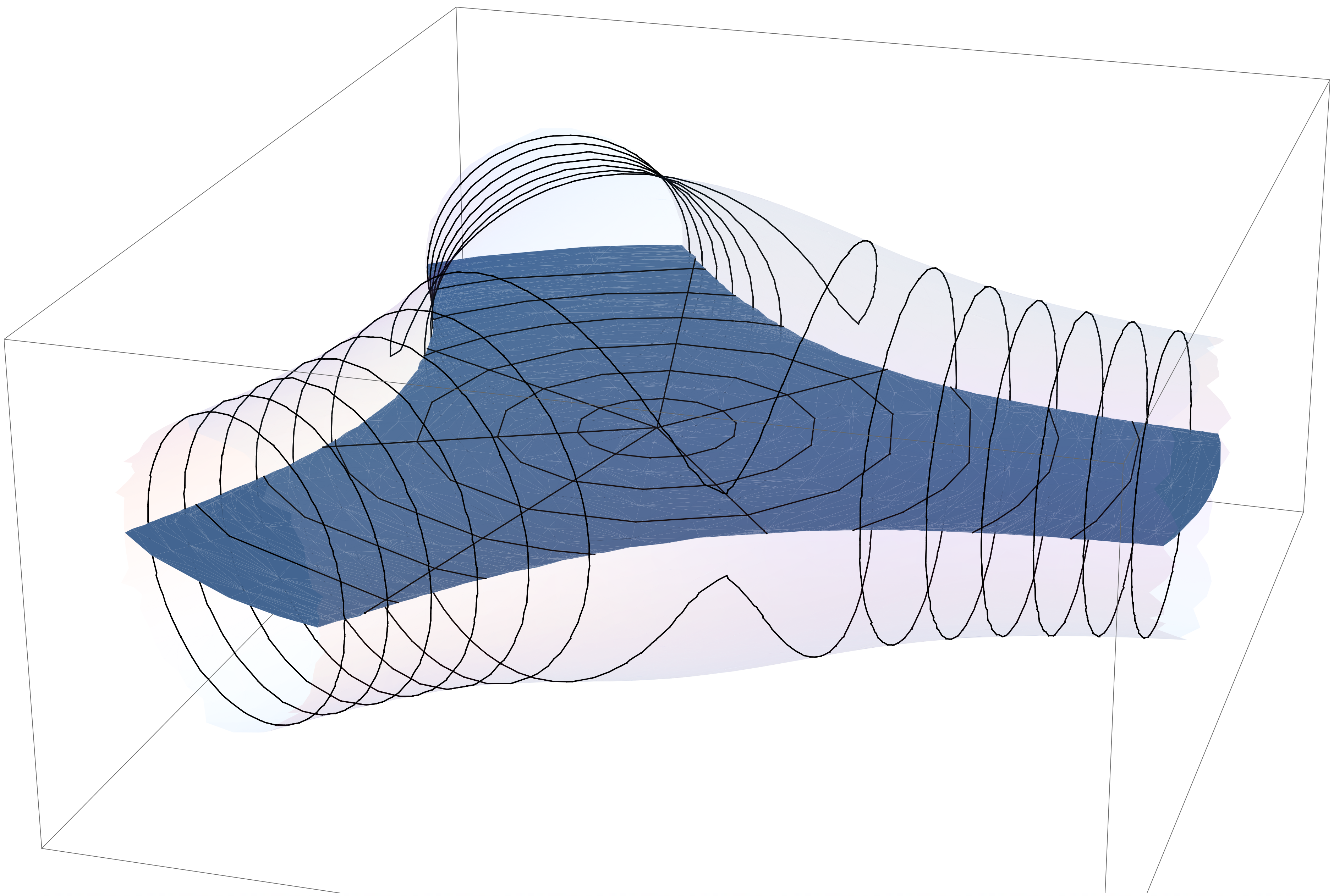
A few recent N-body results

Richard Montgomery
UC Santa Cruz
May 2025 San Luis AMS Conf.



THE NEGATIVE ENERGY N-BODY PROBLEM HAS FINITE DIAMETER

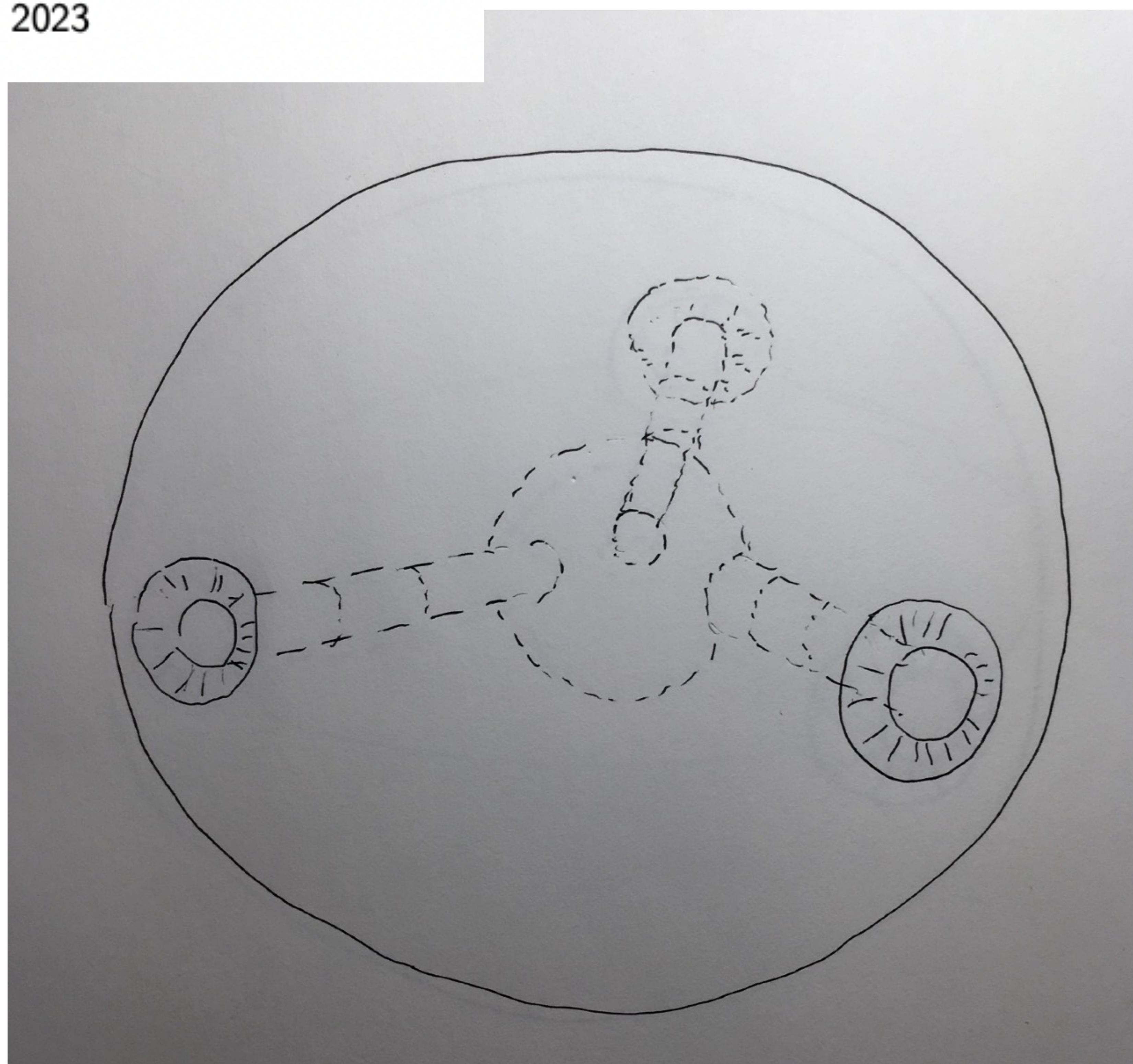
R. Mont. ;
arXiv: 2406.05563; Cel. Mech. and Dyn. Astr: Volume 137, article number 11, (2025)

Compactification of the energy surfaces for n bodies

Andreas Knauf* Richard Montgomery†

August 15, 2023

arXiv:2307.03837;
RCD. Oct 2023



Scaling Symmetries, Contact Reduction and Poincaré's dream

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A symplectic Hamiltonian system admitting a scaling symmetry can be reduced to an equivalent contact Hamiltonian system in which some physically-irrelevant degree of freedom has been removed.

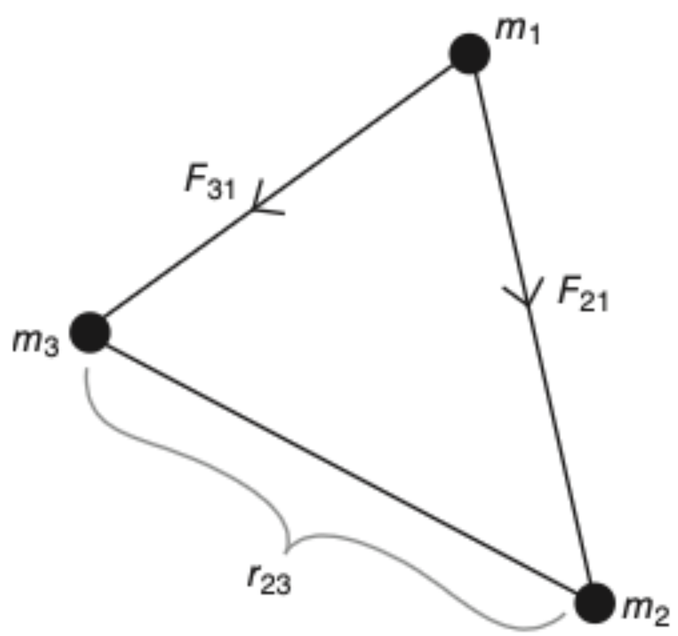
arXiv:2206.09911

`Hammers looking for nails'

These papers represent ``optical instruments"
which combined, give a picture of the Newtonian
N-body problem as an analytic flow
on a compact stratified analytic space.

Will this picture
solve any interesting problems?

Problem 1: the `oldest question' of Newton-Herman and parabolic infinity



$$m_1\ddot{q}_1 = F_{21} + F_{31},$$
$$m_2\ddot{q}_2 = F_{12} + F_{32},$$
$$m_3\ddot{q}_3 = F_{23} + F_{13},$$

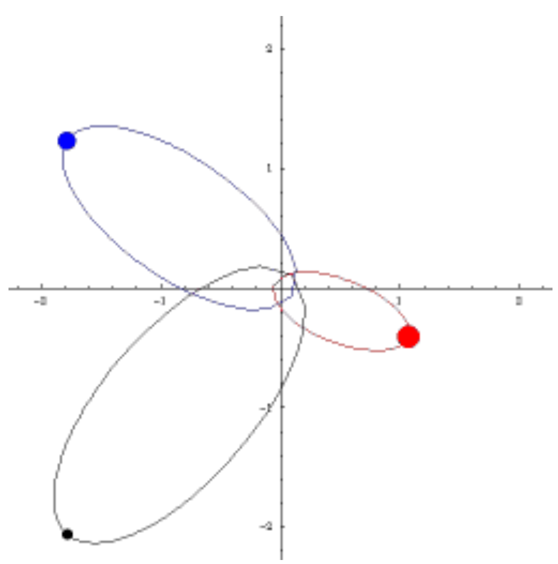
Bounded orbits: all $r_{ab}(t)$ are bounded functions of time

Unbounded orbits: some $r_{ab}(t) \to \infty$

OLDEST QUESTION: ¿ Do unbounded orbits pass arbitrarily close to any bound orbit in phase space?

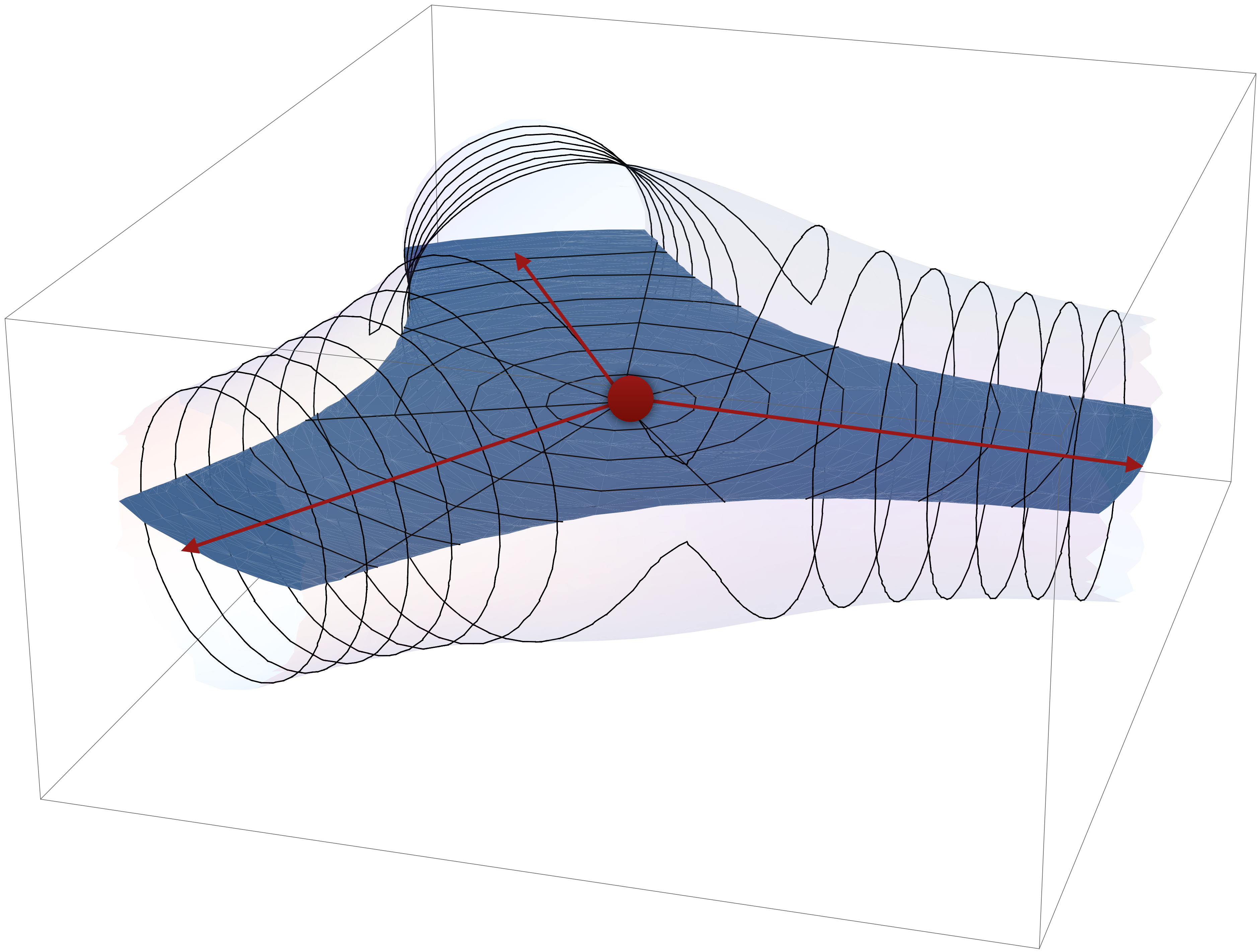
QUESTION 2: For any given mass ratios, and ZERO angular momentum do there exist periodic orbits?

If the angular momentum is NON-ZERO:
Yes! Lagrange and Euler's solutions.

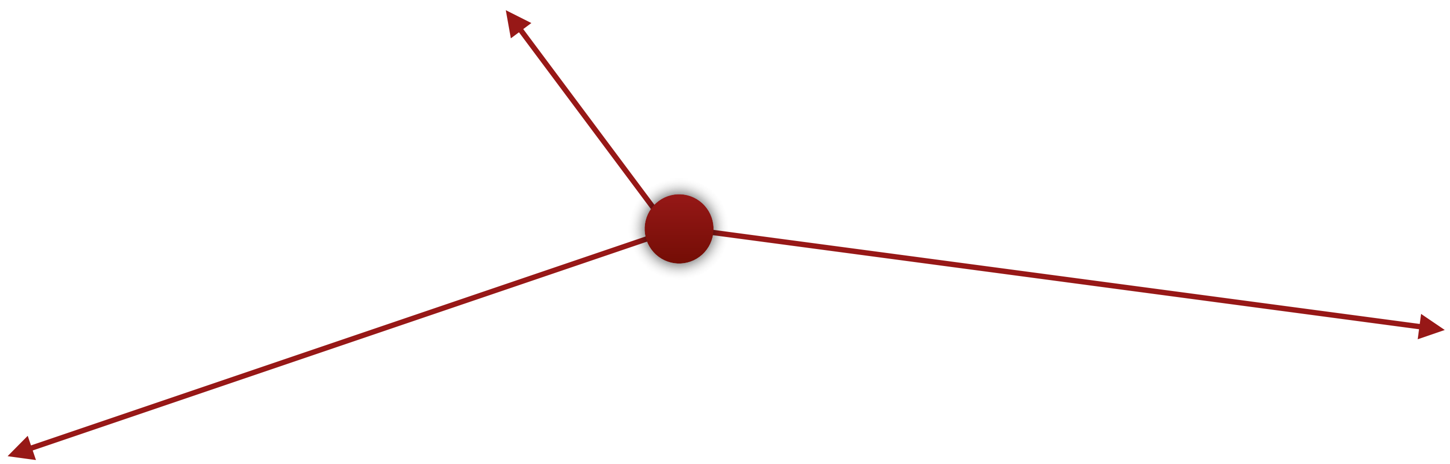


Recall: bounded $\implies$ energy is NEGATIVE.

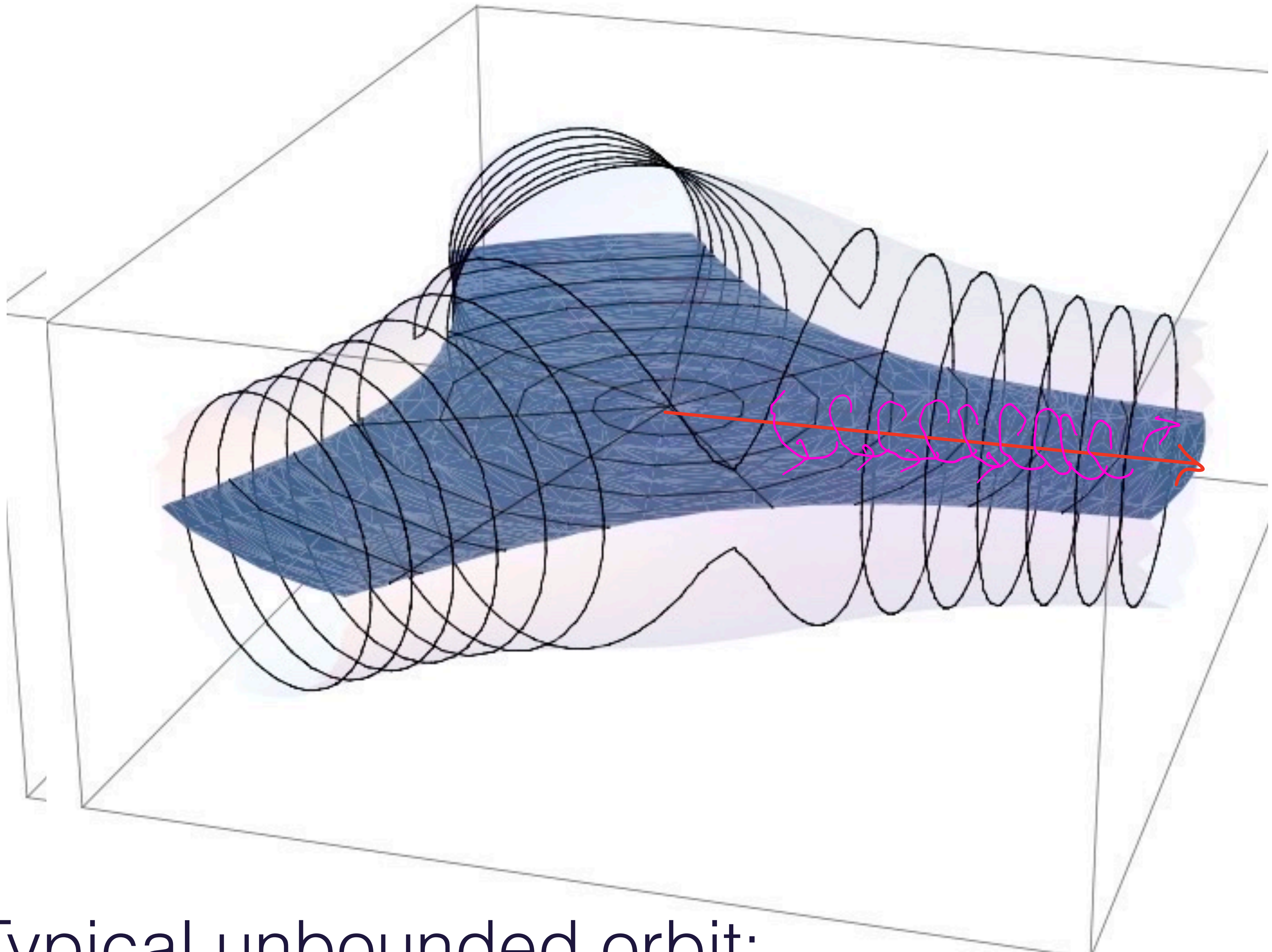
So, for both questions, we are interested in the flow when the energy E is NEGATIVE.



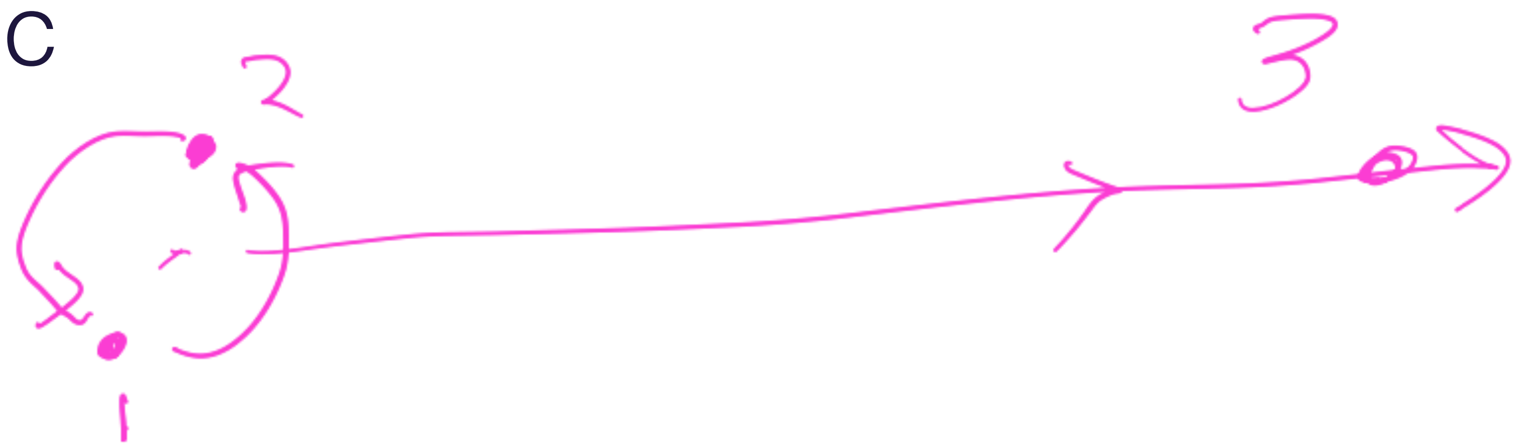
The Hill region at negative energy,
for the planar 3- body problem
projected to shape space



Collision locus=
where potential is negative infinity



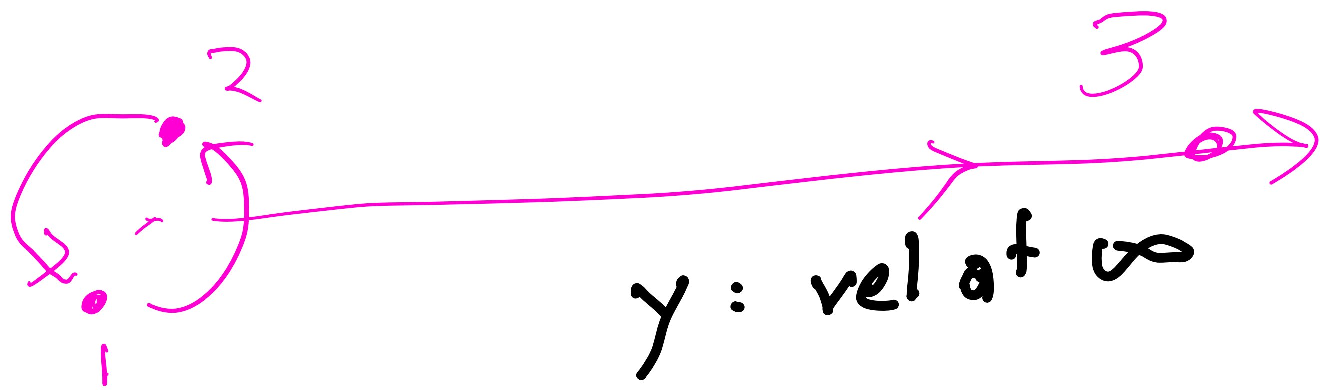
Typical unbounded orbit:
hyperbolic-elliptic



$(E < 0)$

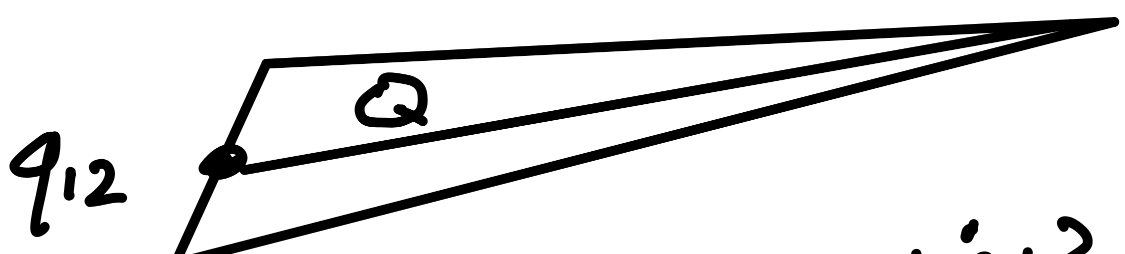
'P' : $\to y = 0$: parabolic-elliptic

'H' : $\to y > 0$: hyperbolic-elliptic

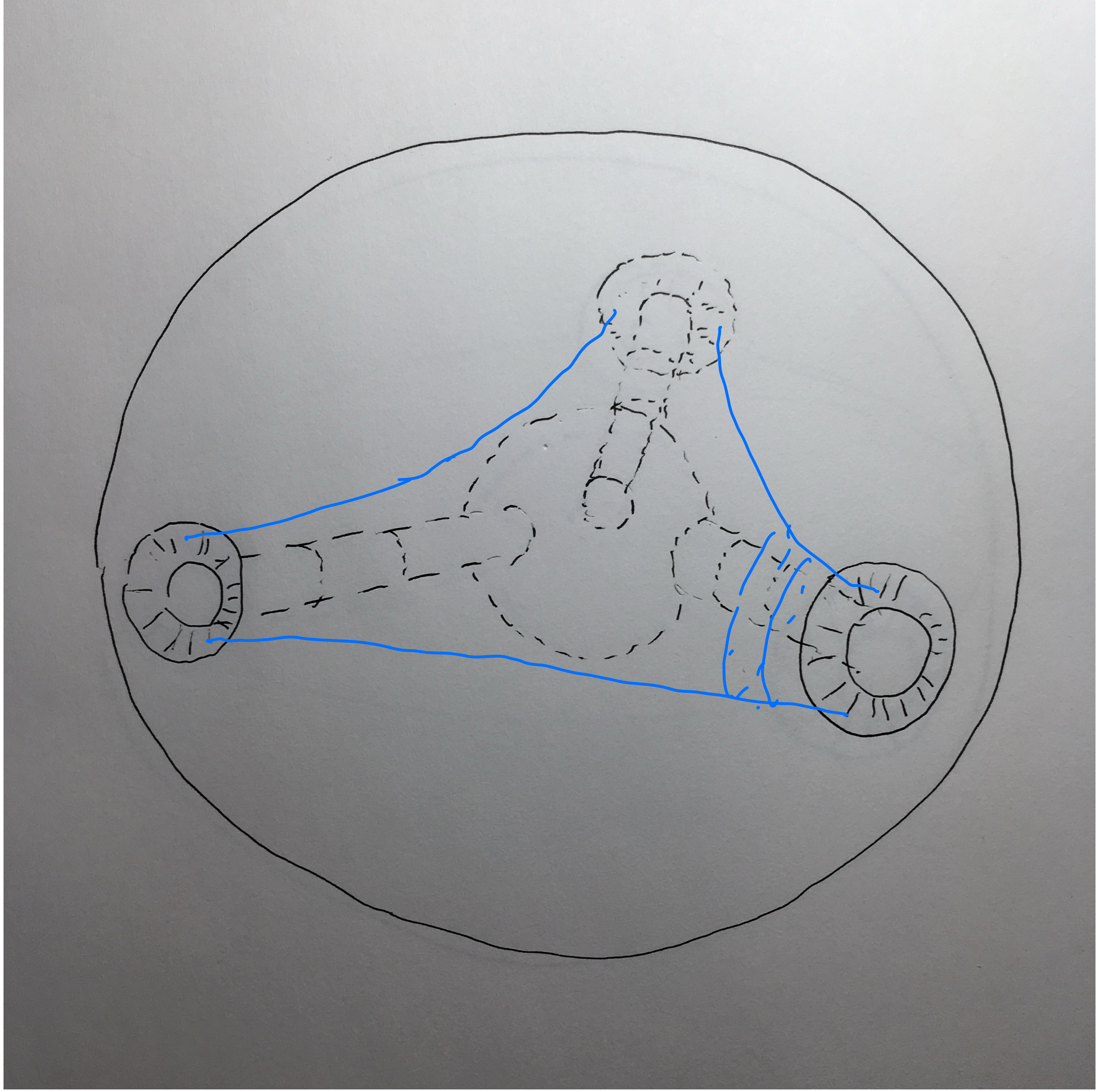


y : vel at ∞

$= |\dot{Q}|$



$$K = \frac{1}{2}\left(\mu_1 |\dot{q}_{12}|^2 + \mu_2 |\dot{Q}|^2\right)$$



After compactification.

The const. neg. energy hypersurface modulo rigid motions forms a two-sphere bundle over the interior of the blue pair of pants. These spheres degenerate to points on the pants boundary.

Notation and Pictures

Form the non-compact 5-manifold

$$M^5 = M^5(E, J; m_1, m_2, m_3)$$

by fixing the center of mass and linear momentum equal to zero in the planar three-body problem, fixing the energy E and the angular momentum J and forming the quotient of the resulting variety by rotations.

Masses m_1, m_2, m_3 enter as parameters.

Reduced planar 3-body flows live here.

Compactifying M^5 adds boundary strata corresponding to $R \to \infty$ or $R \to 0$ or some $r_{ab} \to 0$.

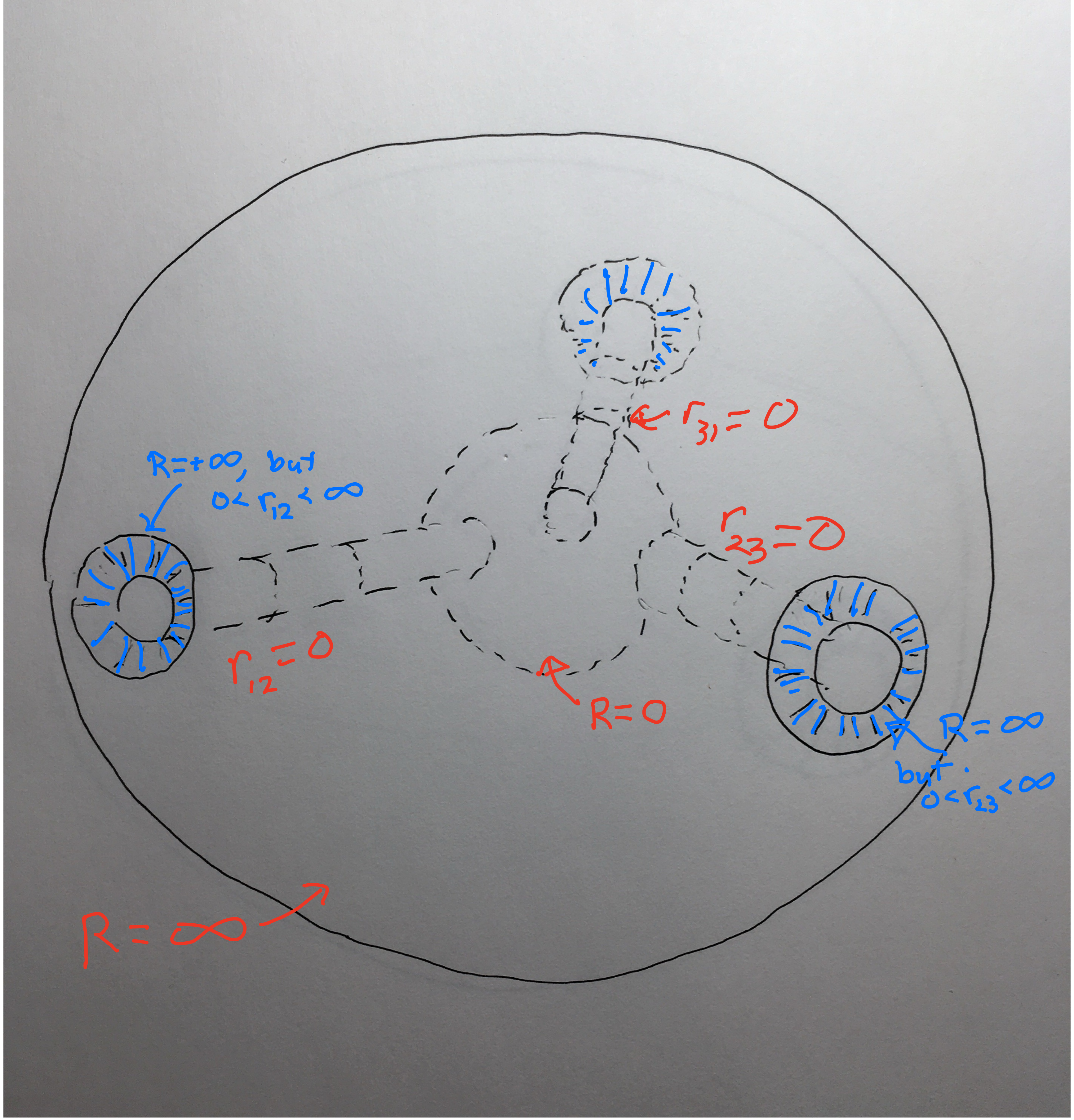
$R^2 = I := \sum m_a |q_a|^2 = \sum m_a m_b r_{ab}^2 / \sum m_a$

Define subsets $B, H, P \subset M^5 = M^5(E, J; m_1, m_2, m_3)$ by:
B = i.c.s of bounded orbits
H = i.c.s for hyperbolic-elliptic orbits
P =i.c.s for parabolic-elliptic orbits.

The answer to the OLDEST Q is 'yes' $\iff$ any of the following equivalent conditions hold
I) $int(B) = \emptyset$
II) $\bar{H} = M^5$
III) $\bar{P} \supset B$.

What we know: $H \neq \emptyset$. $P \neq \emptyset$. $P \subset \bar{H}$. $B \neq \emptyset$ provided the ang. mom. is nonzero. Often (always??) $meas(B) > 0$ (KAM).

What we do not know [Q2 !] is $B \neq \emptyset$ when ang. mom. is zero, regardless of mass ratios?



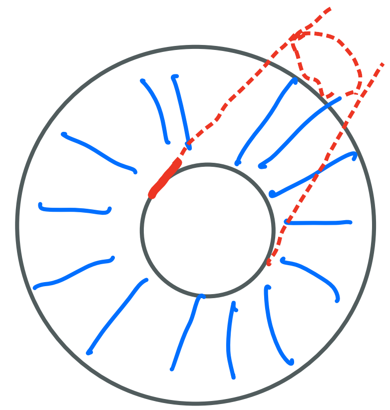
$$R = \sqrt{\Sigma m_a m_b r_{ab}^2 / \Sigma m_a}$$

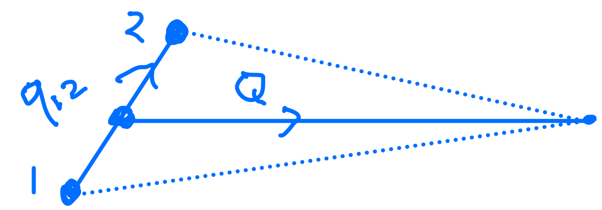
$$= \sqrt{\Sigma m |a|^2 \ldots}$$

JOURNAL OF DIFFERENTIAL EQUATIONS 52, 356–377 (1984)

Homoclinic Orbits and Oscillation for the Planar Three-Body Problem

CLARK ROBINSON*



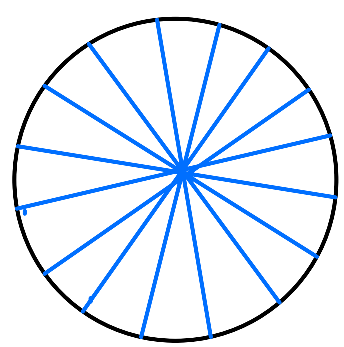


$R = 0$:

$0 < r_{12} < \infty$

$E = H_{12}^{Kep} + H_3^{Kep} + O\left(\frac{1}{R^2}\right)$

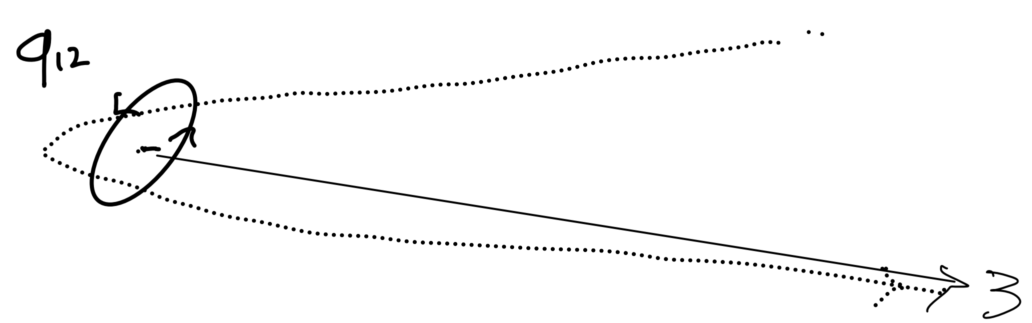
L-C regularize $\Downarrow$ or Moser



flow @ ∞:

$S^3 \times \mathbb{R}$

reg. Kepler $\quad$ y



$|\dot{Q}| = y$

The following two theorems give the smoothness of the manifolds $W^s(S^3)$ and $W^u(S^3)$ and the smoothness of the assignment of the asymptotic binary motion. Both are trivially true for the model equations used in [3] where the equations completely decouple for $x^2 + y^2$ small.

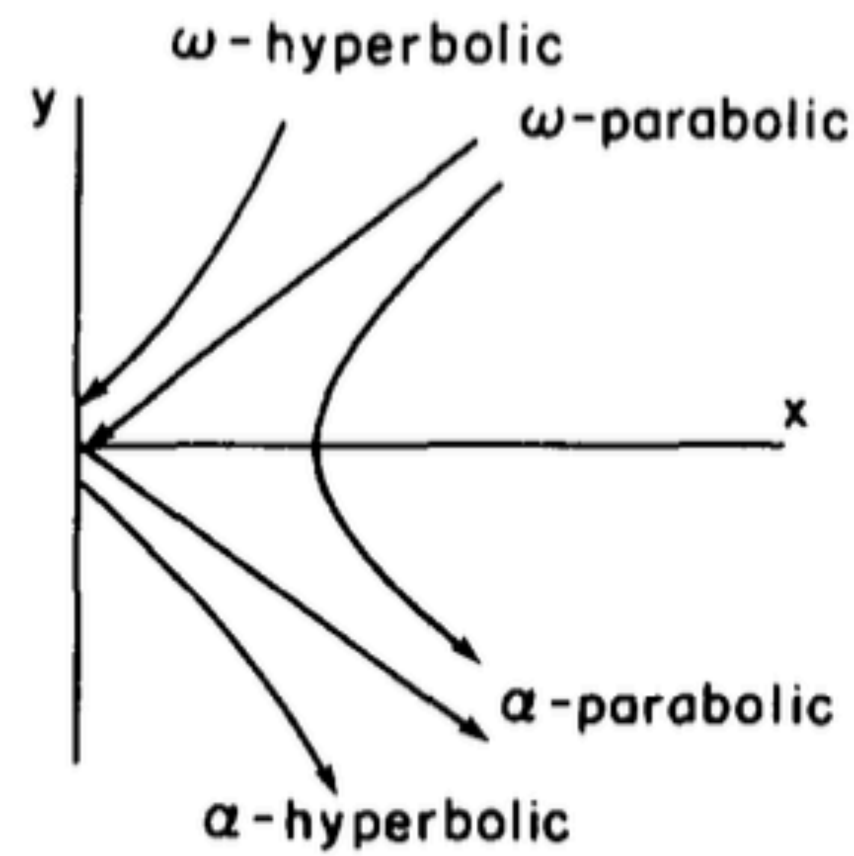
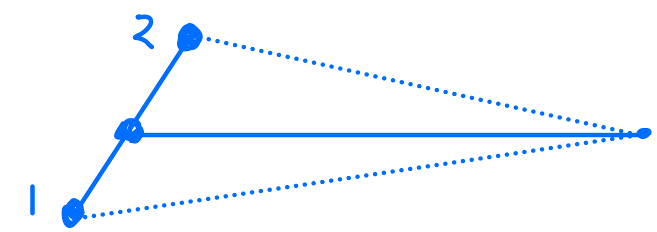
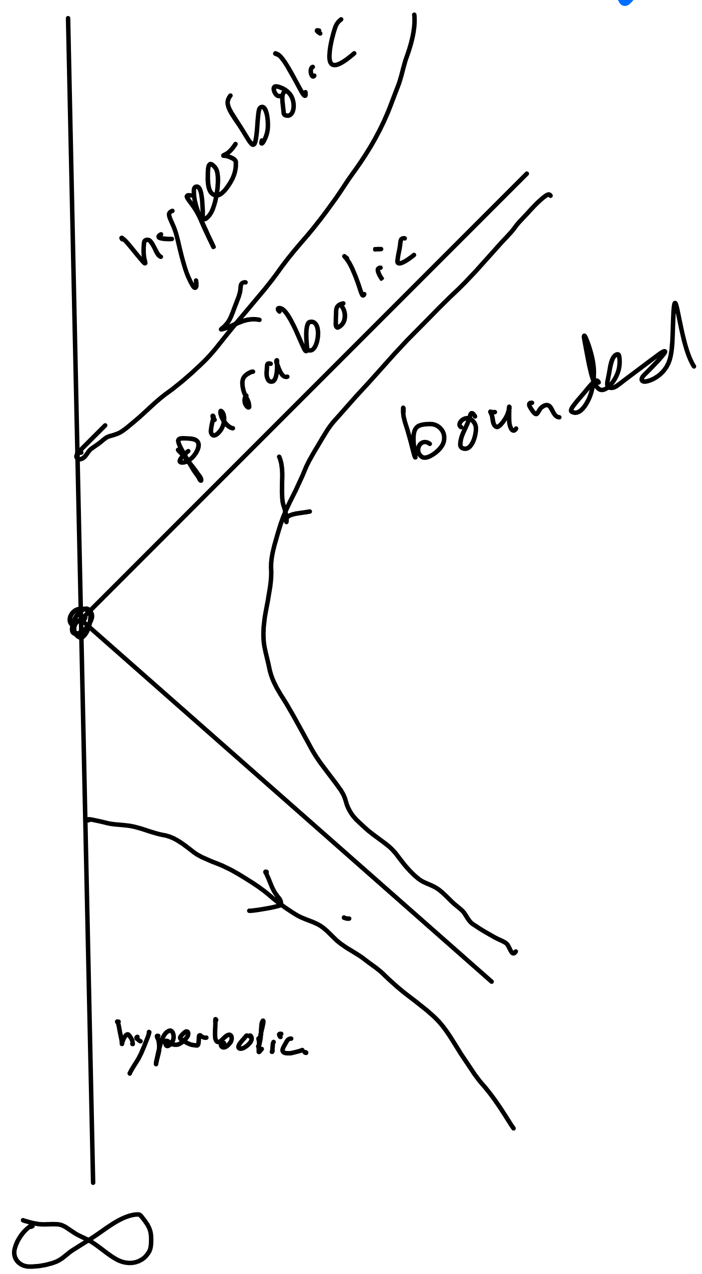


FIGURE 3



2
1



hyperbolic
parabolic
bounded
hyperbolic
∞

- Parabolic ∞: $S^3 \times \{0\}$
  hopf flow
  all orbits S^1's

- $W^s(\text{Parab.}) = P = P^4 \hookrightarrow M^5$
  Robinson: analytic

- Where does $P^4 = W^s(\text{parab})$ go?

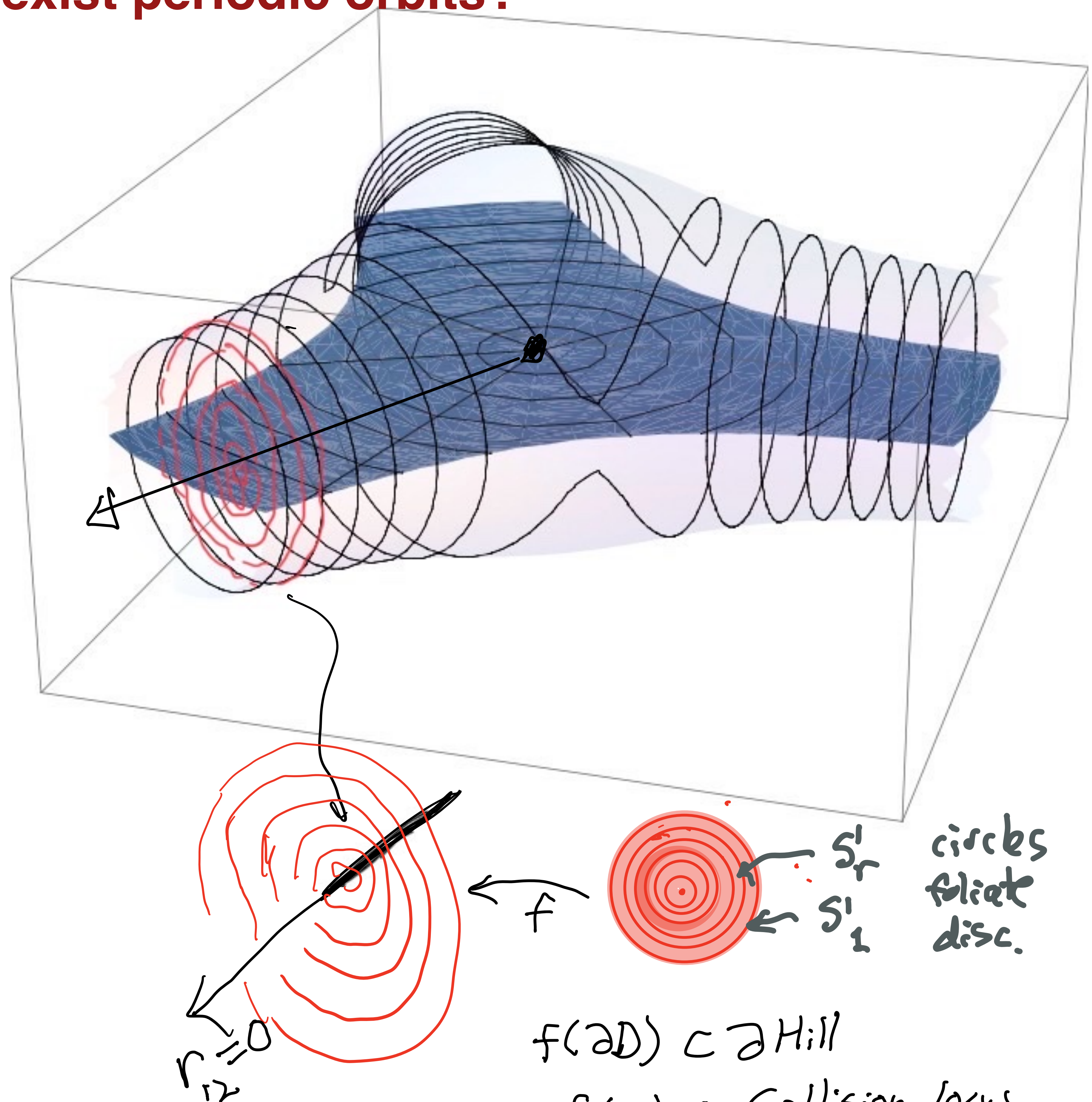
- Does it enter into the interstices of my favorite KAM stable orbits?

- How so?

- Are variational methods of any use here?

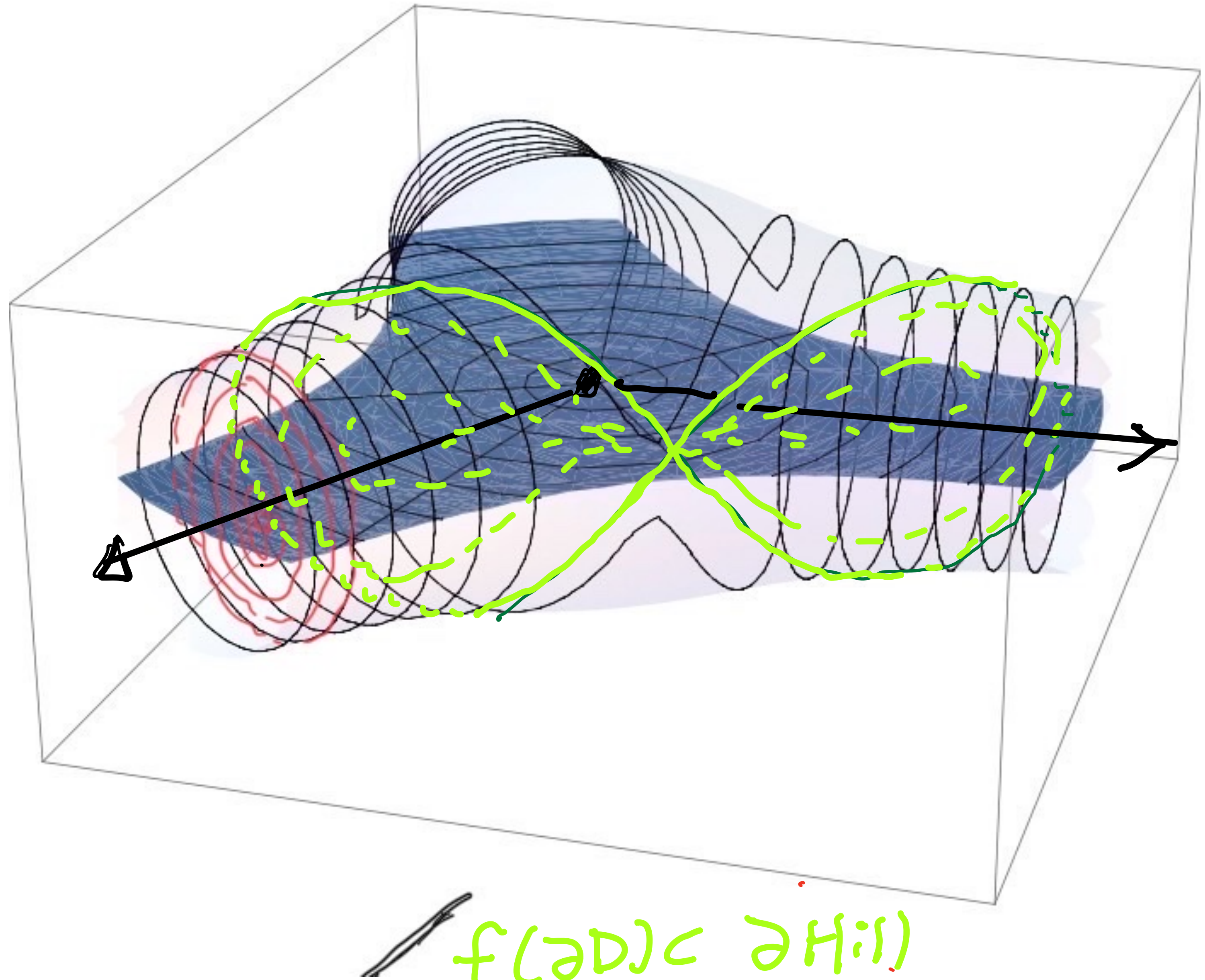
On to Q 2

QUESTION 2: For any given mass ratios, and ZERO angular momentum do there exist periodic orbits?



a min-max attack:

or :



$f(\partial D) \subset \partial Hill$

$f(D) = 0$

Bu t! $f(\partial D)$ realizes green free homotopy class on ∂Hill!

Now try min max:

$$\min_f \max \ell_{JM}(f(S_r^1))$$

some Guesses ...

eg ones tight binary class

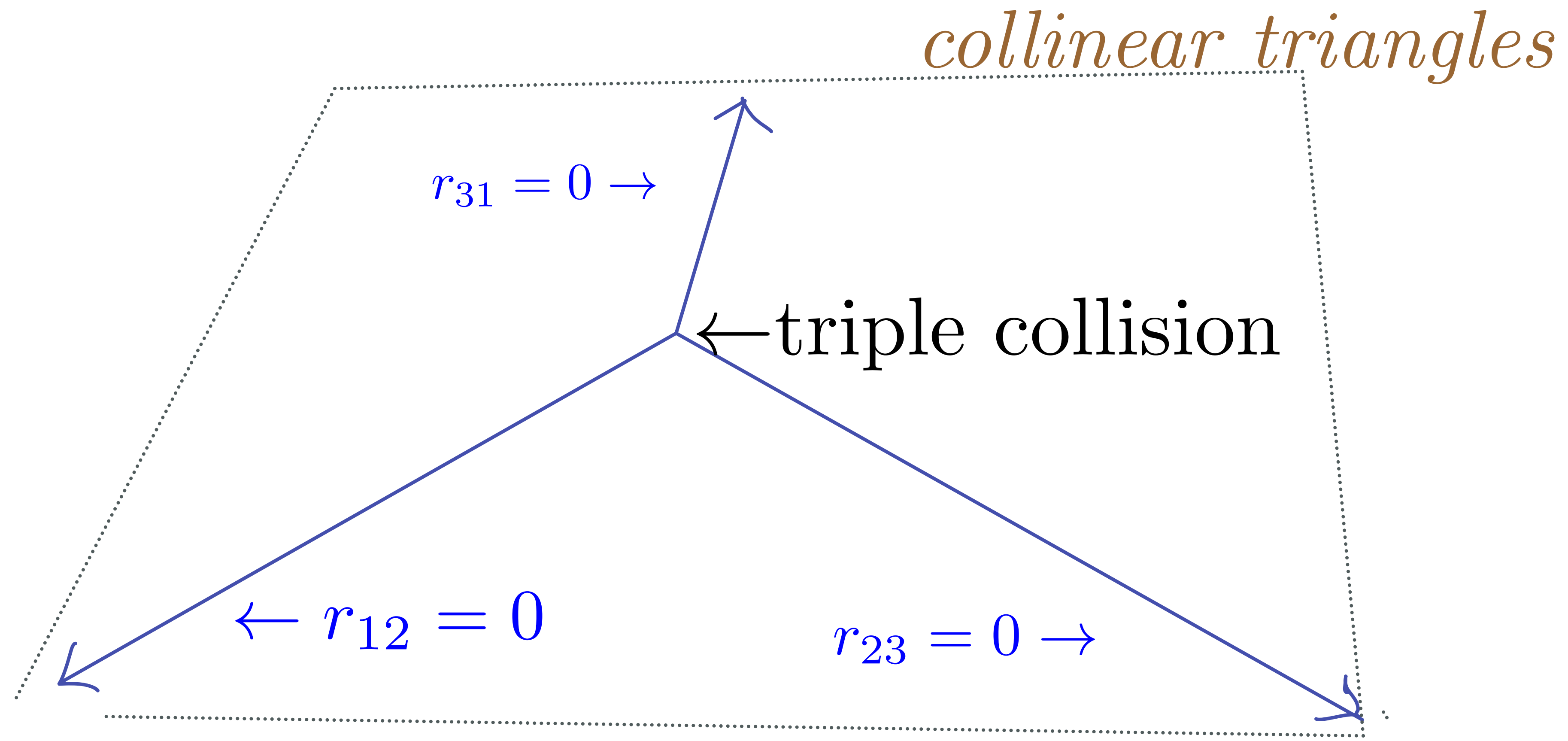
End.

if more time:

a) details of papers …
b) speculations.
c) audience Qs of course!

SHAPE SPACE.

an $\mathbb{R}^3$ realizing the quotient
of the planar three-body configuration space
by rigid motions.



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