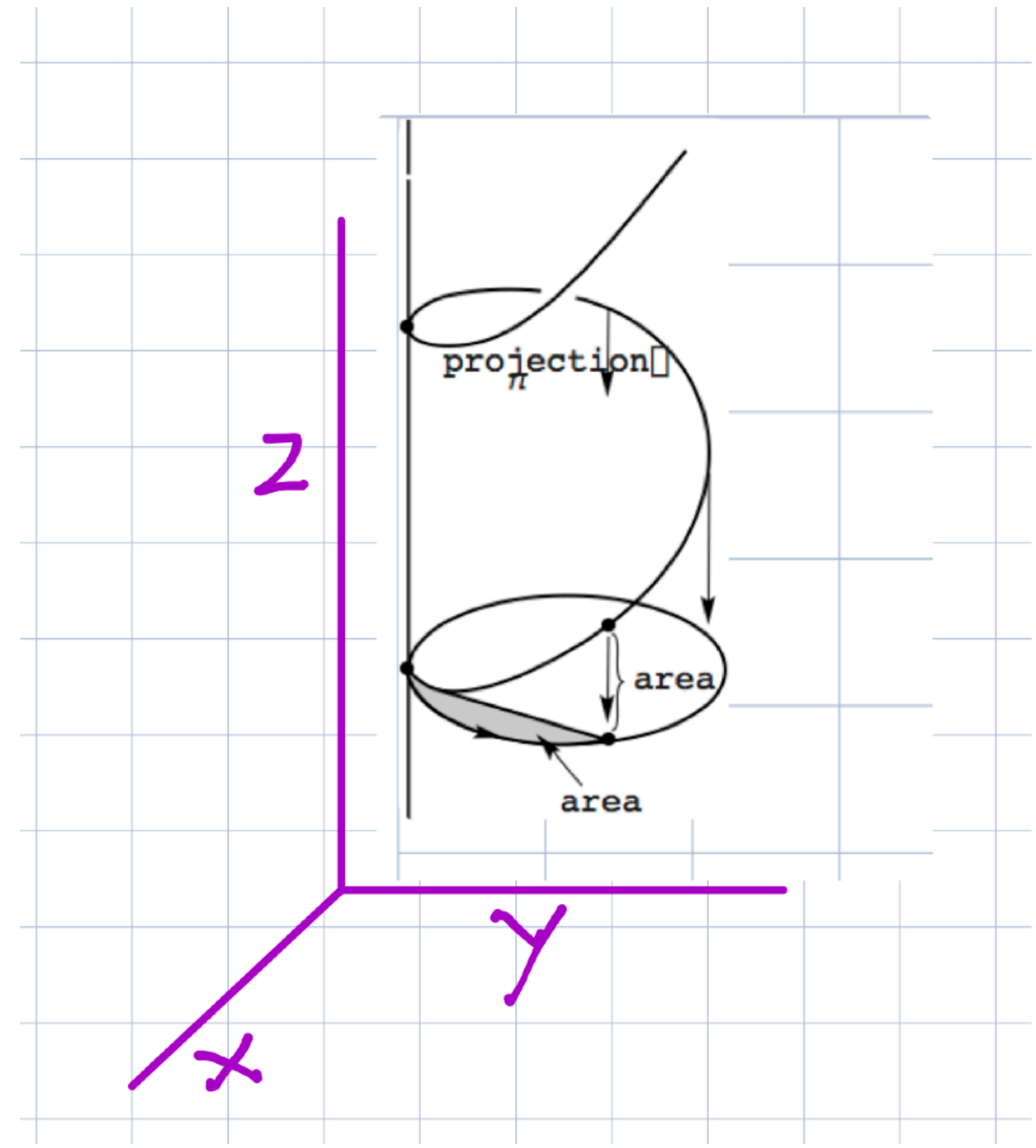
for any horizontal curve γ

Question [LeDonne]: are there any other metric lines besides those whose projections are straight lines ?

Case B(x,y) = 1. ``Heisenberg group''

Eqns for projected geod.: $\kappa = \lambda$





Theorem. **No:** The only metric lines for the Heisenberg group are those projecting onto Euclidean lines in the plane.

`Martinet case': B(x,y) = x.

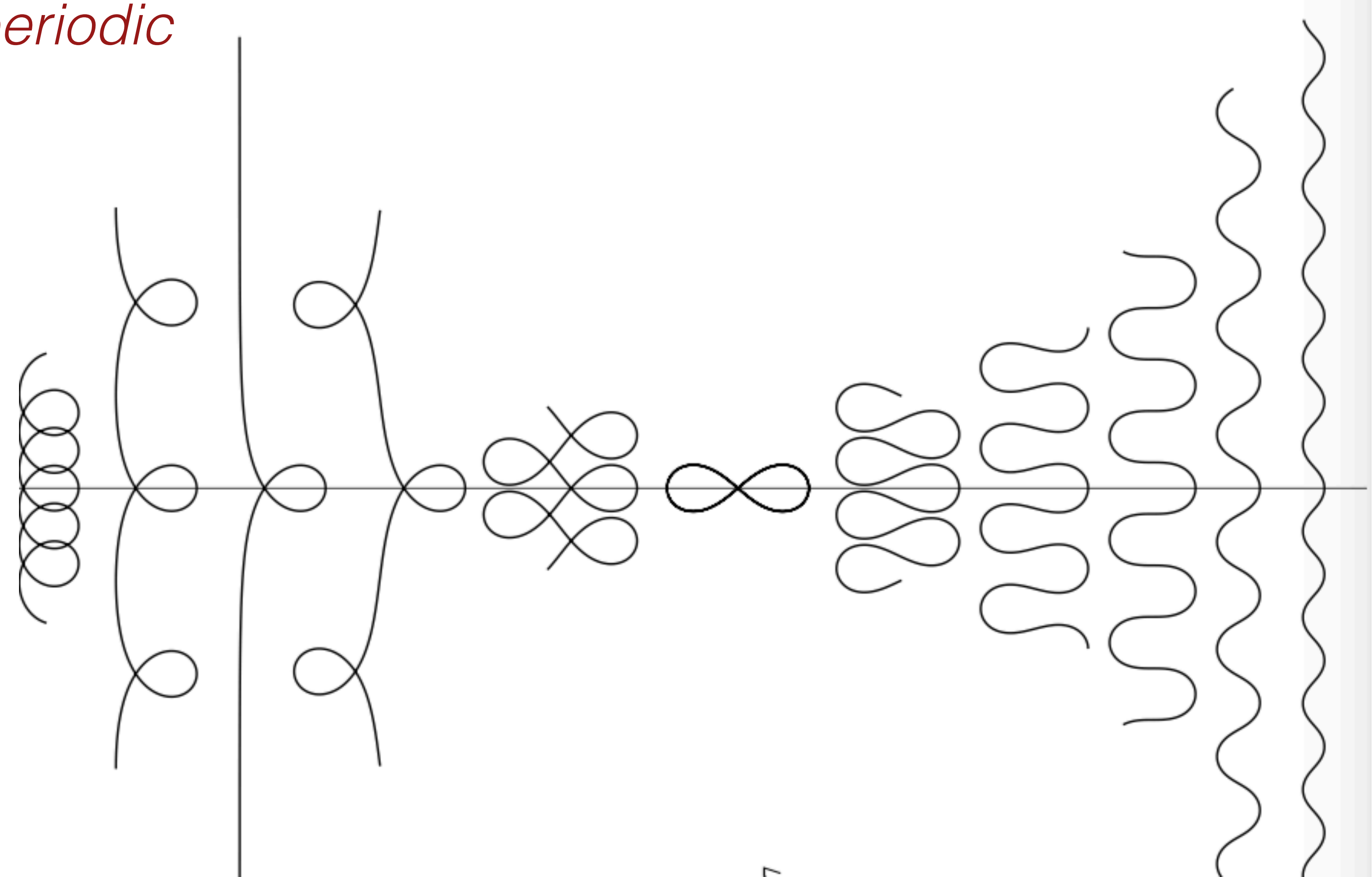
Geod eqns: $\kappa = \lambda x$

Theorem. **[Ardentov-Sachkov] Yes.**

The Euler kinks correspond to the other metric lines.

These are the full list of projected geodesics. They are the Euler elastica aligned to have y-axis (x = 0) as directrix.

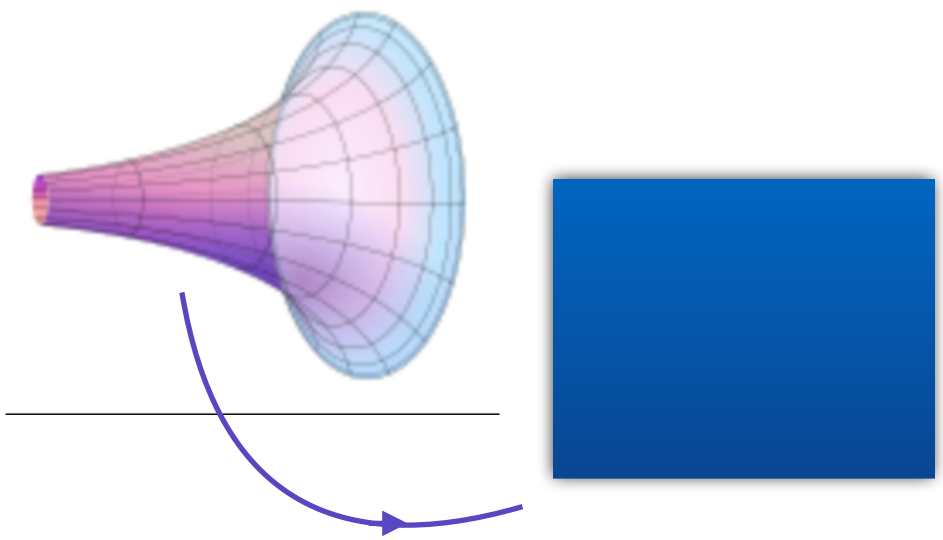
All but the kink are periodic in the x direction



Elastica also arise as the projections of the geodesics for:

rolling a ball (sphere) on the plane

[Jurdjevic-Zimmerman,..]



rolling a hyperbolic plane on the Euc. plane

[Jurdjevic-Zimmerman,..]

bicycling

[Ardentov-Bor-LeDonne-M., Sachkov]

and two Carnot groups:

Engel: (2,3,4) [**Ardentov-Sachkov]**

\`Cartan': (2,3,5) [**Moiseev-Sachkov]**

For all of these: $Q = \mathbb{R}^2 \times G$

$$X = \frac{\partial}{\partial x} + \xi_1(x,y) \qquad Y = \frac{\partial}{\partial y} + \xi_2(x,y)$$

$$\xi_1, \xi_2 : \mathbb{R}^2 \to \mathfrak{g}$$

so $\pi : Q \to \mathbb{R}^2$

satisfies $\ell(\gamma) = \ell_{\mathbb{R}^2}(\pi \circ \gamma)$ for any horizontal curve γ

In all these , only Euclidean lines and Euler kinks correspond to metric lines.

(For the rolling ball not all kinks that arise as projections
correspond to metric lines upstairs)

Why do the kinks give the only additional lines**?**

a) exclude all the other Elastica

b) verify kinks are metric lines

a) **Prop. [Hakavuouri-LeDonne]** : any curve which is the horizontal lift of a planar curve periodic in one direction, **cannot** be a metric line unless the planar curve is a Euclidean line

`periodic in x direction' : means $(x(s), y(s))$ satisfies

$$x(s + L) = x(s), \quad y(s + L) = a + y(s)$$

All non-kink elastica are periodic in the direction orthogonal to their directrix, ie. in the x direction for the Martinet case

Pf of **Prop**. : metric blow-downs .

b): By hand [`optimal synthesis'] sorting out ***all*** *cut and conjugate points*
in all case
except **bicycling**, where we have a simple conceptual proof inspired by `bicycling mathematics':
by Ardentov, Bor, LeD, M-, Sachkov

Why?

a) excluding all the other Elastica

b) verifying the kinks

a) **Prop. [Hakavuouri-LeDonne]** A plane curve which is *periodic in one direction* cannot be the projection of a metric line unless that plane curve is a line

$c(s) = (x(s), y(s))$ is *periodic in x* means that there is a constant $L > 0$ [the x-period] such that

$$x(s + L) = x(s), \quad y(s + L) = a + y(s)$$

All elastica except the kink are periodic in the direction orthogonal to their directrix.

Pf of **prop**. : metric blow-downs. The blow-down of a periodic in-one-direction curve is a line NOT parameterized by arc length…

b): by hand [`optimal synthesis']
in all case **except** bicycling where the proof is simple and inspired by `bicycling mathematics':
Ardentov, Bor, LeD, M-, Sachkov

Geodesics and metric for the simplest jet spaces

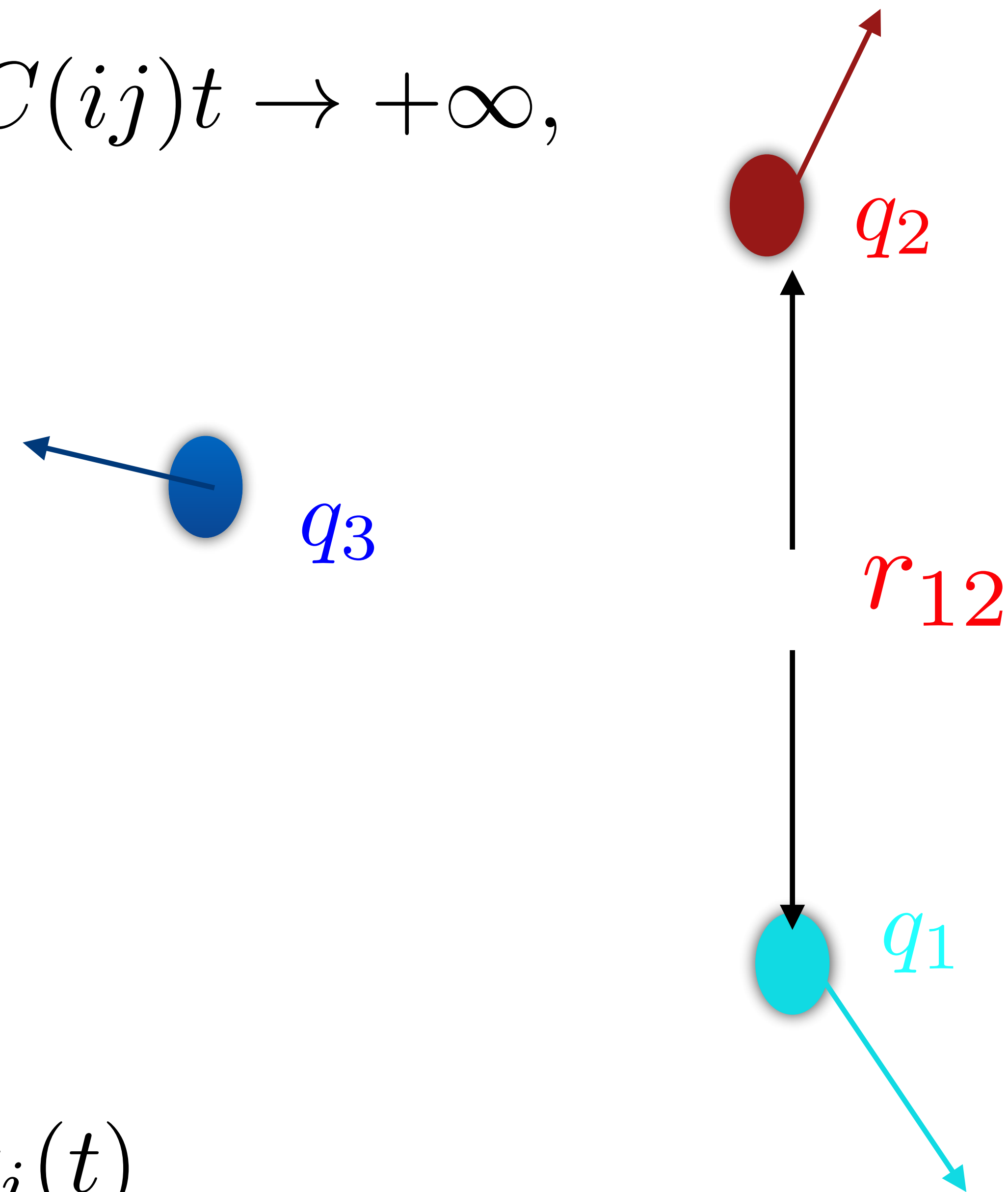
A. Anzaldo-Meneses-Felipe Monroy Perez, 2005;
B. Doddoli, 2019-2020

change gears

Scattering in the N body problem

Def a solution is ***hyperbolic*** iff

$$r_{ij}(t) \sim C(ij)t \to +\infty,$$



equivalently: $\dfrac{q_i(t)}{t} \to a_i$ or $\dot{q}_i(t) \to a_i \neq 0$

$$a_i \neq a_j, i \neq j$$

Think of **a = (a_1, … a_N)** as initial ``positions'' at infinity.

Question: Can we join a given **a** for t = -\infty to a given **b** for t = + \infty by a collision-free hyperbolic solution?

Necessary conditions:
K(**a)** = K(**b**) [conservation of energy],
P(**a**) = P(**b)** [conserv. of Lin. Momentum]

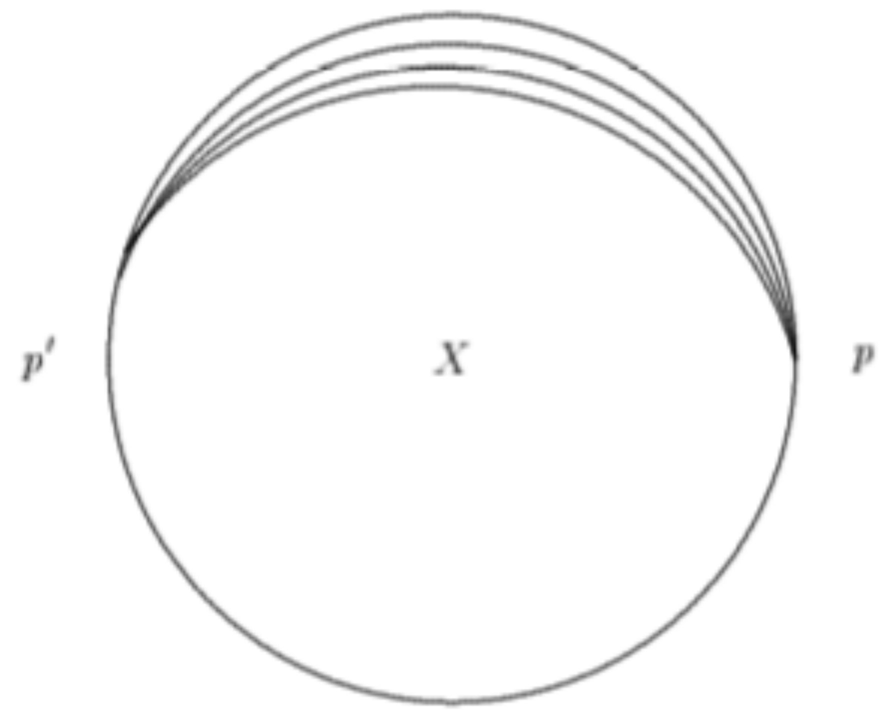
$$K(a) = \frac{1}{2}\sum m_i |a_i|^2$$

$$P(a) = \sum m_i a_i$$

and **a**, **b** collision -free.

Kepler case (N=2) : Yes! as long as **a** \ne \pm **b.**

General case: ??.
Thm. [Duignan, Moeckel, M-, Yu]
Yes, **provided b** lies in a small open punctured nbhd of **a.**



p. 80. Geometric Scattering Theory -Melrose.

Fig. 11. Geodesic of a scattering metric.

`Spherical ' change of var's :

$$r^2 = I(q) = \|q\|_m^2$$

$$\rho = 1/r$$

$$dt = r d\tau$$

$$s \in \mathbb{S} \cong S^{Nd-1}$$

$$\mathbf{q} = r\mathbf{s}$$

$$\dot{\mathbf{q}} = v\mathbf{s} + \mathbf{w}, \mathbf{w} \perp \mathbf{s}$$

ENERGY: $\frac{1}{2}v^2 + \frac{1}{2}\|w\|^2 - \rho U(s) = h.$

Newton's eqns $\Longleftrightarrow$

$$\rho' = -v\rho$$

$$s' = w$$

$$v' = |w|^2 - \rho U(s)$$

$$w' = \rho\tilde{\nabla}U(s) - vw - |w|^2 s$$

Spatial Infinity : $\Longleftrightarrow$ $\rho = 0$

, an invariant submanifold

Flow at infinity. Set $\rho = 0$.

$$s' = w$$
$$w' = -vw - \|w\|^2 s$$
$$v' = \|w\|^2$$

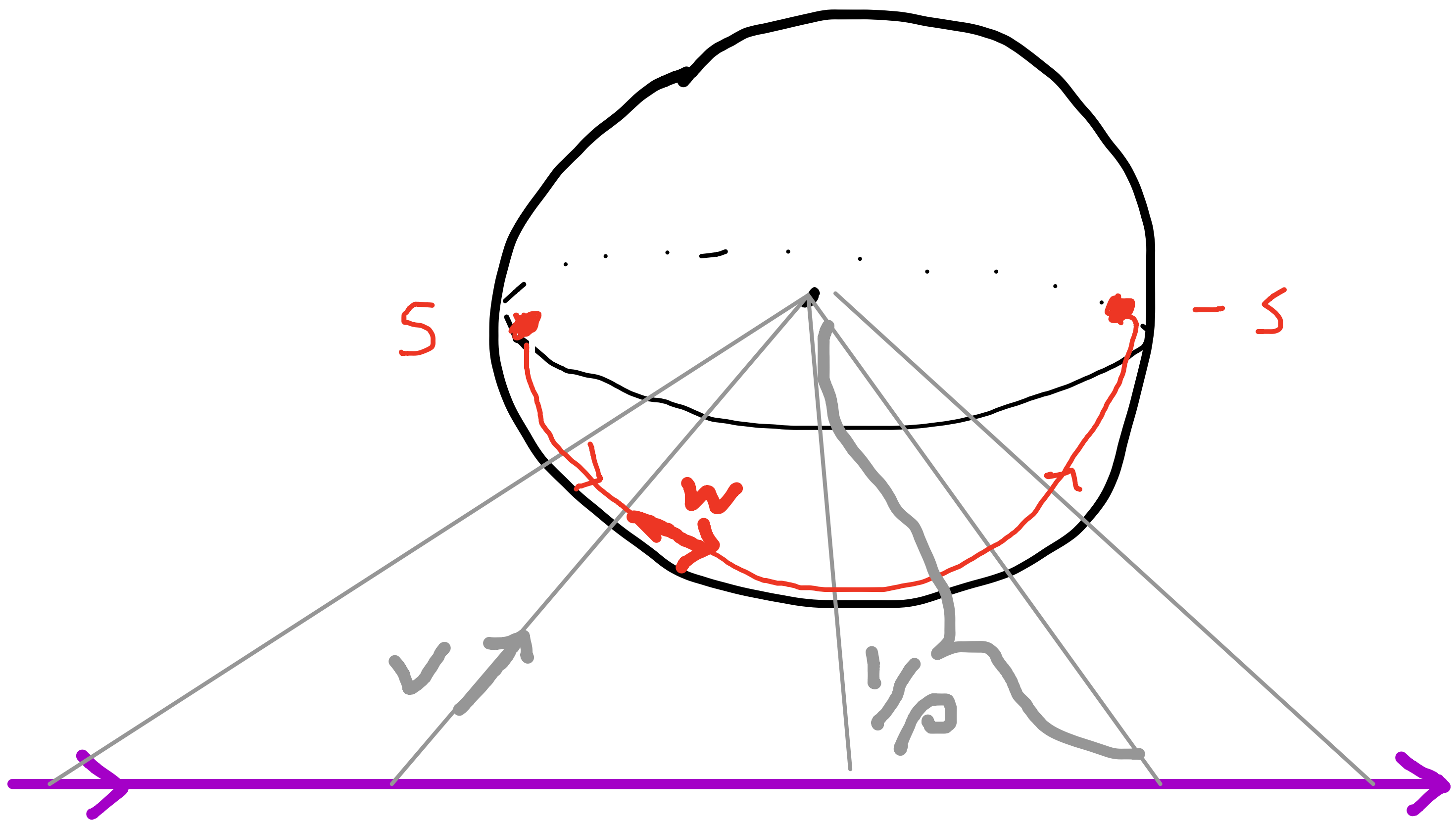
$$s \in \mathbb{S} \cong S^{dN-1}$$

$$v \in \mathbb{R}, v \neq 0$$

Energy at infinity: $$\frac{1}{2}v^2 + \frac{1}{2}\|w\|^2 = h.$$

Flow *at infinity is* ***independent*** *of U.*

Set U = 0 to understand the dynamics at infinity.
Flow = reparam. of free motion! :



s, -s become equilibria! ; flow is gradient like between them…

Equilibria!
Only at infinity.
Given by:

$$(\rho, s, v, w) = (0, s, v, 0)$$
$$s \in \mathbb{S} \cong S^{dN-1}$$
$$v \in \mathbb{R}, v \neq 0$$

Energy of an equilibrium: $h = \frac{1}{2}v^2$ so $v = \pm\sqrt{2h}$

$$\text{Equilibria} = \Sigma_- \cup \Sigma_+$$

- branch: v< 0 . Incoming. LINEARLY UNSTABLE mfd of fixed points.
+ branch: v > 0 . Outgoing: LINEARLY STABLE mfd of fixed points

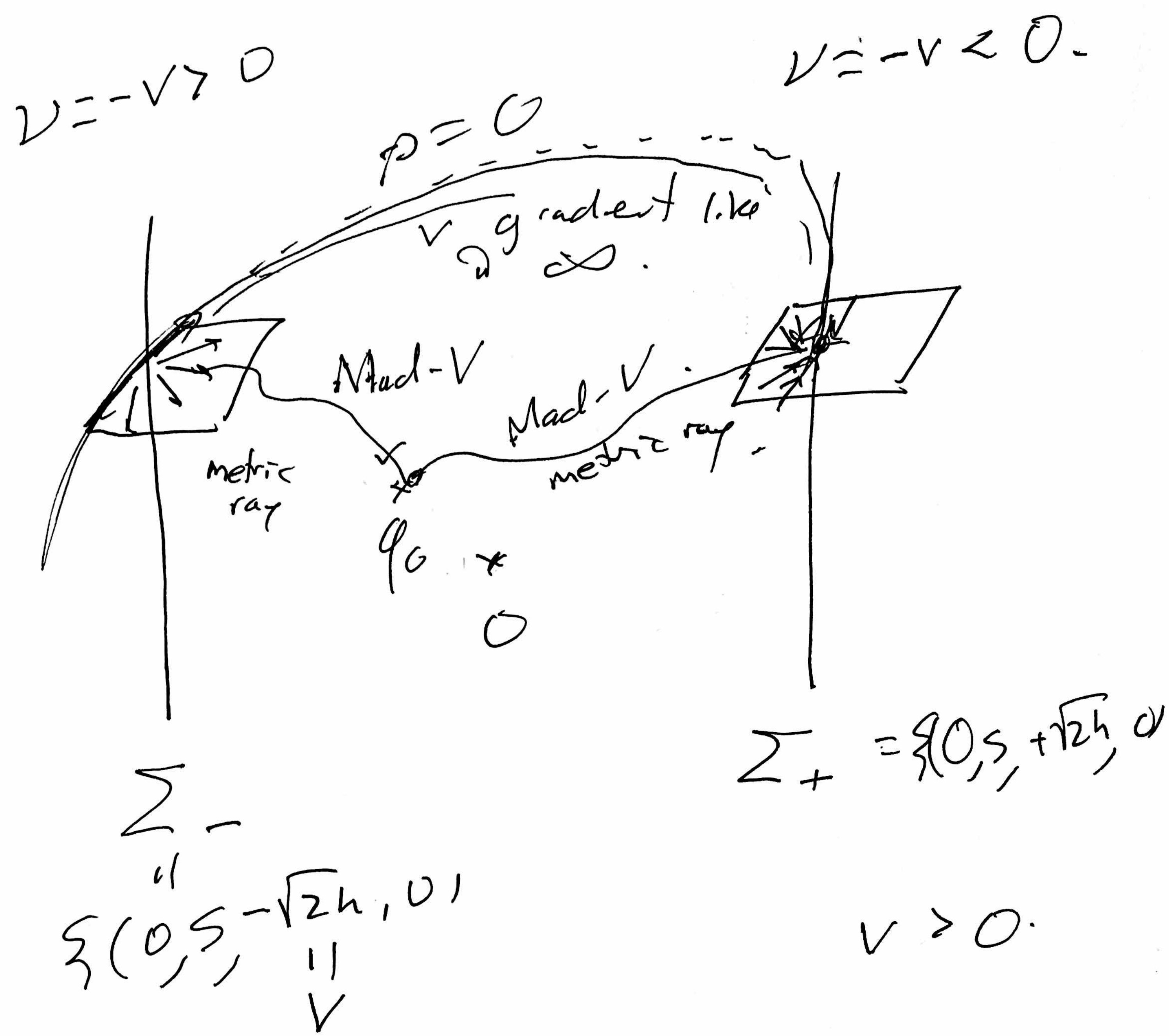
eigenvalues: 0 in the s and v directions, ie along **Equilibria**
-v in the w-direction.

… in the ρ direction..?

Push in to ``bulk'' — the real N-body phase space by turning on $\rho > 0$.

$\Sigma_{\pm}$ are **normally hyperbolic !**

Generalized eigenvector corresponding to $\delta\rho$



Summarizing :

Linearization at an equilibrium e :

e = $(\rho, s, v, w) = (0, s, v, 0)$

Spectrum of linearization at e: 0, -v

$espace = T_e\Sigma_{\pm}$

Un/stable manifold of an equilibrium e

$$W_{\mp}(e), e \in \Sigma_{\pm}$$

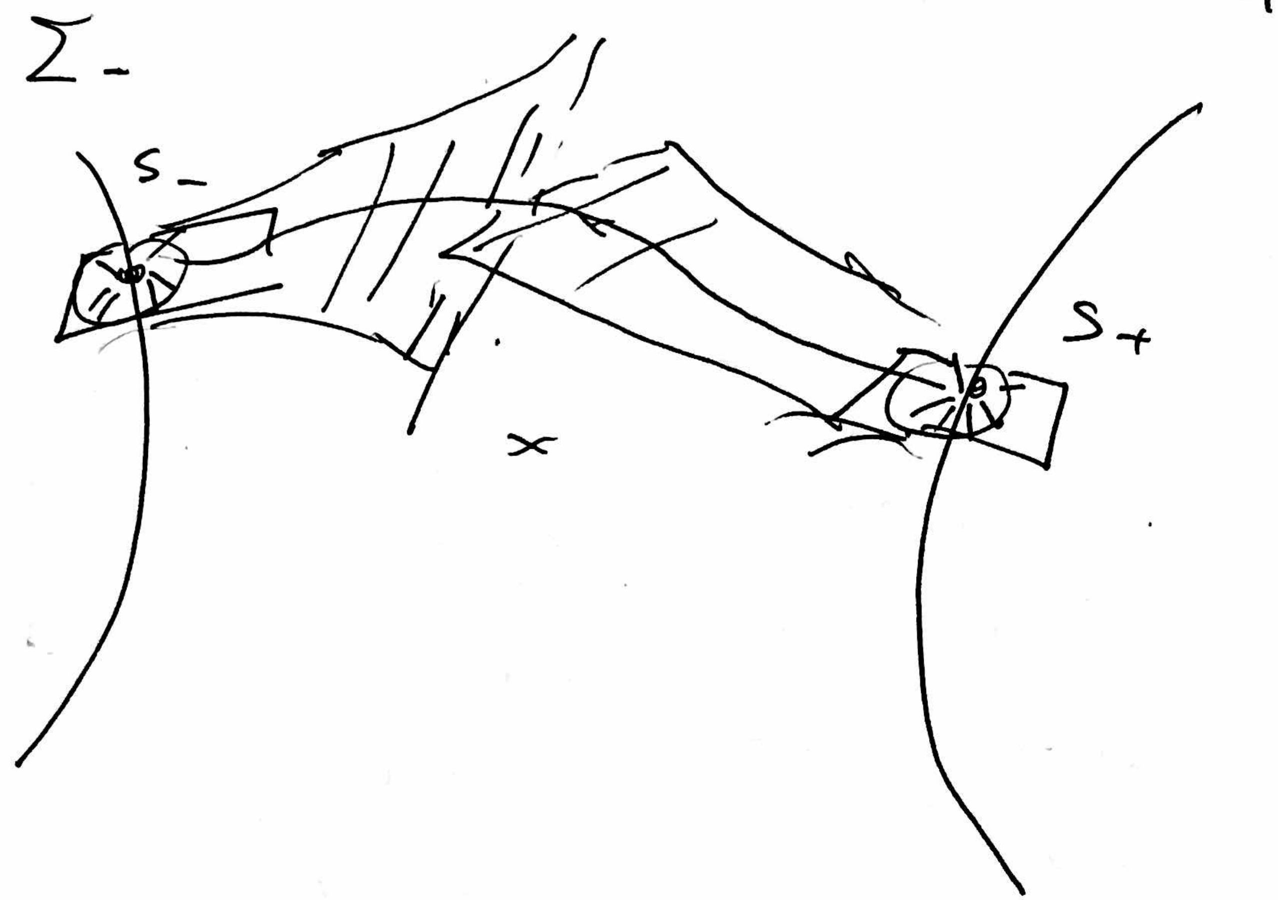
is Lagrangian in the bulk,
transverse to the equilibrium manifold

its tangent space at e is the nonzero generalized eigenspace for -v at e,

$$dim(W_{\mp}(e)) = dim(\Sigma_{\pm}) = dN =$$ half dim of phase space.

Normally hyperbolic
"b-Lag" submfds
of equilibria.

Σ_- Σ_+



$$s_- \overset{\gamma}{\rightsquigarrow} s_+$$

$$\Leftrightarrow W_u(s_-, -v_0) \cap W_s(s_+, +v_0) \neq \emptyset$$

26 MARIAN GIDEA, RAFAEL DE LA LLAVE, AND TERE SEARA

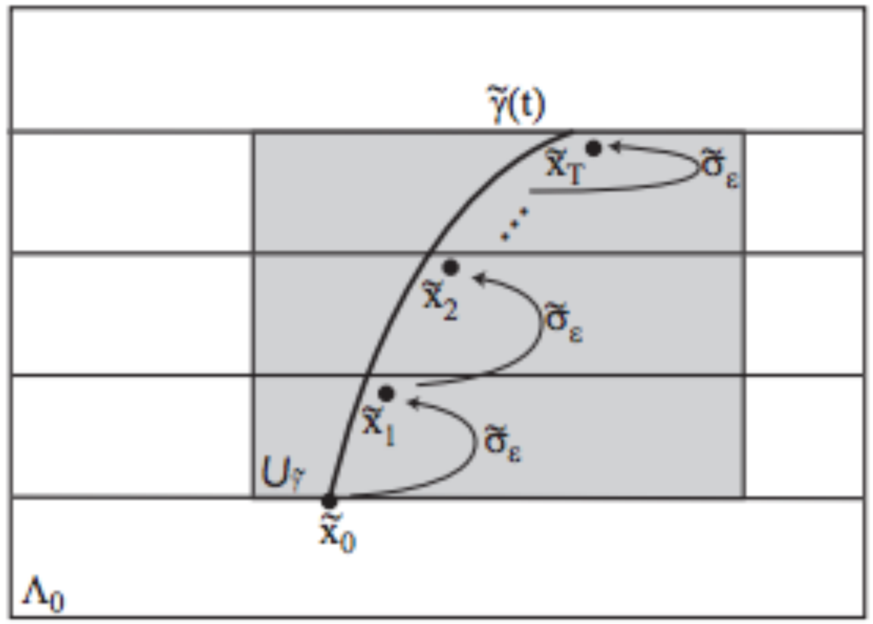


FIGURE 1. A scattering path and a nearby orbit of the scattering map.

A. Delshams, Tere Seara, R de la Llave, M Gidea, ….

Our scattering map is the same as their `scattering map' !
except **that their stable/unstable intersections**
are (1) typically homoclinic
and (2) they have a center manifold with a slow dynamics
in place of our manifold of equilibria

Fini !