N-body scattering & billiards

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remote AMS session on
Geometric Dynamics, organized by
Levi & Tabachnikov, Oct 3, 2020

Description of works with

Jacques Fejoz & Andreas Knauf

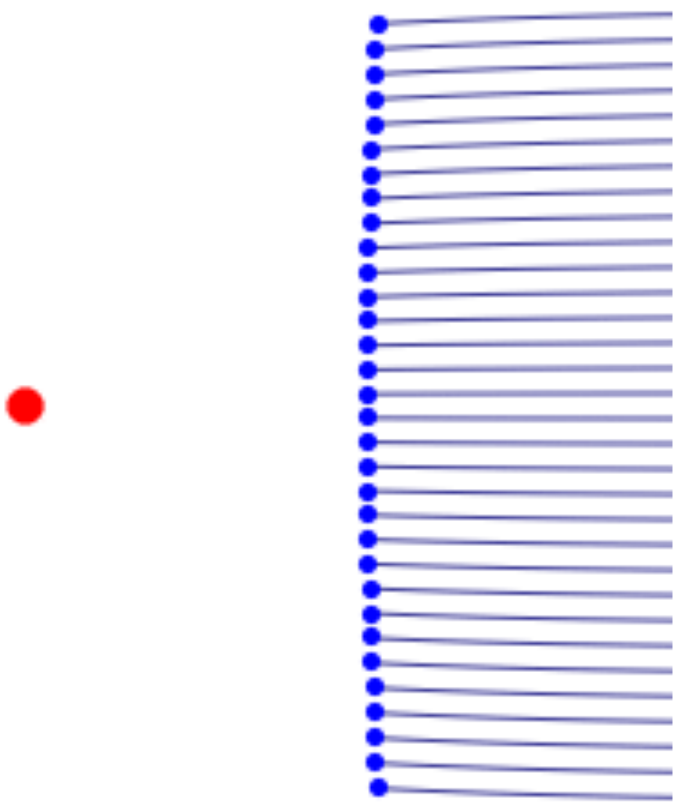
and with

Nathan Duignan, Rick Moeckel and Guowei Yu

many thanks to: **Gil Bor, Rick Moeckel, Rafe Mazzeo, Maciej Zworski**

(*) :retired. Health care still working though…

Rutherford:



Dynamics? $$\ddot{q} = Z\frac{q}{|q|^3} \qquad q(t) \in \mathbb{R}^2$$

first movie: Z > 0; repulsive. orig. Rutherford.

second movie: Z < 0; attractive. eg: parallel comets impinging on sun

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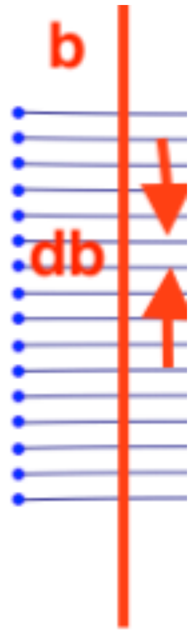
Movies also for **two-body scattering.** Use
:

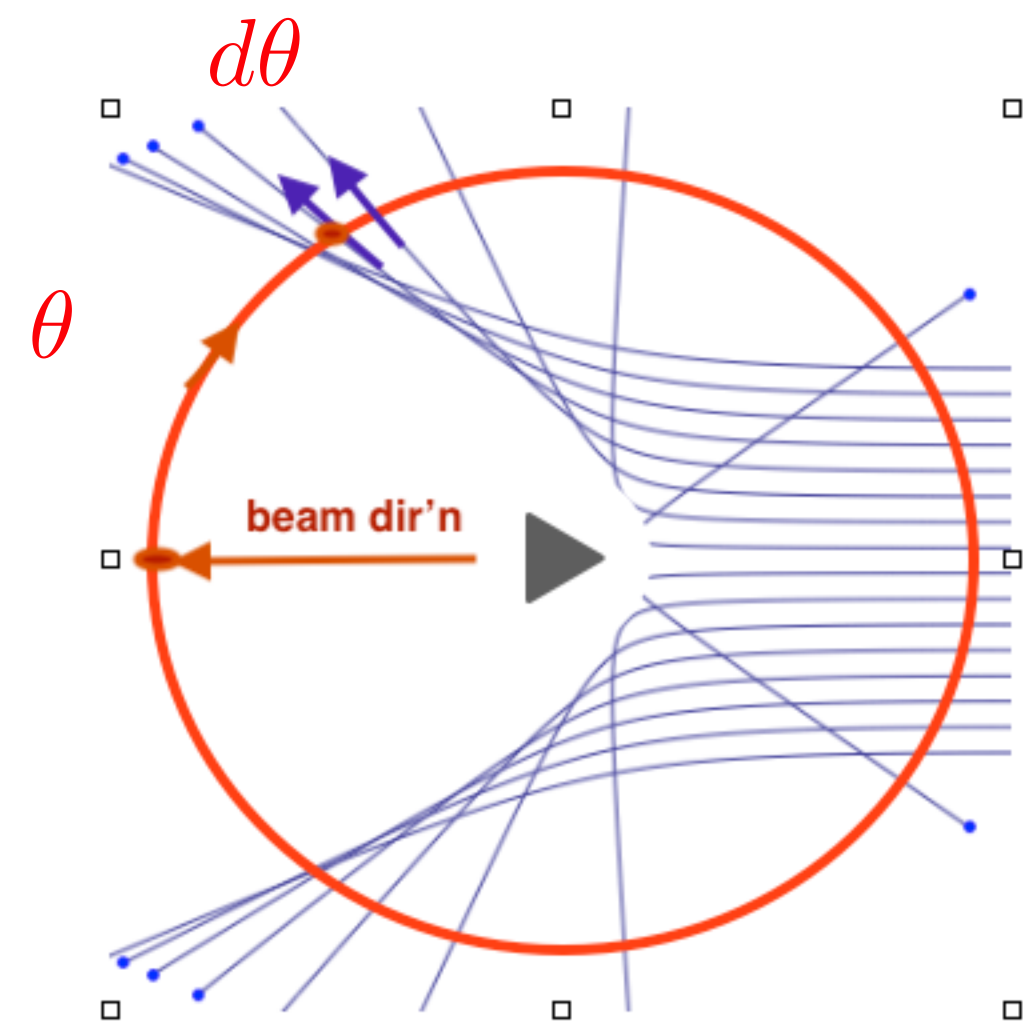
$$q = q_2 - q_1$$

(*c. of mass travels a straight line:* $m_1 q_1 + m_2 q_2 = Pt + Q_0$)



``Scattering map"

$$b \in \mathbb{R} \mapsto \theta \in S^1$$



denoted:

$$\pi : \mathbb{R} \to S^1$$

($$\pi(b) = 2Arctan(\frac{Z}{2Eb})$$

with $$\pi(\pm\infty) = 0$$)

Main result of Rutherford Scattering

$$\Pi_* \, db = \underbrace{\left(\frac{Z}{4E}\right)^2 \left(\frac{1}{\sin^2 \frac{\theta}{2}}\right)^{*}}_{\text{" } \frac{d\sigma}{d\Omega} \text{ "}} d\theta$$

For $q \in \mathbb{R}^2$

if $q \in \mathbb{R}^d$

$$\Pi_* \underbrace{d^d b}_{\text{Leb. on } \mathbb{R}^d\text{'s of impact}} = \left(\frac{Z}{4E}\right)^2 \left(\frac{1}{\sin^2 \frac{\theta}{2}}\right)^{d-1} \underbrace{d\Omega}_{\text{std spher. measure}}$$

same whether attractive (1st movie) or repulsive (2nd) !

sugg. ref: Knauf, Mathematical Physics :Classical Mechanics, chapter 12

What about 3-body scattering?

QUESTIONS:

What is the analogue of the scattering map $\pi : \mathbb{R} \to S^1$?

What is its domain - the space of ``impact parameters b's'?

What is its range - the space of `outgoing directions', theta's?

Is it smooth? open? invertible? almost onto?

Can we say anything quantitative or meaningful regarding its induced ``differential cross-section'' $\pi_*(Leb) = f(\Omega) d\Omega$

To begin to answer, return to …

3-body scattering? It is anisotropic.

2-body scattering is **isotropic:** the scattering map is independent of the direction of the incoming beam. It only depends on θ , the angle between the beam and outgoing particle paths

3-body scattering is **anisotropic. D**ifferent directions of incoming beams will lead to different outgoing scattering maps

`Direction'?

The config space of three bodies in the plane, center-of-mass fixed, is a 4 dimensional Euclidean vector space.
Its space of directions is a 3-sphere.

Modulo rotations, this 3-sphere becomes a 2-sphere -the `shape sphere'.

An incoming ``Lagrange beam '' (equilateral triangles) will lead to a different scattering map then an incoming ``Euler beam'' (a particular degenerate collinear triangle)

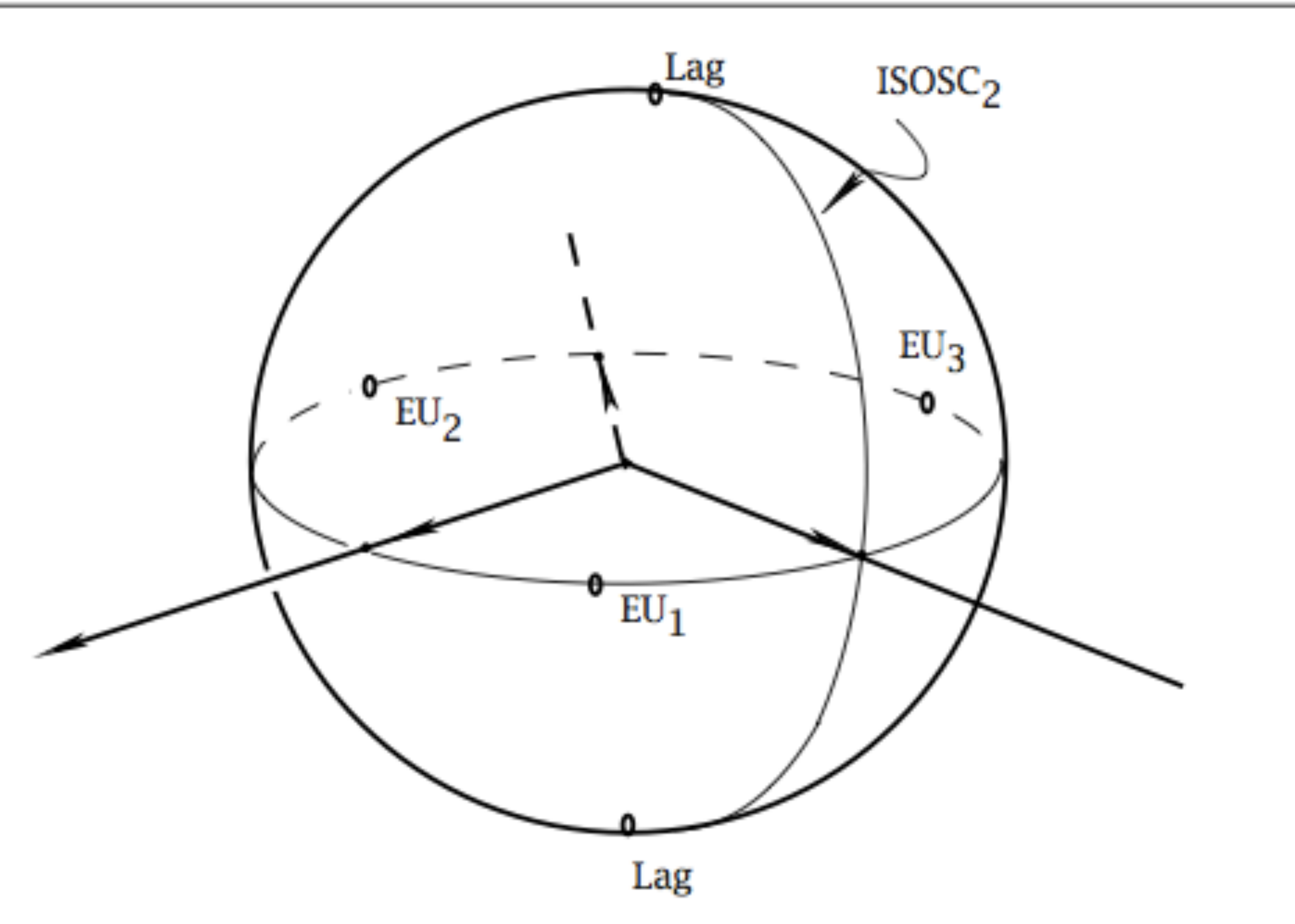
Shape space and shape sphere.

$$(\mathbb{R}^2)^3 = \mathbb{C}^3 \to \mathbb{C}^2 \to \mathbb{R}^3$$

see eg:

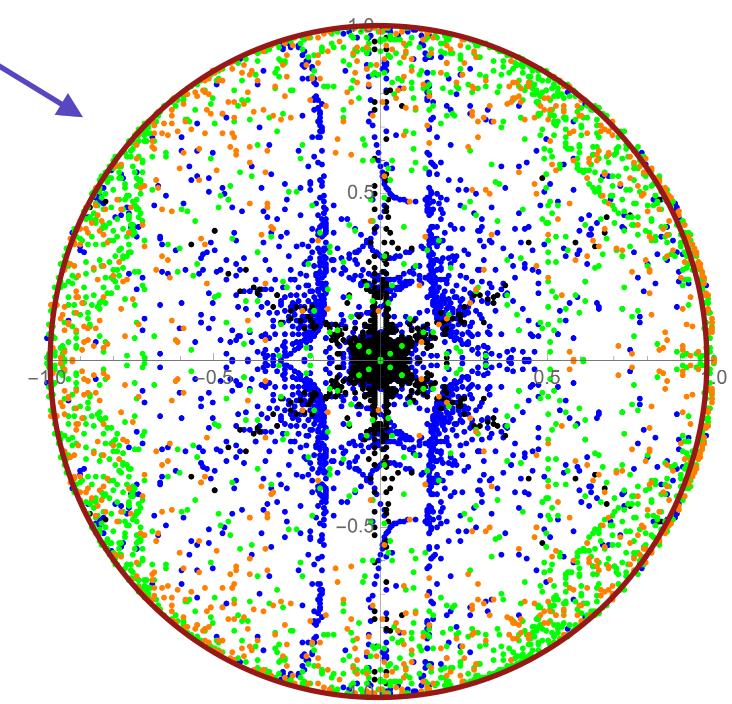
[The Three-Body Problem and the Shape Sphere]

my web p., arXiv, or Amer Math Monthly, 2015

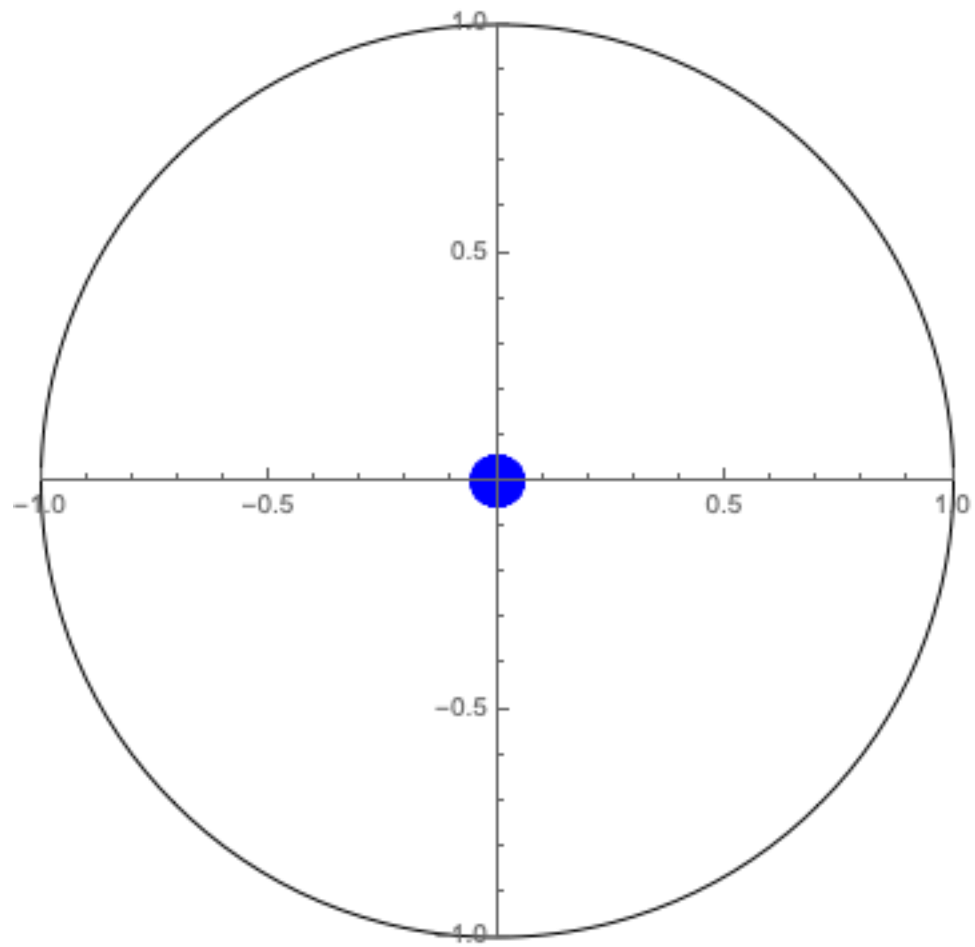


The **shape sphere**: the quotient of 3-body configuration space (minus triple collision) by the group of (orient. pres. isometries) x (scaling).

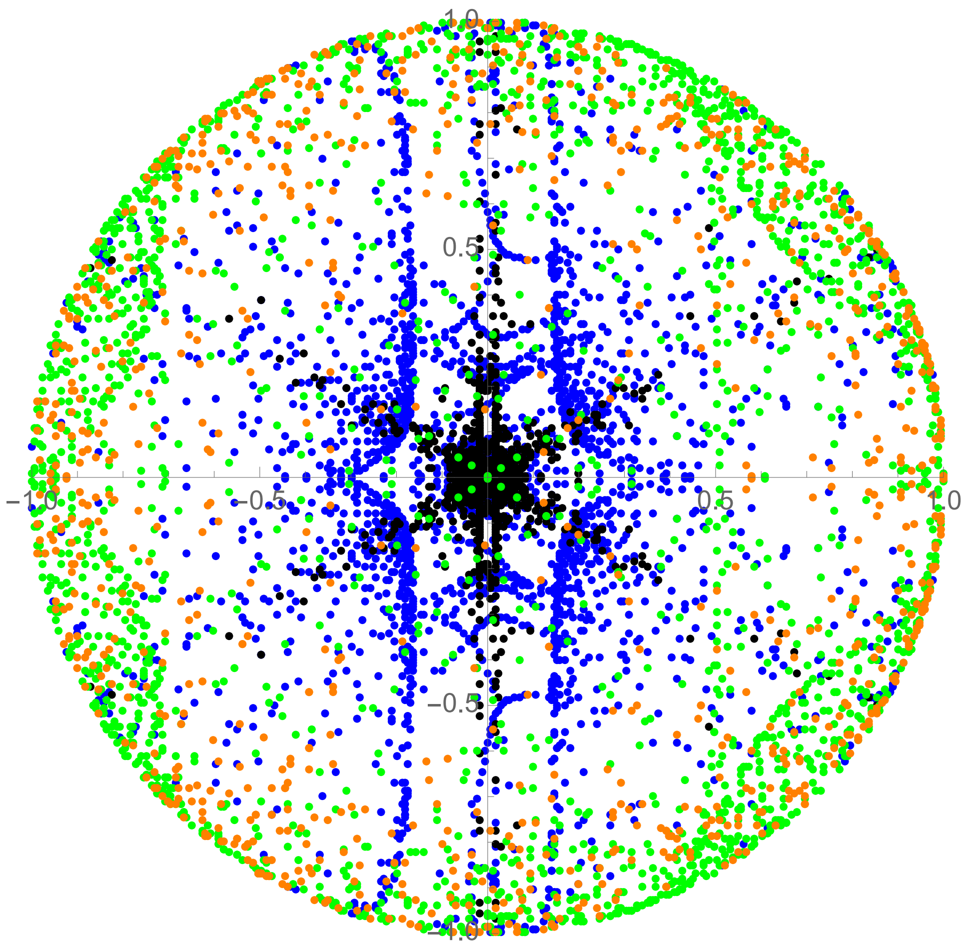
mod reflection



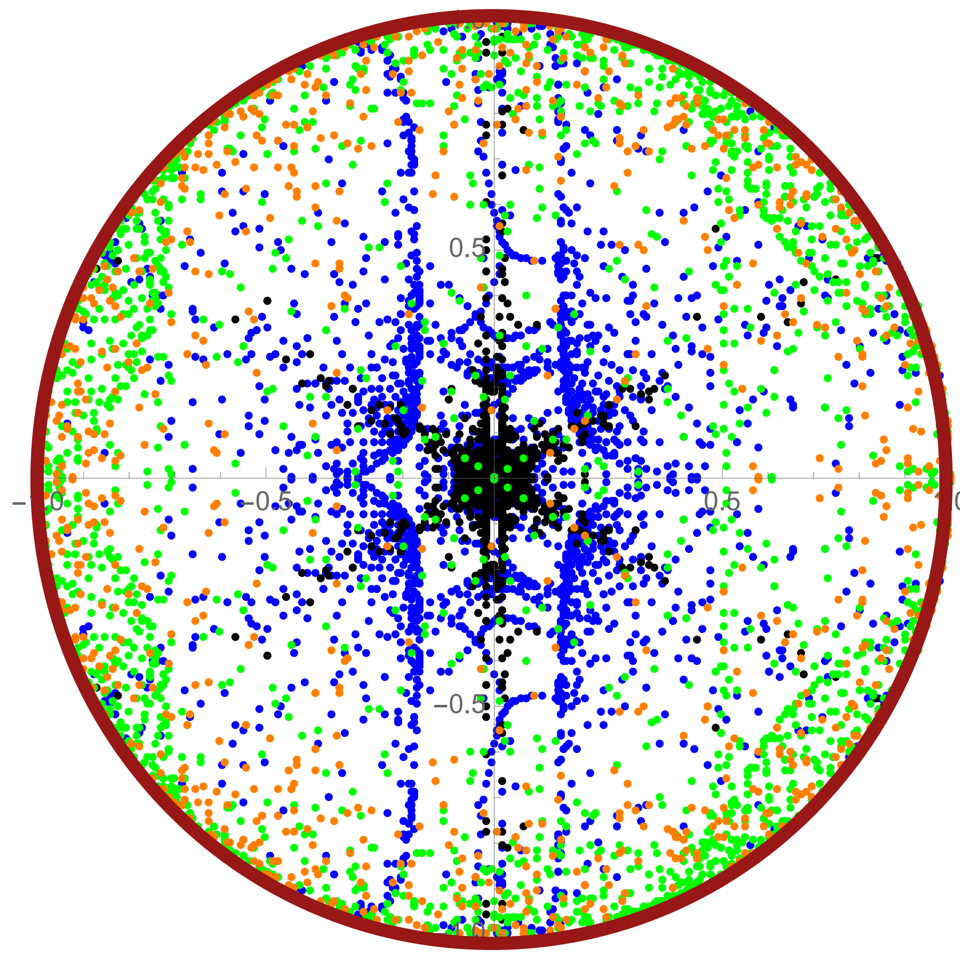
The **shape disc**: the quotient of 3-body configuration space (minus triple collision) by (all isometries) x (scaling) and equals the shape sphere modulo reflection about its collinear equator.


1.0
0.5
−0.5
−1.0
−1.0
−0.5
0.5
1.0

A picture Rick Moeckel made of the **image** of the scattering map for an incoming equilateral triangle (Lagrange) beam projected onto the **shape disc**



colors indicates how close the trajectories stays to infinity



``The equilateral shape is at the center and the collinear shapes are at the outer edge. The isosceles shapes form three diameters of the disk. The collision shapes are at the third roots of unity on the diameter.

The **unstable manifold is a 3D disk whose boundary is a 2 sphere** in the infinity manifold. The points to follow are chosen from other 2D spheres in this disk. **Black points are near the infinity manifold** and blue, green orange farther from infinity. Very crude experiment so far, but encouraging.   How to prove ?
-- Rick''

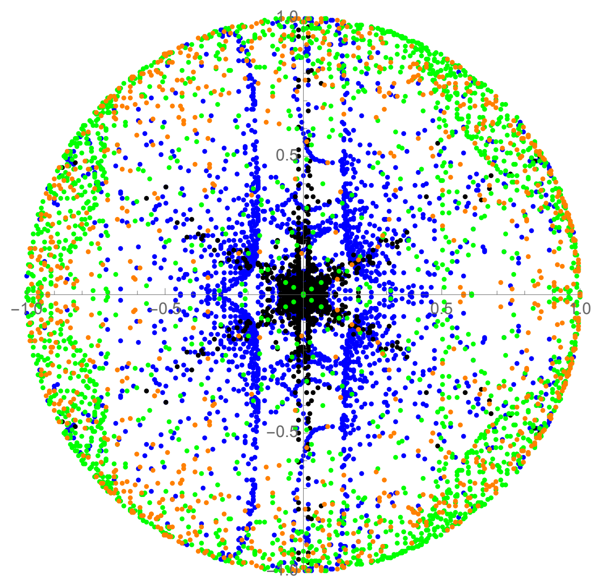
Q. **Unstable manifold?** Of what?

A. Of an equilibrium point on the `**infinity manifold**'' .

These points represent incoming (unstable) and outgoing (stable) asymptotic hyperbolic directions

Q. Who or what is the **infinity manifold**? A. Patience, patience…

Q. **Why those black diameters of `near infinity points''?**

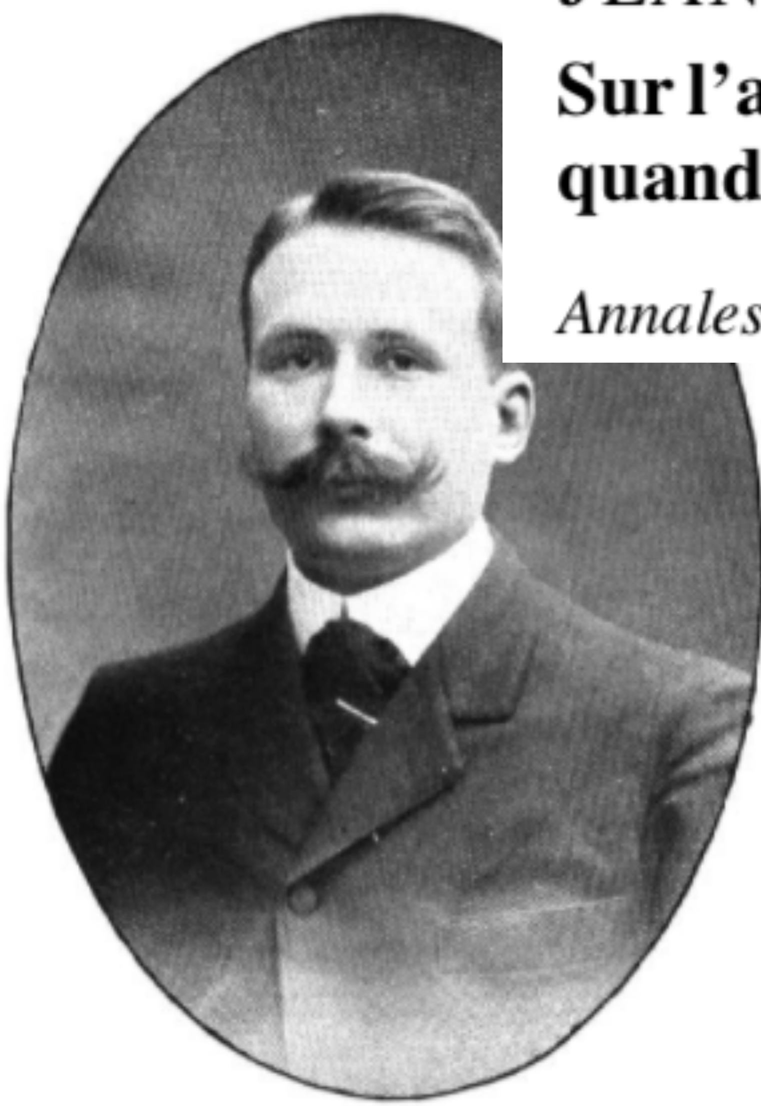


A. The image of `scattering orbits'' that `stay near infinity'.

Q. Why are the diameters located as they are, arranged in a hexagon in the shape disc?

A. ***-That is where billiards come in.*** *Explanation coming at the end…*

Our methods and Inspiration



JEAN CHAZY

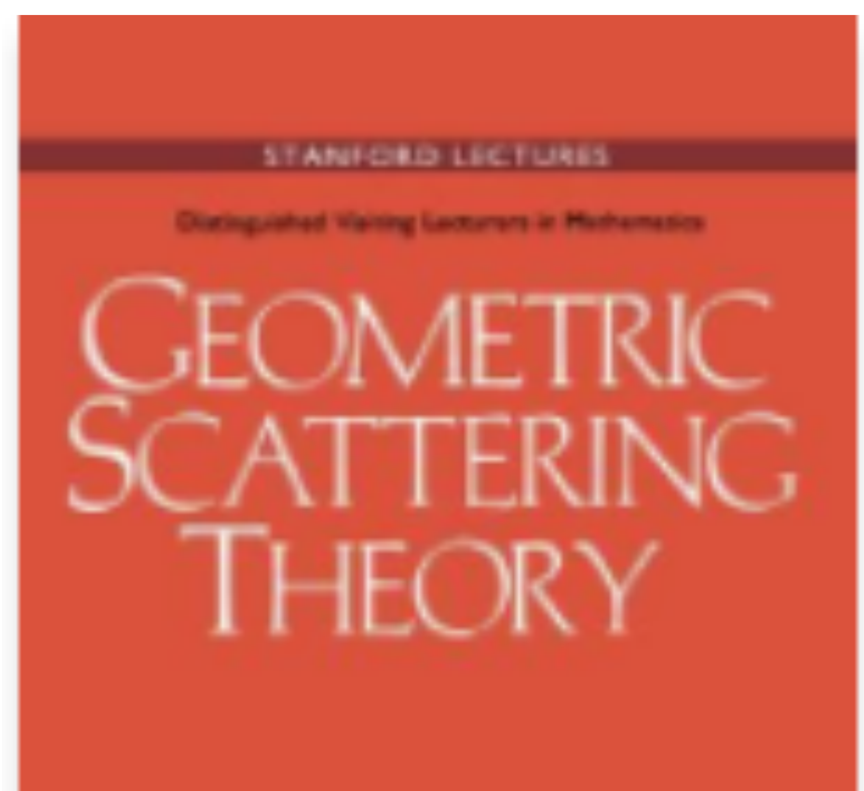
JEAN CHAZY

Sur l'allure du mouvement dans le problème des trois corps quand le temps croît indéfiniment

Annales scientifiques de l'É.N.S. 3^e^ série, tome 39 (1922), p. 29-130.

McGehee's **blow-up**

Melrose's view of:



May as well do the N-body problem
with bodies moving in d-dimensional Euclidean space

body positions: $q_a \in \mathbb{R}^d, a = 1, \ldots, N$

interbody distances: $r_{ij} = |q_i - q_j|$

DEF. Call the motion hyperbolic if

$r_{ij}(t) \to \infty$ **at a linear rate as t goes to infinity**

forward hyperbolic: $t \to +\infty$

backward hyperbolic **t goes to** $-\infty$

Set-up and eqns N bodies in d-dimensional Euc. space:

Newton's eqns: $\iff \ddot{q} = \nabla_m U(q)$

$$q = (q_1, \ldots, q_N) \in \mathbb{E} := \mathbb{R}^{Nd} \qquad q_a \in \mathbb{R}^d, a = 1, \ldots, N$$

Conserved energy

$$E(q, \dot{q}) = \frac{1}{2}\langle \dot{q}, \dot{q} \rangle_m - G \sum \frac{m_a m_b}{r_{ab}}$$

$$= h.$$

$$= K(\dot{q}) - U(q)$$

where

$$2K(\dot{q}) = \langle \dot{q}, \dot{q} \rangle_m = \sum m_i \|\dot{q}_i\|^2 =$$

and

$$U(q) = G \sum \frac{m_a m_b}{r_{ab}}$$

$\nabla_m = \nabla =$ gradient relative to mass metric.

`Spherical ' change of var's :

$$\mathbf{q} = r\mathbf{s}$$
$$r = \|\mathbf{q}\|_m$$
$$\dot{\mathbf{q}} = v\mathbf{s} + \mathbf{w}, \mathbf{s} \perp \mathbf{w}$$

$$s \in \mathbb{S} \cong S^{Nd-1}$$

$$\rho = \frac{1}{r}$$
$$d\tau = r dt$$

ENERGY: $\frac{1}{2}v^2 + \frac{1}{2}\|w\|^2 - \rho U(s) = h.$

Newton's eqns $\iff$

$$\rho' = -v\rho$$
$$s' = w$$
$$v' = |w|^2 - \rho U(s)$$
$$w' = \rho\tilde{\nabla}U(s) - vw - |w|^2 s$$

($\tilde{\nabla}U(s) = \nabla U(s) + U(s)s =$ tangential proj of $\nabla U(s)$ by Euler's ident.)

Spatial Infinity : $\rho = 0$, an invariant submanifold

the **infinity manifold.**

Flow at infinity. Set $\rho = 0$.

$$s' = w$$
$$w' = -vw - \|w\|^2 s$$
$$v' = \|w\|^2$$

$$s \in \mathbb{S} \cong S^{dN-1}$$

$$v \in \mathbb{R}, v \neq 0$$

Energy at infinity: $\frac{1}{2}v^2 + \frac{1}{2}\|w\|^2 = h.$

***Flow** at infinity is **independent** of U !*

Equilibria! $$(\rho, s, v, w) = (0, s, v, 0)$$

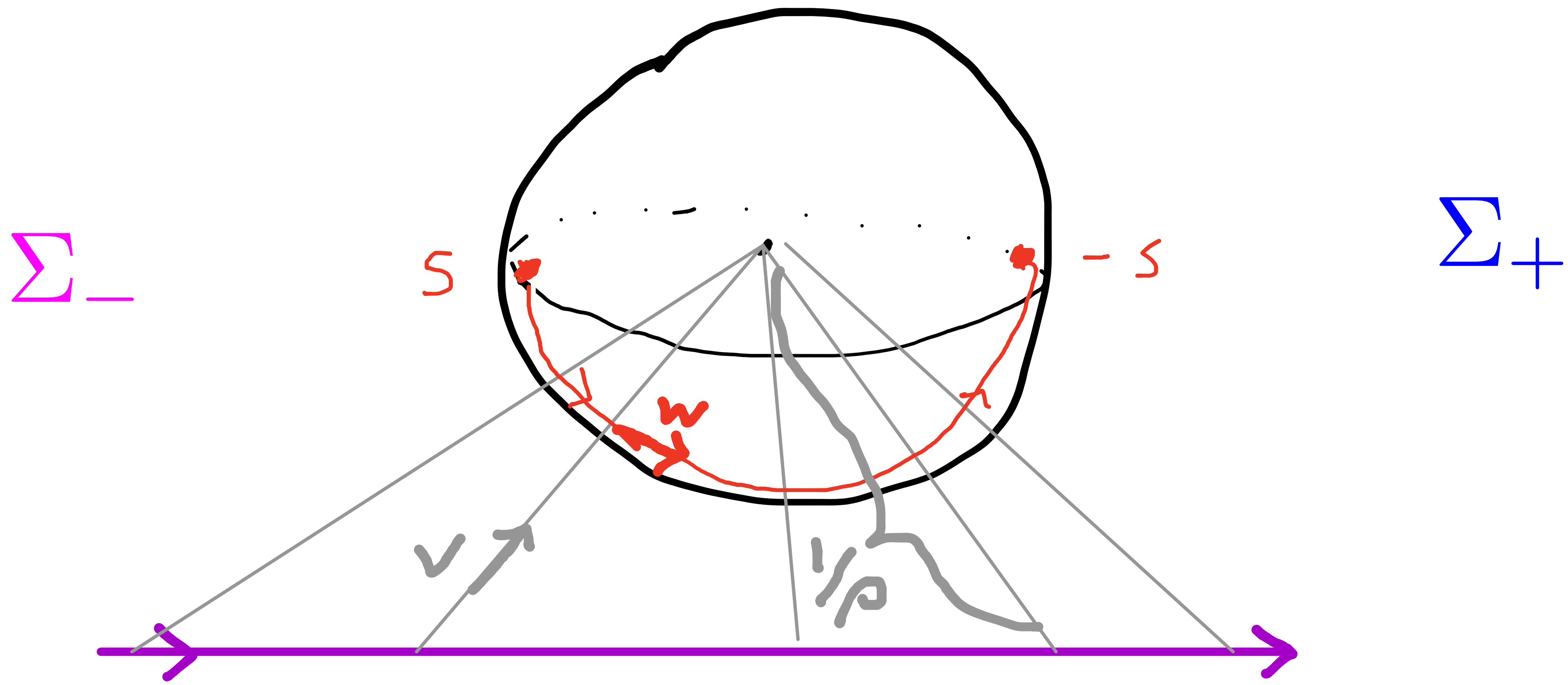
form a normally hyperbolic manifold of equilibria within the full phase space.

$$\Sigma = \Sigma_- \cup \Sigma_+$$

disjoint union of unstable (v > 0) and stable (v < 0) equilibria
representing past and future end shapes

Flow *at infinity is* ***independent*** *of U.*

Set U = 0 to understand the dynamics at infinity.
Flow = reparam. of free motion -
projected onto the sphere !:

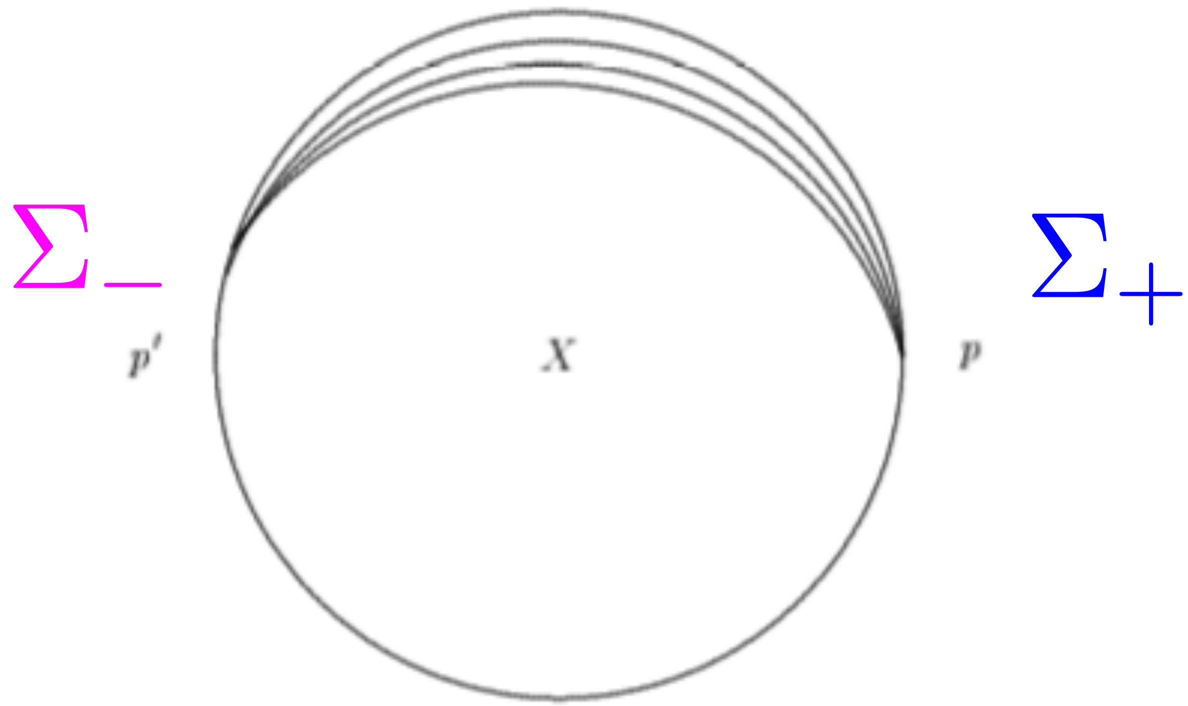


v >0
rays incoming
from infinity

v <0

rays outgoing
to infinity

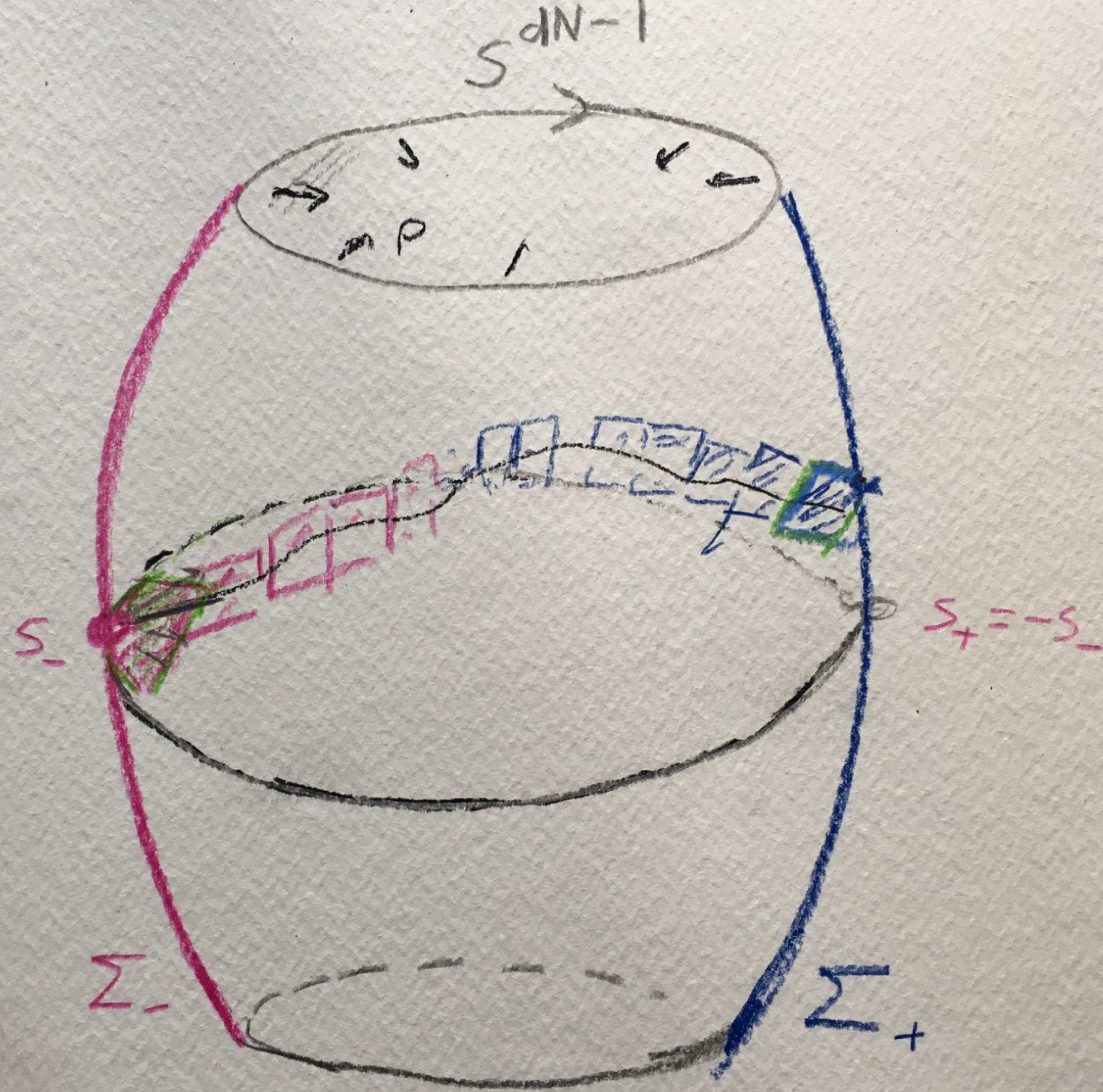
s, -s become equilibria! ; flow is gradient like between them…



p. 80. Geometric Scattering Theory -Melrose.

Fig. 11. Geodesic of a scattering metric.

``time pi geodesic flow on the sphere''



S^{dN-1}
s_-
$s_+ = -s_-$
Σ_-
Σ_+

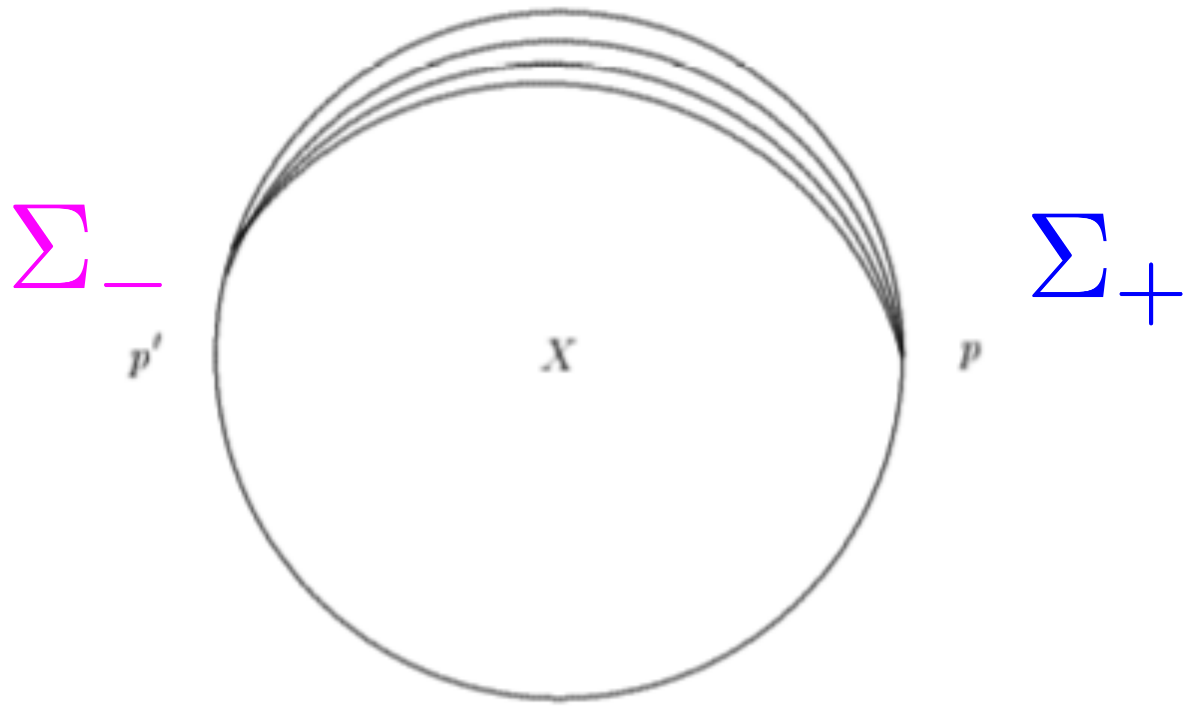


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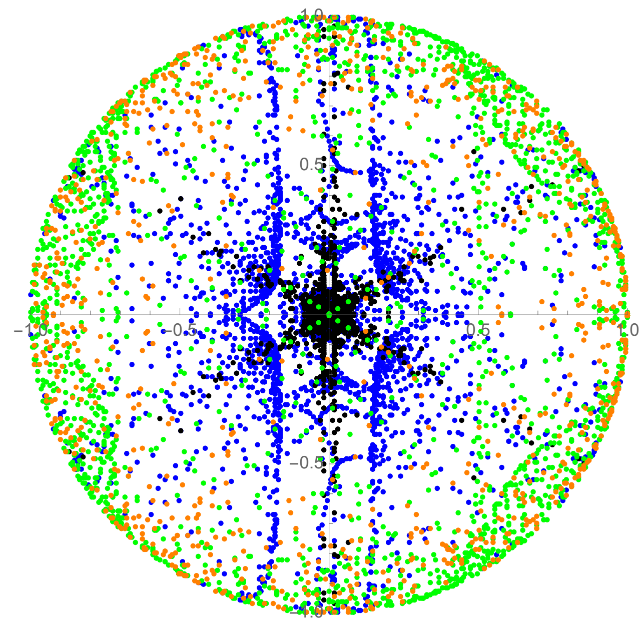
Fig. 11. Geodesic of a scattering metric.

``time pi geodesic flow on the sphere''

ASIDE: This picture becomes much more accurate when we go to the ``Jacobi-Maupertuis'' version of Newtonian dynamics.

Hang on ..

If this flow at infinitely accurately captured the near-infinity dynamics we would not see the three black diameters, but instead just a centered small black blob near the origin rep. (-)Lag.

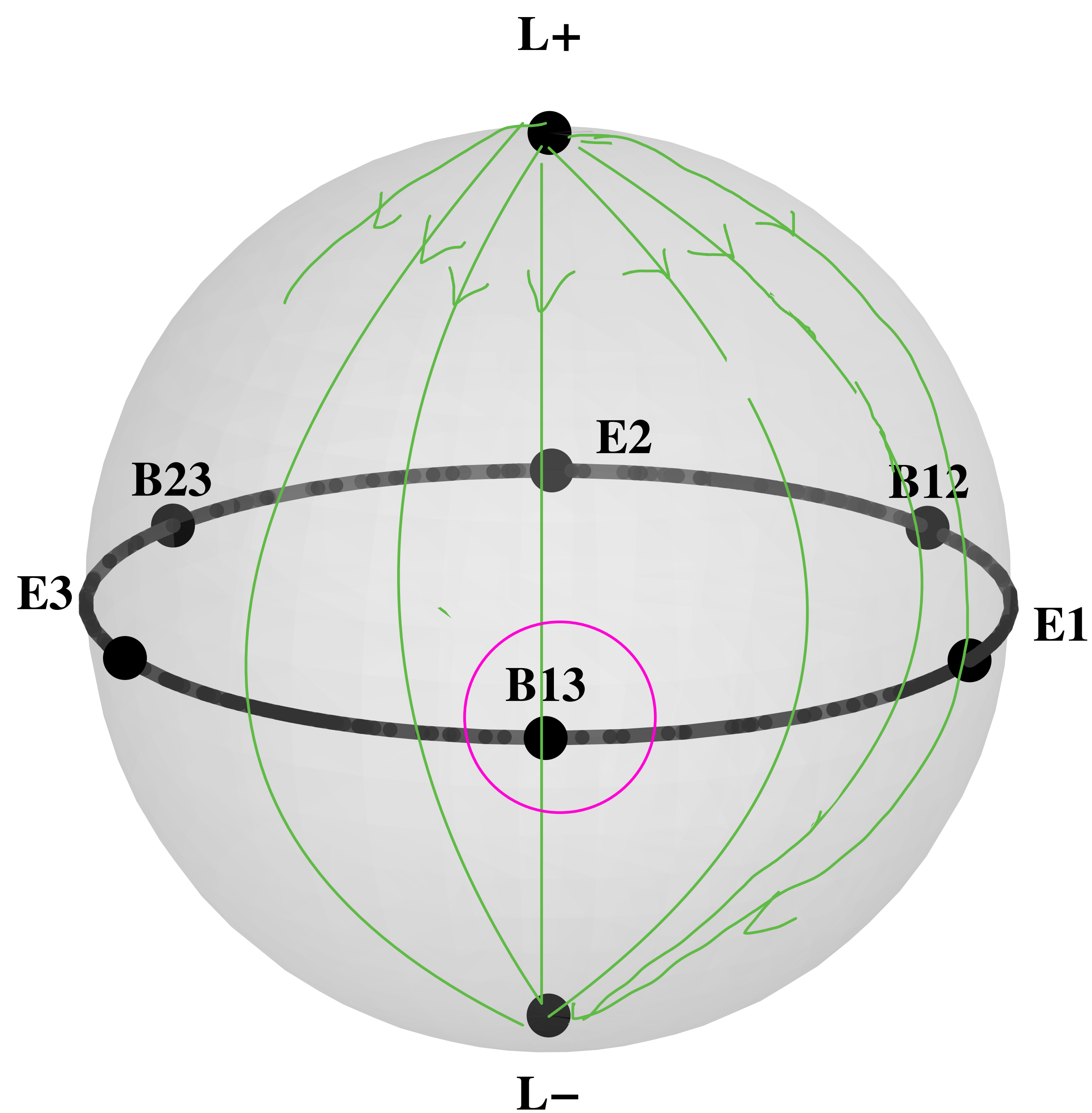


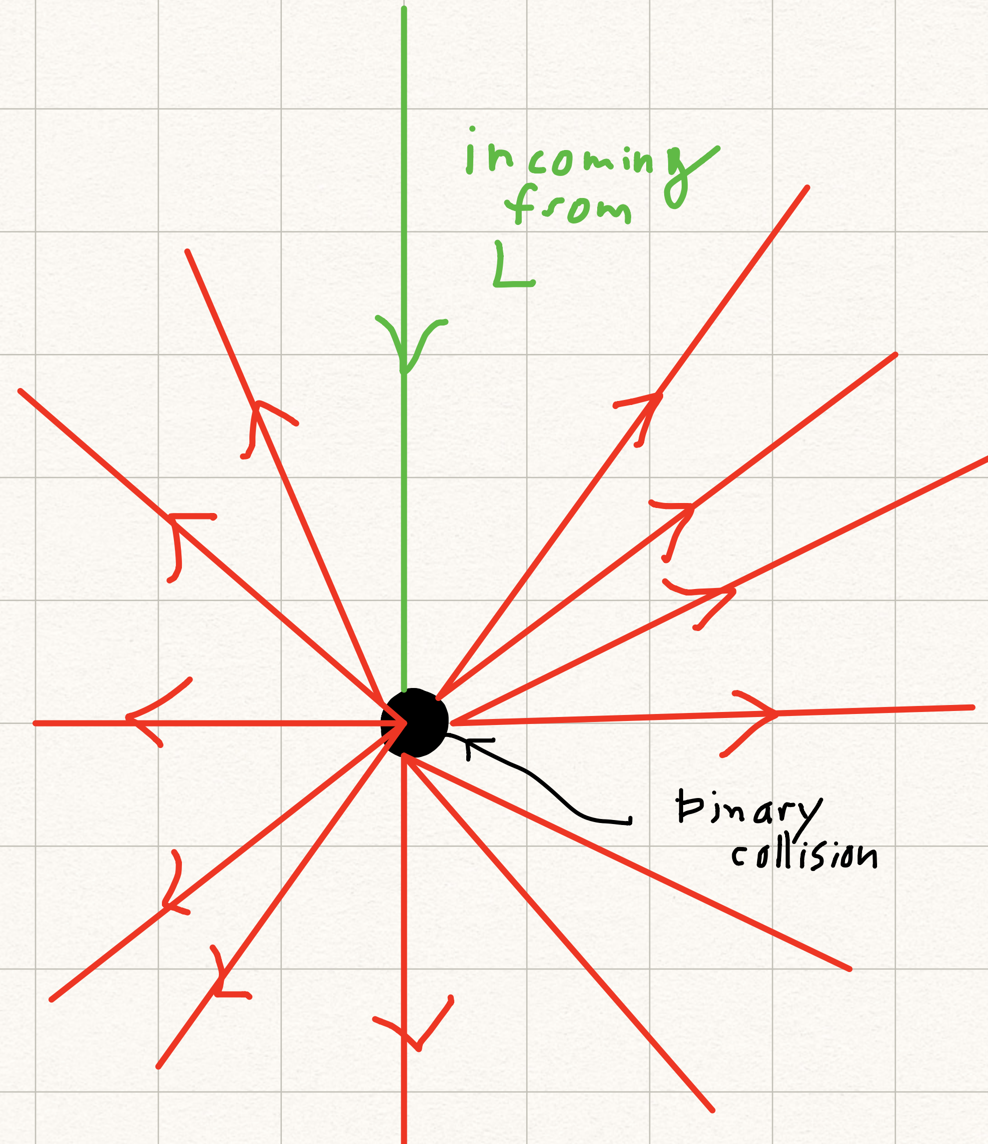
We must modify `scattering flow' at infinity so that the binary collision loci at infinity act as `perfect codimension 2' reflectors.

rationale 1:

rationales, 2 (Knauf) + 3 (Vasy et al):
modify so collisions at infinity
act as ``perfect reflectors'
and the total path length
continues to be π

$$\rho' = -v\rho$$
$$s' = w$$
$$v' = |w|^2 - \rho U(s)$$
$$w' = \rho\tilde{\nabla}U(s) - vw - |w|^2 s$$

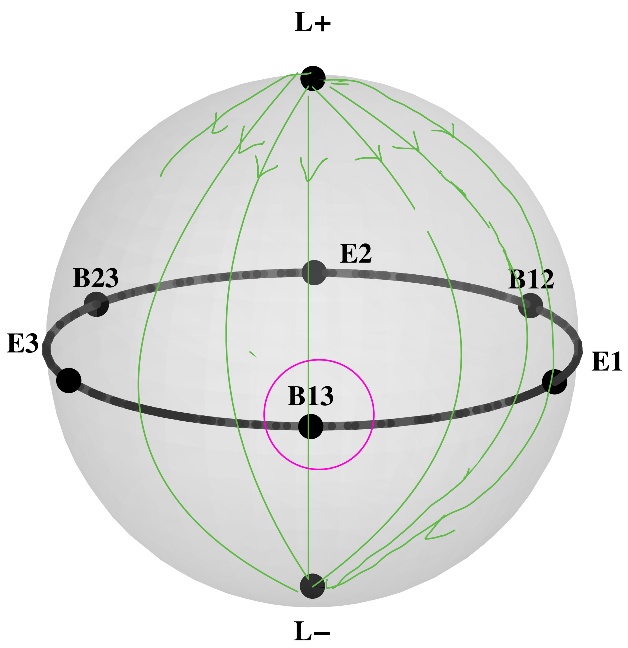

L+
E2
B23
B12
E3
E1
B13
L−



`Broken' geodesic flow: the collision loci on the sphere act as `perfect reflectors'

Non-deterministic!

If a geodesic hits a point on the collision locus it bounces off in a random direction, continuing until either it hits another , continuing in this manner `flowing' for a total time = spherical arclength of π



Lag-> Collision = Lag -> Collinear Equator = $\frac{1}{2}\frac{\pi}{2} = \frac{\pi}{4}$

$$\frac{\pi}{4} + x = \pi \implies x = \frac{3\pi}{4} = \frac{1}{2}\frac{3\pi}{2}$$

Scenario: Leave binary. Hit collision locus at a point B. Go 3/2 away around the sphere in any direction and mark the resulting points:

Circle of radius $\frac{3\pi}{2}$ about B on standard unit sphere ircle of radius

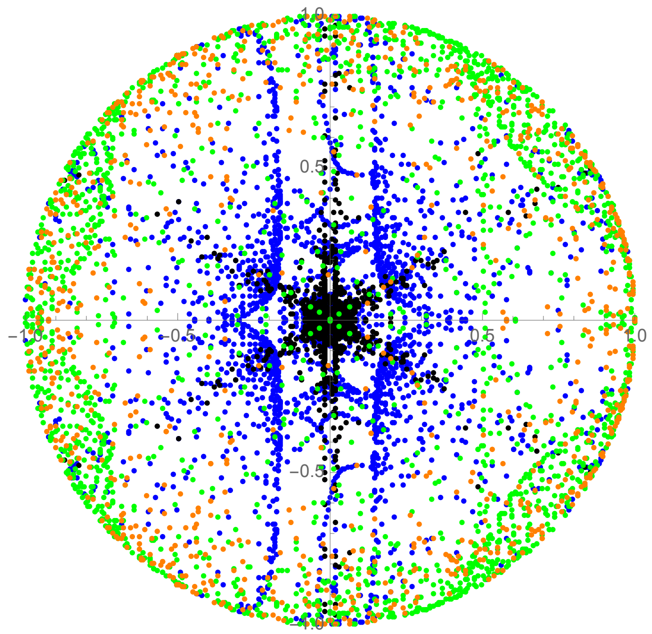
= circle of radius $\frac{\pi}{2}$ = great circle midway between B and -B.

SUMMARY:

Q. What is the **infinity manifold**?

A. the invariant manifold $\rho = 0$

Its equilibria represent asymptotic states: limits q(t)/t or q(t)/|q(t)| as $t \to \pm\infty$

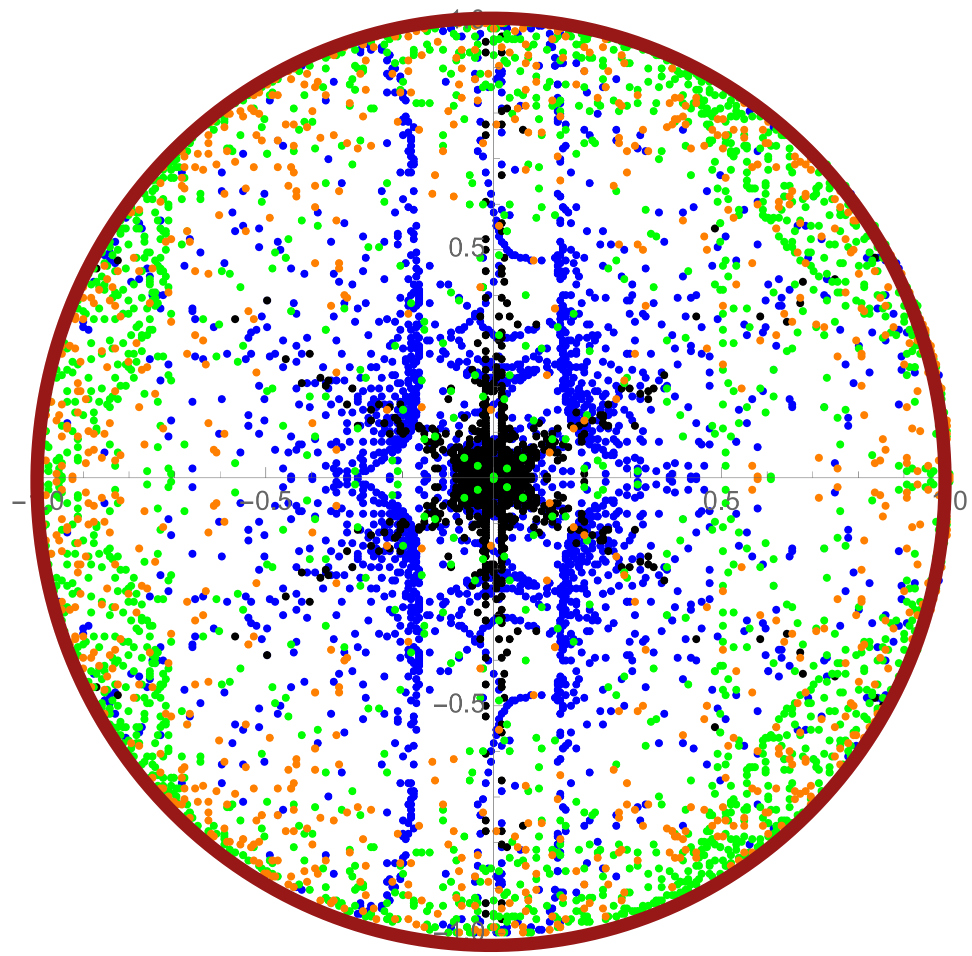


Q. **What are those black diameters of `near infinity points''?**

A. The image of `scattering orbits'' in the unstable manifold of the Lagrange state at infinity and which `stay near infinity' for all time ($\forall t \quad \rho(t) < \epsilon$)

Q. Why are these black diameters arranged as they are?

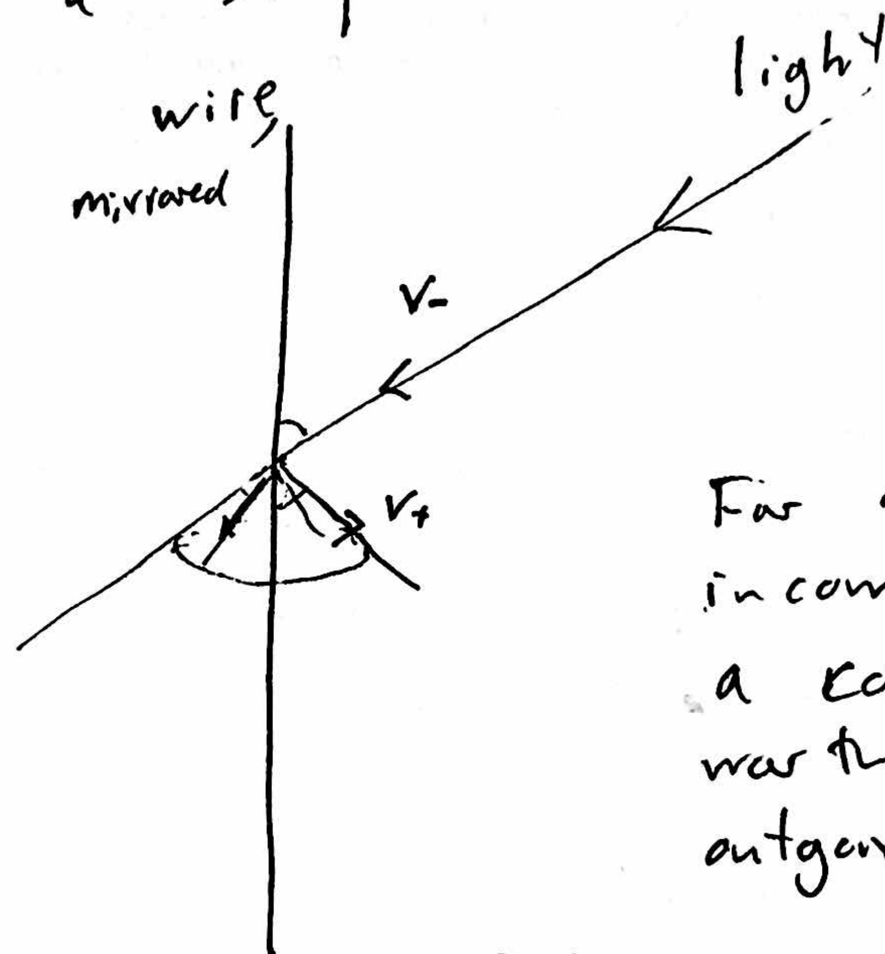
A. **JUST explained.** ***That is where the billiards (broken geod. flow) comes i***



0.5
−0.5
−0.5
0.5

How we $\boxed{\text{think}}$ "time Π broken geodesic nondeterministic flow" arises.

1st a simple model.



For a given incoming ray, a cone's worth of outgoing rays.

Rule:

1) $|v_-| = |v_+|$

2) $\Pi_{wire} v_- = \Pi_{wire} v_+$.

Now in N-body problem in d-dim'ns

'space' $\longleftrightarrow$ $(\mathbb{R}^d)^N := \mathbb{E}$

"wire" $\longleftrightarrow$ collision locus

$= \{ q : q_a = q_b, \text{ some } a \neq b \}$

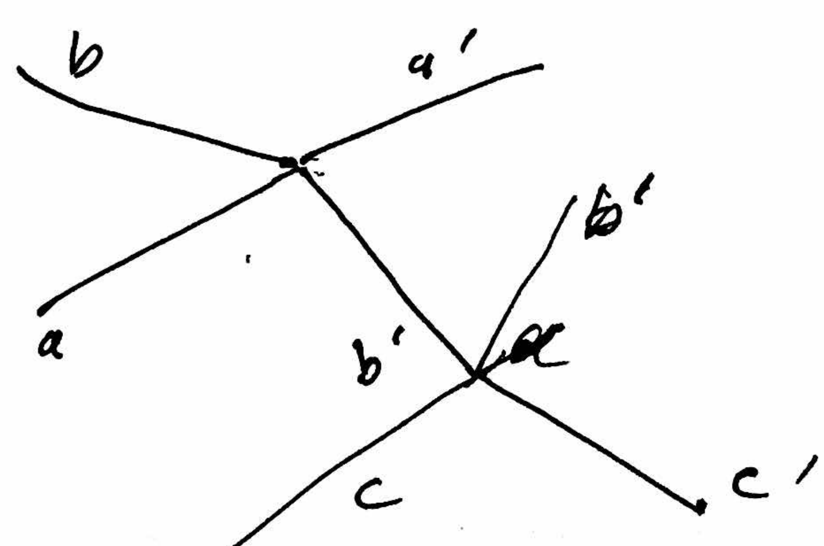
$= \bigcup_{\text{pairs}} \Delta_{ab}$.

1) $\Leftrightarrow$ conserv. of Kinetic energy

2) $\Leftrightarrow$ " of linear mom.

Call resulting non-deterministic trajectories "linear point billiards"

(Knauf & train tracks)



in $\mathbb{R}^d$.

c

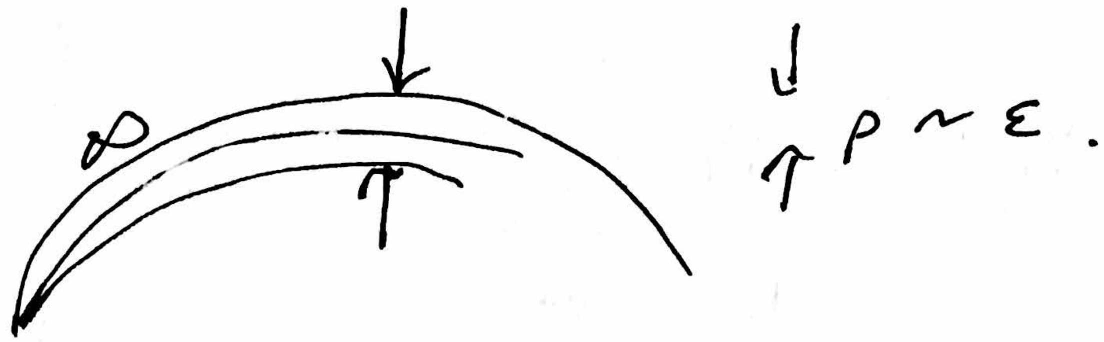
Prop the spherical proj

$\mathbb{E} - 0 \to S(\mathbb{E}) = S^{dN-1}$

of any linear pt billiard traj is a broken geod, broken at $S \cap \Delta$.

& conversely.

Q becomes: how do linear pt billiards arise out of Newton.



trajectory $q(t)$ w/

$$\rho(t) = \frac{1}{r(t)} < \varepsilon$$

so $r(t) := |q(t)| \geq \varepsilon$

$\forall t$.

Scaling: $q_\varepsilon(t) = \varepsilon q(t/\varepsilon)$

Verify: $\ddot{q} = F(q)$

$\Leftrightarrow \ddot{q}_\varepsilon = \varepsilon F(q_\varepsilon)$.

where $F = \nabla U$.

& $\underbrace{\underbrace{\frac{1}{2}|\dot{q}_\varepsilon|^2 - \varepsilon U(q_\varepsilon)}_{\text{time } t/\varepsilon} = \underbrace{\frac{1}{2}|\dot{q}|^2 - U(q)}_{\text{time } t.}}$

so "$H_\varepsilon = H$"

E

Now $U = \sum \frac{G m_a m_b}{r_{ab}}$

this process is equivalent

to $G \to \varepsilon G$.

"weak coupling"

---

weak coupl limit: let $\varepsilon \to 0$.

---

so For such limits

$\ddot{q} = 0$ outside Δ.

what happens @ Δ.?

We say: any traj sat (1)

& (2) can occur.

No proof written yet.

EITHER: DONE,

or to OPEN PROBLEMS

or to Chazy
Maderna-Venturelli
JM metric
then OPEN PROBLEMS

depending on time

Thm: [Chazy, 1922]: any hyperbolic solution q(t) satisfies

$$q(t) = at + (\nabla_m U(a)) \log t + c + f(t) \qquad \text{as t} \rightarrow \infty$$

with f(t) = O(log(t)/t), and f(t) = g(1/t, log(t)), g analytic in its two variables.

and $a \in \mathbb{R}^{Nd} \setminus \{ \text{ collisions } \}$

a = asymptotic position at infinity = element of

$$a \in \Sigma_- \subset \text{unstable equilibria} = \mathbb{S}^{dN-1} \subset \text{ infinity manifold}$$

Question: Given a , q_0 in $\mathbb{R}^{Nd}$ with **a** not a collision configuration.

Does there exist a hyperbolic solution connecting q_0 at time 0 to **a** at time ∞?

Thm [Maderna-Venturelli; 2019]. YES. Moreover this solution is a metric **ray** for the JM metric with energy h = K(a)= (1/2) |a|^2 .

Thm: [Chazy, 1922]: any hyperbolic solution q(t) satisfies

$$q(t) = at + (\nabla_m U(a)) \log t + c + f(t) \qquad \text{as t } \to \infty$$

with f(t) = O(log(t)/t), and f(t) = g(1/t, log(t)), g analytic in its two variables.

and $a \in \mathbb{R}^{Nd} \setminus \{ \text{ collisions } \}$

Energy of q(t) must be $\frac{1}{2}\|a\|^2$

Our re-interpretation of Chazy:

$$a \in \Sigma_- \subset \text{unstable equilibria} = \mathbb{S}^{dN-1} \subset \text{ infinity manifold}$$

c = `impact parameter' = affine coord on `projectivized' tangent space to unstable manifold of unstable eq. point **a**

Question: Given a , q_0 **in** $\mathbb{R}^{Nd}$ with **a** not a collision configuration.

Does there exist a hyperbolic solution connecting **a** at time - ∞ to q_0 at time t = 0 ? (and having energy $\frac{1}{2}\|a\|^2$)

Thm [Maderna-Venturelli; 2019]. YES!
Moreover this solution is a metric **ray**
for the JM metric with the given energy $\frac{1}{2}\|a\|^2$

Method of proof ``weak KAM'' a la Fathi

for the Jacobi-Maupertuis metric associated to Newton's eqs and this energy

relevant PDE: H(q, dS(q)) = h

tools: calculus of variations + some PDE + some metric geometry

Metric input: Buseman, Buseman functions as
solutions to the (weak) Hamilton-Jacobi eqns
some Gromov ideas re the boundary at infinity

Jacobi-Maupertuis reformulation of Newtonian mechanics:

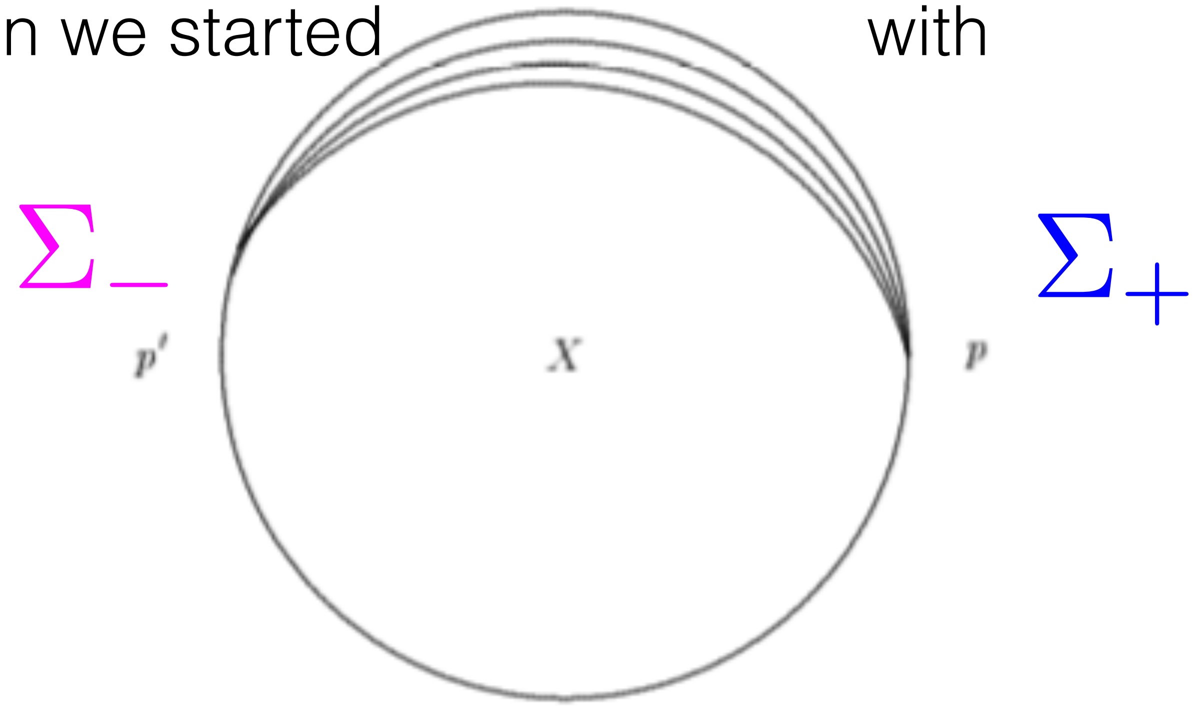
Newton's eqns: $\iff \ddot{q} = \nabla_m U(q)$

where $\langle \nabla_m U(q), w \rangle_m = dU(q)(w)$

Solutions for fixed E = h are reparam's of geodesics for the JM -metric:

$ds_h^2 = 2(h + U(q))|dq|_m^2$ on $\Omega_h = \{q : h + U(q) \geq 0\}$

is closer in spirit to how Melrose and co. look at things
than the standard Newton
formulation we started with



p. 80. Geometric Scattering Theory -Melrose.

Fig. 11. Geodesic of a scattering metric.

REMARK.

Ω_h is a complete metric space.
Riemannian **except** at the Hill boundary h + U(q) = 0
and at the collision locus h + U(q) = + ∞

Solutions to Newton at energy h are metric geodesics
up until they hit the Hill boundary
or **the collision locus**
beyond which instant they cannot be continued
as geodesics.

$$h \geq 0 \implies \Omega_h = \mathbb{R}^{Nd}$$

Open problems …

What about 3-body scattering?

QUESTIONS:

What is the analogue of the scattering map $\pi : \mathbb{R} \to S^1$?

What is its domain - the space of ``impact parameters b's'?

What is its range - the space of `outgoing directions', theta's?

Is it smooth? open? invertible? almost onto?

Can we say anything quantitative or meaningful regarding its induced ``differential cross-section'' $\pi_*(Leb) = f(\Omega)d\Omega$

To begin to answer, return to …

notations

Σ_- and Σ_+ are the unstable/ stable equilibria, both identified with $S^{M-1}\backslash$ (collisions).

Scattering map :

$$Sc : T^*\Sigma_- - \to T^*\Sigma_+$$

where the broken arrow means the domain is not all of the space, but rather an open subset thereof.

The fiber

$$T_b^*\Sigma_- \cong D^{M-1}(b)$$

Fix an ‘incoming beam direction” $b \in \Sigma_-$

Project the restriction map $w \mapsto Sc(b, w)$ onto Σ_+. Call this map

$$\pi_b : D^{M-1}(b) - \to S^{M-1}.$$

EG: Rutherford: ‘ $N = 2$, $d = 2$. $M - 1 = 1$

$$\pi_b : D^1 \cong \mathbb{R} \to S^1$$

EG: $M = 4$, the planar 3-body problem:

$$\pi_b : D^3 - \to S^3$$

Theorem. π_b is analytic on its domain and its image has nonempty interior.

Q1. Is π_b onto?

No. …

So..

Modified Q1s.

Is the image of π_b open ? dense?

What is the complement of its range?

Is it one-to-one? If ‘no’ , one-to-one on an open dense set?

By Maderna and Venturelli,
there is a backward hyperbolic orbit flowing from any noncollision **a** to total collision $q_0 = 0$

This orbit is represented by flowing from a certain \`impact parameter' **c** in D(**a**).

Q2. Is this point **c** unique, or, does more than one orbit leave **a** and end in total collision?

If it is not unique then…

some pictures…

End.

thank you for your attention
and
QUESTIONS

26 MARIAN GIDEA, RAFAEL DE LA LLAVE, AND TERE SEARA

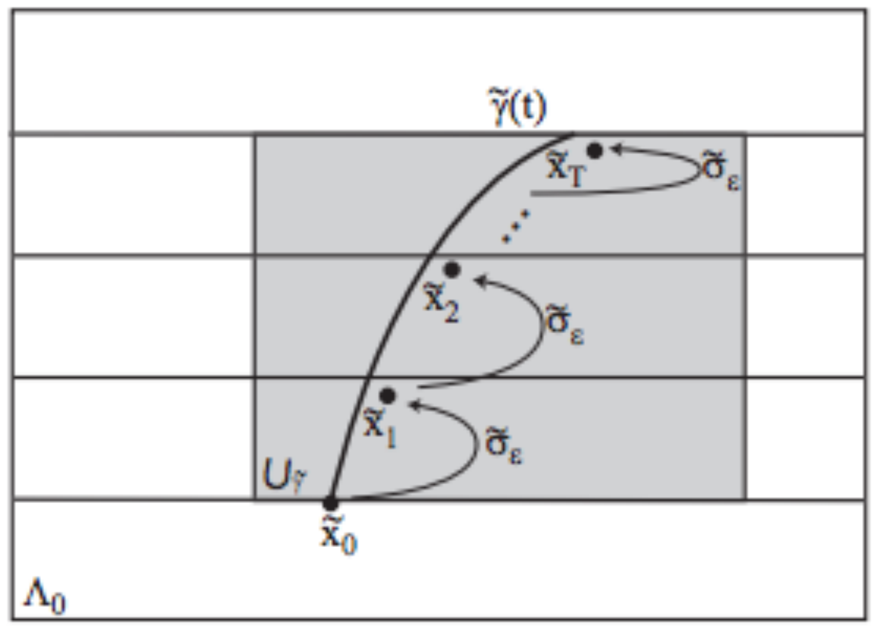


FIGURE 1. A scattering path and a nearby orbit of the scattering map.

A. Delshams, Tere Seara, R de la Llave, M Gidea, ….

Our scattering map is the same as their `scattering map' !
except **that their stable/unstable intersections**
are (1) typically homoclinic
and (2) they have a center manifold with a slow dynamics
in place of our manifold of equilibria