A Magnetic Playground for SubRiemannian Geodesics

Richard Montgomery
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Enrico's Int'l sR seminar

via Zoom, April 29, 2020

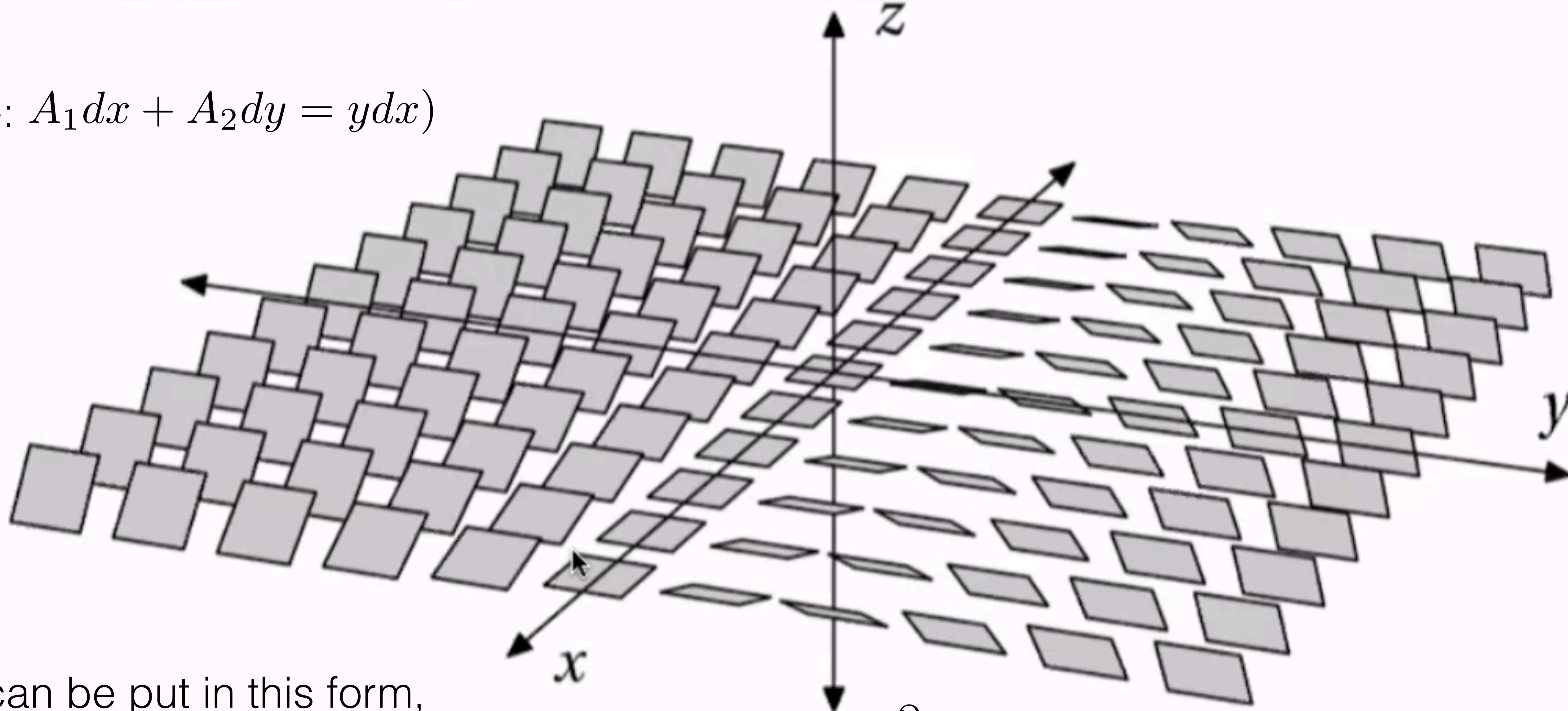
(*) : am retiring, July 1, 2020:
so - keep me in mind for post
C-virus longish term invites, eg, for 2021..

Two-plane fields in 3-space: $\{dz - A_1(x,y)dx - A_2(x,y)dy = 0\}$

`distribution', D

one-form, θ

(here: $A_1 dx + A_2 dy = y dx$)



D can be put in this form,

provided: the two-planes don't go vertical: $\frac{\partial}{\partial z} \notin D(x,y,z)$

and they are invariant under z-translations

Getting there: **PROBLEM :** to join (x_0, y_0, z_0) to (x_1, y_1, z_1) by a horizontal path. `horizontal' = tangent to D.

Write horiz. paths as control system:

`controls'

$$\dot{x} = u_1$$

$$\dot{y} = u_2$$

$$\dot{z} = u_1(t)A_1(x, y) + u_2(t)A_2(x, y).$$

or:

$$\dot{q} = u_1(t)X(q(t)) + u_2(t)Y(q(t))$$

with:

,

$$X = \frac{\partial}{\partial x} + A_1(x, y)\frac{\partial}{\partial z}$$

$$Y = \frac{\partial}{\partial y} + A_2(x, y)\frac{\partial}{\partial z}$$

Strategy:

1.Line up x and y coordinates , using a line segment

2. Fiddle around at the final (x_1, y_1) using planar loops c.

Step 1. u_1 (t) = x_1 - x_0 =const ;
u_2 (t) = y_1 - y_0,

0 < t < 1.

with i.c.: x(0) = x_0, y(0) = y_0, z(0) = z_0.

yields: x(1) =x_1, y(1) = y_1 but

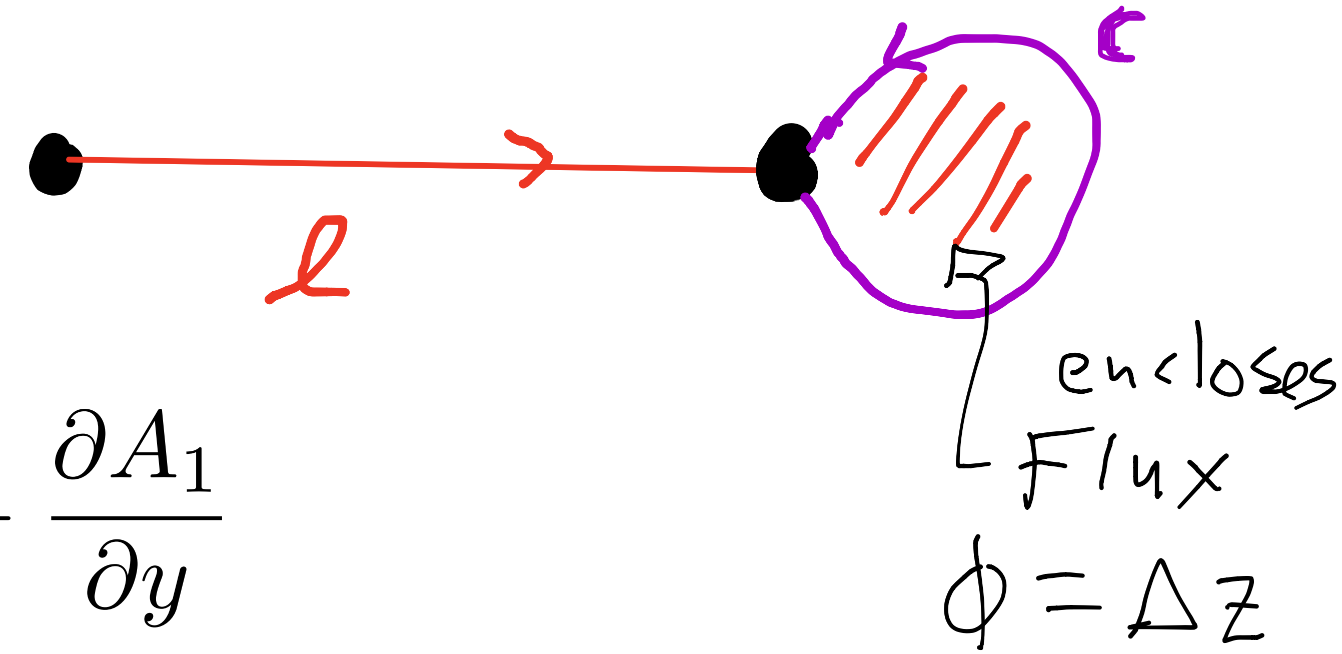
$$z(1) = \int_{\ell} A_1 dx + A_2 dy \quad \neq z_1$$

Step 2. Try moving around ina planar loop c based at (x_1, y_1) .
Then our height z changes according to :

$$\dot{z} = A_1(x, y)\dot{x} + A_2(x, y)\dot{y}$$

or…

$$\Delta z = \int_c A_1 dx + A_2 dy = \int\int_D B(x,y) dx dy$$ = Flux of Magnetic Field



$$B(x,y) = \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}$$

= Magnetic Field

So, choose c so that flux = z_1 - z(1) .

DONE!

(via Lie brackets: [X, Y] = B(x,y) Z; $Z = \frac{\partial}{\partial z}$

Next. Getting there **optimally**:

Join (x_0, y_0, z_0) to (x_1, y_1, z_1)

by the **shortest** horizontal path connecting them.

`**Shortest?": Let the** length of a horiz. path = length of its proj. to xy plane.

(we need that D does not go vertical for this def. to work)

$$\iff \ell_{sR}(\gamma) = \ell_{\mathbb{R}^2}(c); c = \pi \circ \gamma, \pi(x, y, z) = (x, y)$$

$$\iff sR \text{ struc. is } : D = \{dz - A_1 dx - A_2 dy = 0\},$$

$$\text{and } : \langle \cdot, \cdot \rangle = (dx^2 + dy^2)|_D$$

$$\iff X = \frac{\partial}{\partial x} + A_1 \frac{\partial}{\partial z}, Y = \frac{\partial}{\partial y} + A_2 \frac{\partial}{\partial z}$$

form an orthonormal frame for D

---

complete this frame: $Z = \frac{\partial}{\partial z}$

Deriving sR geodesics. Use $\theta = dz - A_1(x,y)dx - A_2(x,y)dy$

Riem. structure (`penalty metric) tending to

$$ds^2_\epsilon = dx^2 + dy^2 + \frac{1}{\epsilon^2}\theta^2 \quad \rightarrow_{\epsilon \to 0}$$ our sR structure;

$$dx, dy, \theta \quad \leftrightarrow_{dual} \quad X, Y, Z$$

so dually:

$$X^2 + Y^2 + \epsilon^2 Z^2 \rightarrow X^2 + Y^2$$ encodes sR structure.

viewed as:

- -2nd order diff'l operators
- -co-metric [symm. bilinear form on T*]
- -fiber-quadratic f'n (`Hamiltonian'!) on cotangent bundle

Symbol of X: = X, thought of as a fiber-linear Hamiltonian on T*

$$X = \frac{\partial}{\partial x} + A_1(x,y)\frac{\partial}{\partial z} \longrightarrow P_X = p_x + A_1(x,y)p_z$$

$$Y = \frac{\partial}{\partial y} + A_2(x,y)\frac{\partial}{\partial z} \longrightarrow P_Y = p_y + A_2(x,y)p_z$$

$$Z = \frac{\partial}{\partial z} \longrightarrow P_Z = p_z$$

$$(x, y, z, p_x, p_y, p_z) \text{ coord. on } T^*\mathbb{R}^3$$

$$p = p_x dx + p_y dy + p_z dz \in T^*_{(x,y,z)}\mathbb{R}^3$$

$$H_\epsilon = \frac{1}{2}(P_X^2 + P_Y^2 + \epsilon^2 P_Z^2) \to \frac{1}{2}(P_X^2 + P_Y^2)$$

governs (normal) geodesics.

Full disclosure:

__up till now, right out of a review of the book
`A Comprehensive Guide to subRiemannian Geometry'
- by Agrachev, Barilari, and Boscain
which I wrote for the Bulletin of the AMS.

out in a year ?

Geod eqns = Ham'ns eqns =

Poisson bracket of fns

$$\dot{f} = \{f, H\}$$

f runs over fns on T* ; f =x, y, z , P_X, P_Y, P_Z = p_z, good enough

$$\dot{x} = P_X$$

$$\dot{y} = P_Y$$

$$\dot{z} = A_1 P_X + A_2 P_Y + \epsilon \underline{P_Z}$$

$$\dot{P}_X = -(B(x,y)\underline{P_Z})P_Y$$

$$\dot{P}_Y = +(B(x,y)\underline{P_Z})P_X$$

$$\dot{P}_Z = 0$$

$P_Z = const. =$ 'charge '

$= \lambda$, later

Details of computation:

$$\{f, gh\} = g\{f,h\} + h\{f,g\}$$

$$\implies \{f, H_\epsilon\} = P_X\{f, P_X\} + P_Y\{f, P_Y\} + \epsilon^2\{f, P_Z\}$$

$$\{x, p_x\} = \{y, p_y\} = \{z, p_z\} = 1; \qquad \{x, y\} = \ldots = 0 = \{p_x, p_y\} = \ldots = 0$$

$$\implies \text{ for } f = f(x,y,z), \{f, P_X\} = X[f] := df(X); \{f, P_Y\} = Y[f]; ..$$

$$\implies \{f(x,y,z), H\} = P_X X[f] + P_Y Y[f] \implies u_1(t) = P_X(t), u_2(t) = P_Y(t)$$

$$\text{of } \dot{q} = u_1(t)X(q(t)) + u_2(t)Y(q(t))$$

Finally:

$$\{P_Z, P_X\} = \{P_Z, P_Y\} = 0; \{P_X, P_Y\} = -B(x,y)P_Z$$

The x, y, P_X, P_Y eqns decouple from z; P_Z = \lambda, parameter:

regardless of \epsilon !

$$\frac{d}{dt}(x, y) = (P_X, P_Y)$$

$$\frac{d}{dt}(P_X, P_Y) = \lambda B(x, y)\mathbb{J}(P_X, P_Y)$$

where $\mathbb{J}(P_X, P_Y) = (-P_Y, P_X) =$ 90 deg. rotation of (P_X, P_Y)

These planar ODES
are the eqns of charged particle traveling in the plane under the influence of a magnetic field of strength B(x,y) `pointing out of the plane'

WLOG: H = 1/2, so (P_X)^2 + (P_Y)^2 = 1, which says that the plane curve is parameterizes by arc length s, i.e. t= s.

Riem case: $P_X^2 + P_Y^2 + \epsilon^2 P_Z^2 = 1; P_Z = \lambda$

which in turn are equivalent to the geometric eqns:

$$\kappa(s) = \lambda B(x(s), y(s))$$

where: $\kappa = \text{ plane curvature of curve } (x(s), y(s))$

Recall κ

$$\vec{q}(s) = (x(s), y(s)), \frac{d}{ds}\vec{q}(s) = \vec{T}(s); \frac{d}{ds}\vec{T}(s) = \kappa(s)\mathbb{J}\vec{T}(s)$$

Heisenberg Group Case
B(x,y) = const. = 1

Projected geodesic eqs: $\kappa(s) = \lambda$

Solutions: circles and lines

Geodesics: (rough) helices , and lines

for the circles : rate of climbing

$\Delta z/\mathrm{cycle}$ = (signed) area of circle

GEOMETRIC PHASES IN PHYSICS

Alfred Shapere
Frank Wilczek

A nice surprise:

The set of planar curves arising as projections to the xy plane of geodesics is the same for all the Riemannian [penalty] metrics

$$H_\epsilon$$

and for the sR case

$$H = \lim_{\epsilon \to 0} H_\epsilon$$

ABNORMAL GEODESICS.

Horiz. lift of nondeg. *zero locus of magnetic field* = C^1- rigid curve (sense of Bryant-Hsu)

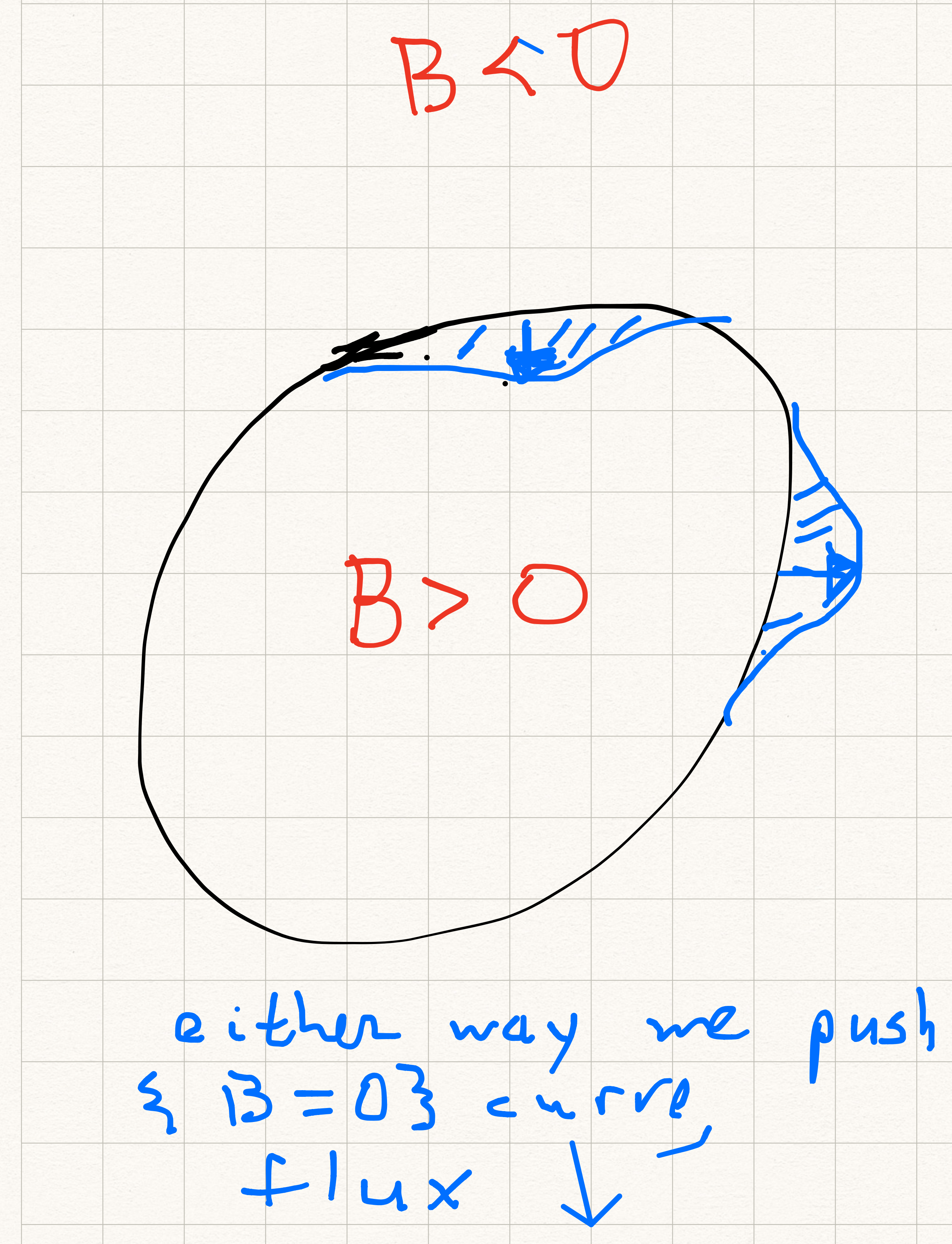
Thm. These curves are geodesics (= loc. length minimizers)

repaired geodesic eq:

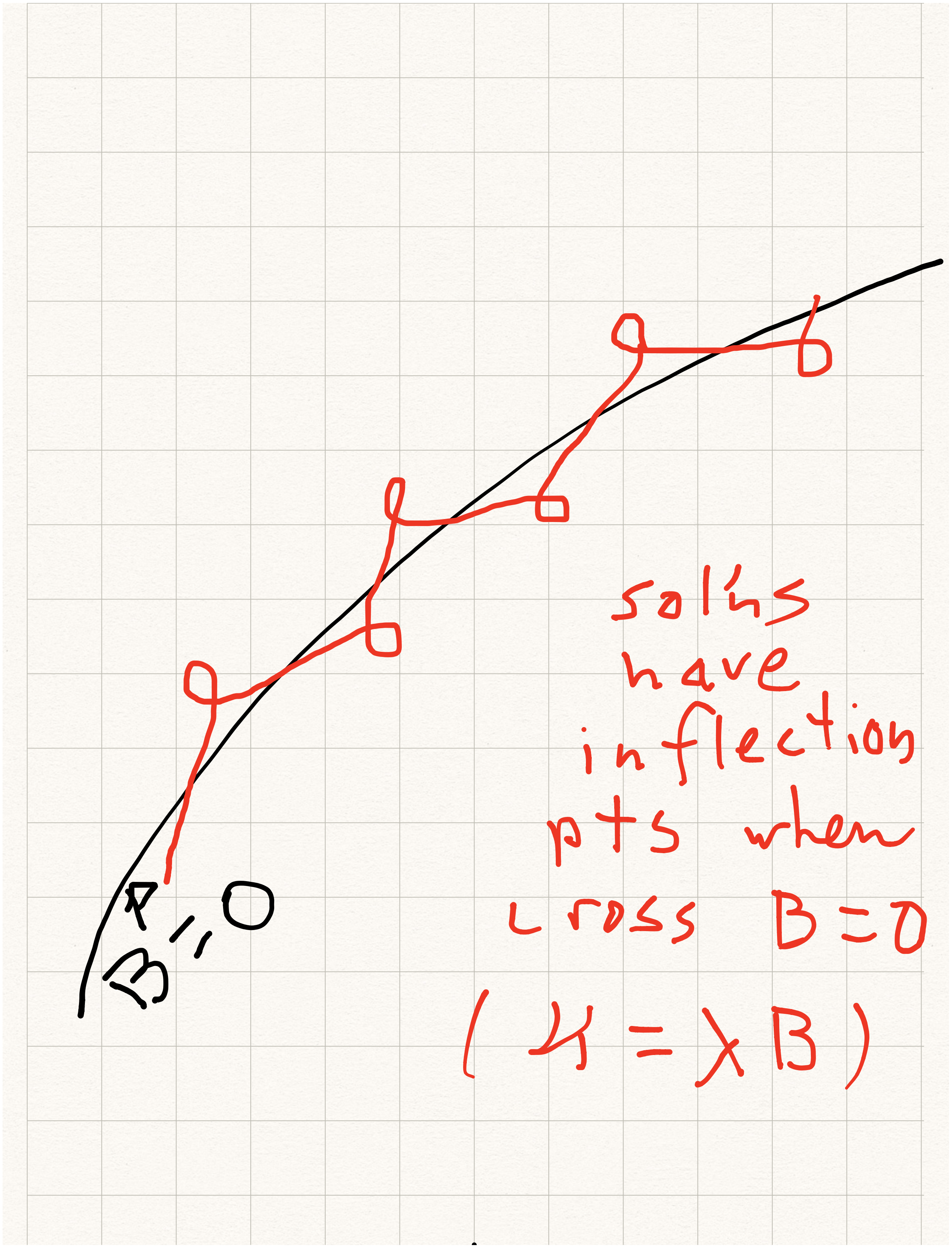
$$\lambda_0 \kappa(s) = \lambda B(x(s), y(s))$$

multiplier for `cost' in Max princ. zero for these abnormal good.

accounts for all possible geod's.



as charge (\lambda) —> infinity
normal geod C^0-converge to
abnormal geod



FLAT MARTINET CASE:.

B = x .

Straighten out zero locus:
B(x,y) = x;

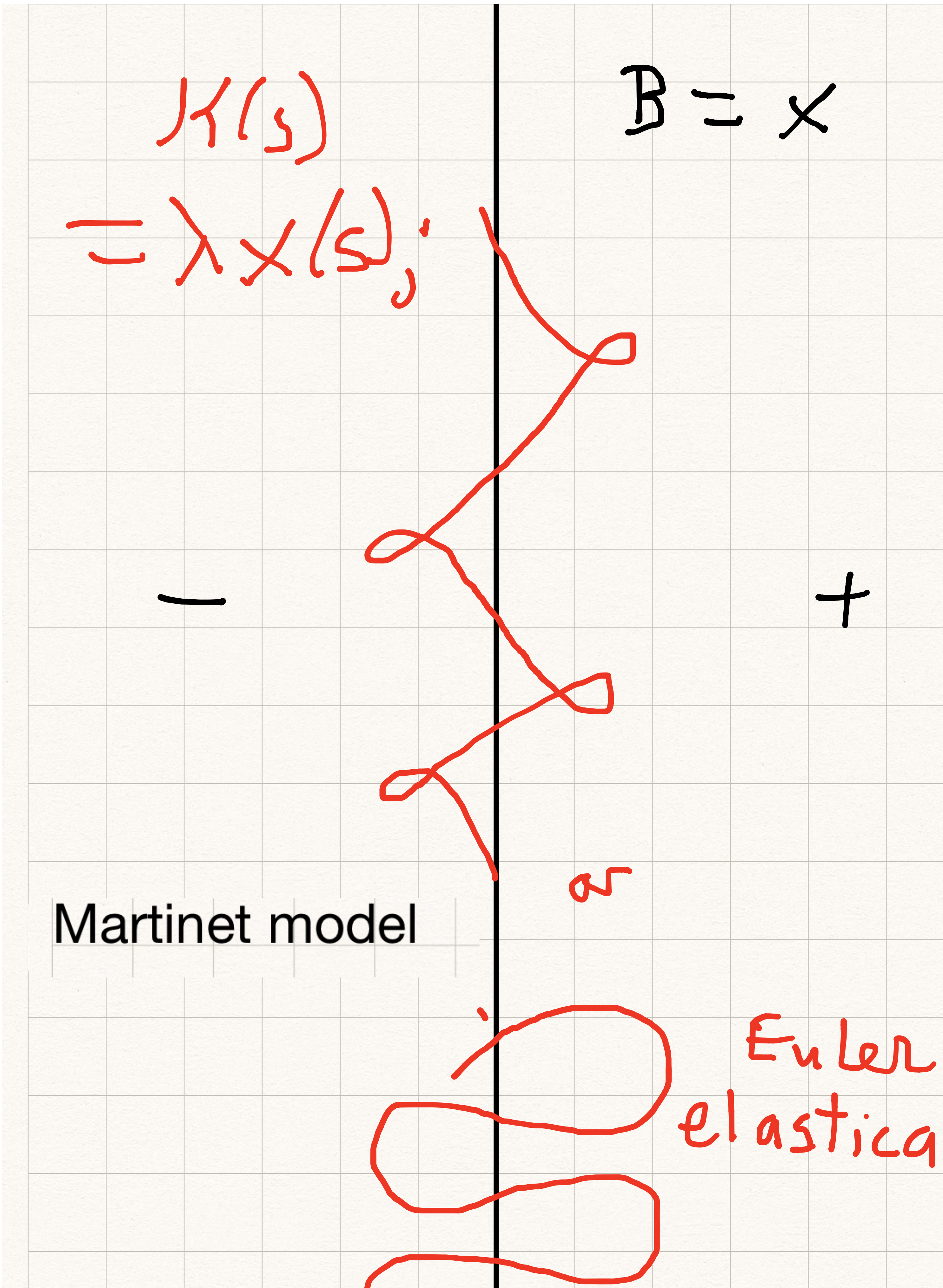
Martinet model for D(x,y,z):
dz - (x^2) dy = 0

Abnormal geod is also Normal

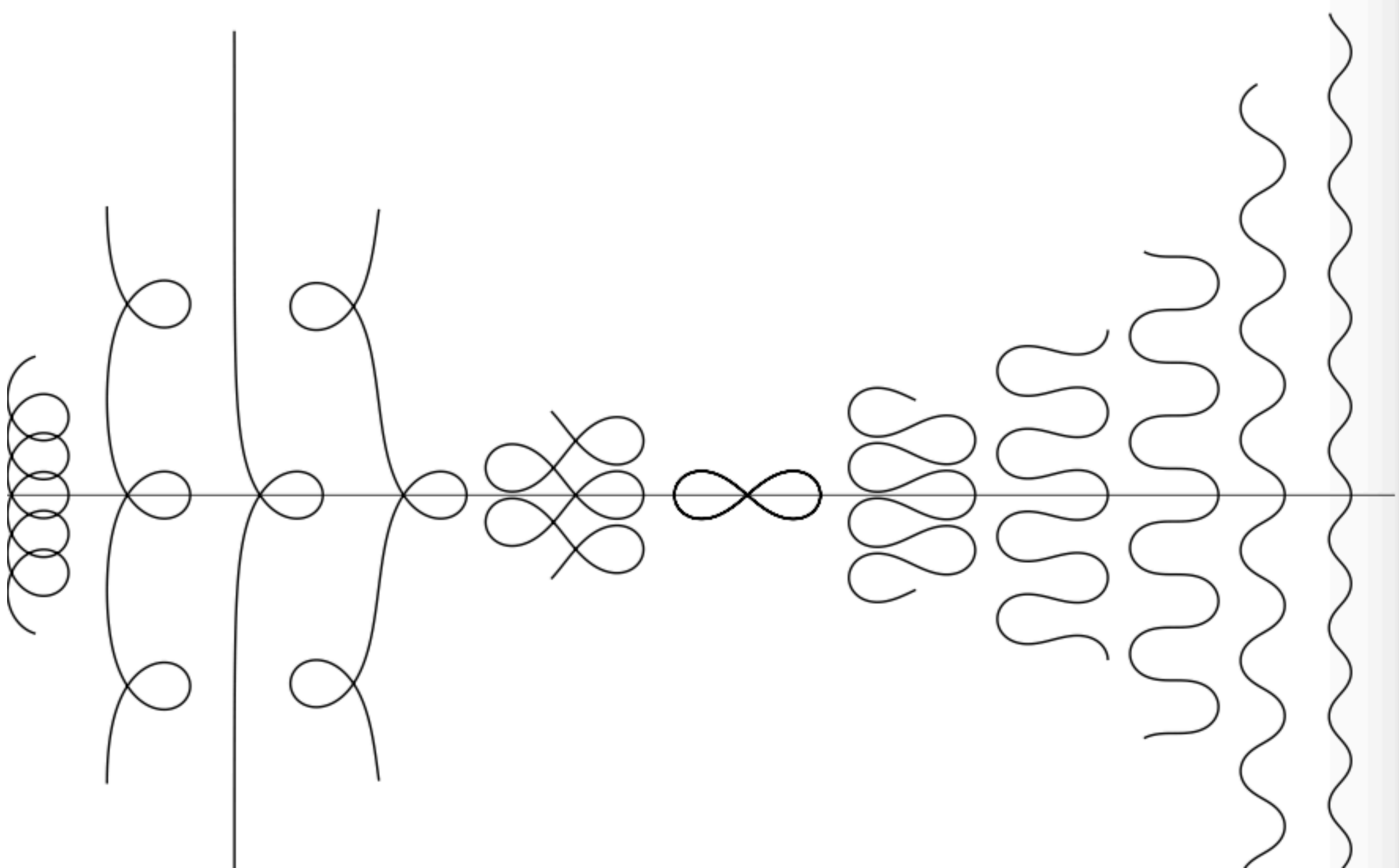
$$\lambda_0 \kappa(s) = \lambda B(x(s), y(s))$$

reads `0 = 0' for all choices of multipliers

Other geodesics:
Euler elastica, given by elliptic fns



thank you Levien ; Ardentov



BRANCHING GEODESICS.
(Meitton-Rizzi, 2019)

$$B = \begin{cases} x, y < 0 \\ 1, y > 1 \end{cases}$$

interpolates between
flat Martinet and Heisenberg

A simpler model (for me)

$$B = x, y < 0$$

and

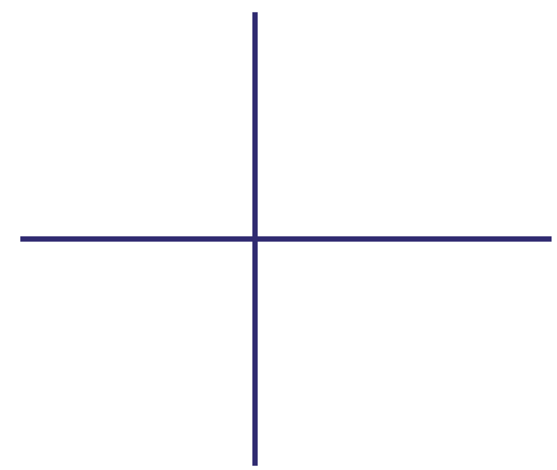
$$B(0, y) > 0, y > 0$$

sR exponential map
`explanation' of branching

Big open problem:
Are all sR geodesics smooth?

Idea for counterexamples:
look at situations where the zero-locus of B(x,y)
is not smooth.

Eg: a) $B(x,y) = xy$ [`normal crossing']

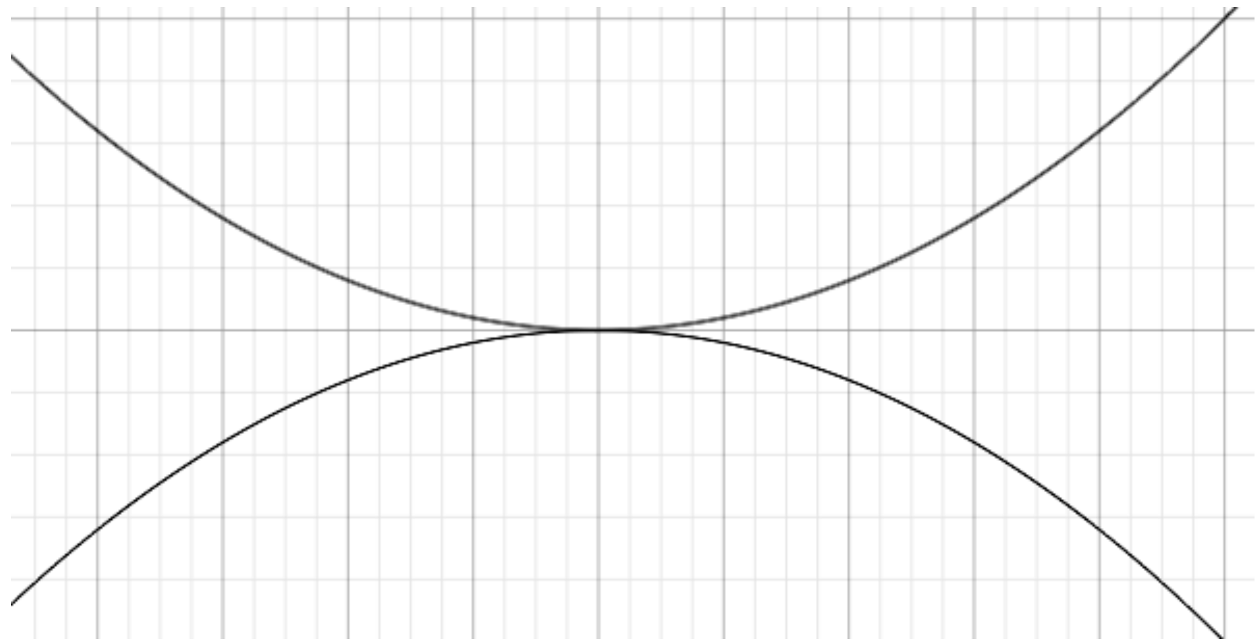


Eero Hakavouri & Enrico Le Donne shoot down this example.
with their ``No corners theorem'' [2016]

try order 2 contact:

Eg (b): Tacnode

$$B = y^2 - x^4$$



1994. Minnesota. Winter. ICM.
Sussmann. Yacine Chitour.
a certain warm crowded cafe across the railroad tracks. Crossing the frozen Mississippi, with hexagonal patterns of ice;
(youngest daughter, heart problem appears..)
Agrachev… looking looking looking for counterexamples…

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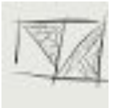
so..

subject: returning to the beginning

Richard Montgomery <rmont@ucsc.edu>
Wed, Apr 22, 9:15 PM (7 days ago)
to sussmann, Andrei…



[..] I hope you are well [...]

I have been thinking about old things,
and realize that you two have likely already pursued these things
and with high likelihood to the bitter end - and found it a dead end. ...
Hence this letter, so either I do not repeat your dead end,
or you give me some nuggets of hope. [....]

Take a magnetic field B whose zero locus is
a tacnode or its higher degree generalizations:
$B(x,y) = y^2 - x^{2k}$; Its zero locus -[...] consists of two branches
$y = x^k$, and $y = -x^k$
with order k-1 of contact at the origin.

[...] Either branch, following until the origin will be a locally minimizing geodesic of Martinet type. So, follow the + branch to the origin, then switch to the other branch.. [to get] a horizontal curve
which is C^{k-1} but not C^k.

QUESTION: is this concatenation a minimizer for any positive integer k ?

Sat, Apr 25, 2020 at 3:32 AM Andrei Agrachev wrote:
Dear Richard,

I am fine, thank you very much.
I do not know the answer but **may be there is a way to reduce this case to the Hakavuori - Le Donne theorem by taking a jet prolongation in the spirit of your and Misha Monster?**

What do you think?
With kindest regards,
Andrei

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Number 956

Points and Curves in the Monster Tower

Richard Montgomery
Michail Zhitomirskii

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Thm: [Agrachev-M-;][2020]
The tacnode example does not minimize.
More generally *any piecewise smooth*
(or piecewise C^k)
sR minimizer is smooth (C^k)

Pf. By induction ,starting w
Hakavouri-LeDonne's k =1.

Tool: Prolongation of distributions AND their curves.

Key facts:
1) the sR struc. also prolongs,

2) the prolongation of a geod is a geod.

Prolonging a distribution
and its horizontal curves

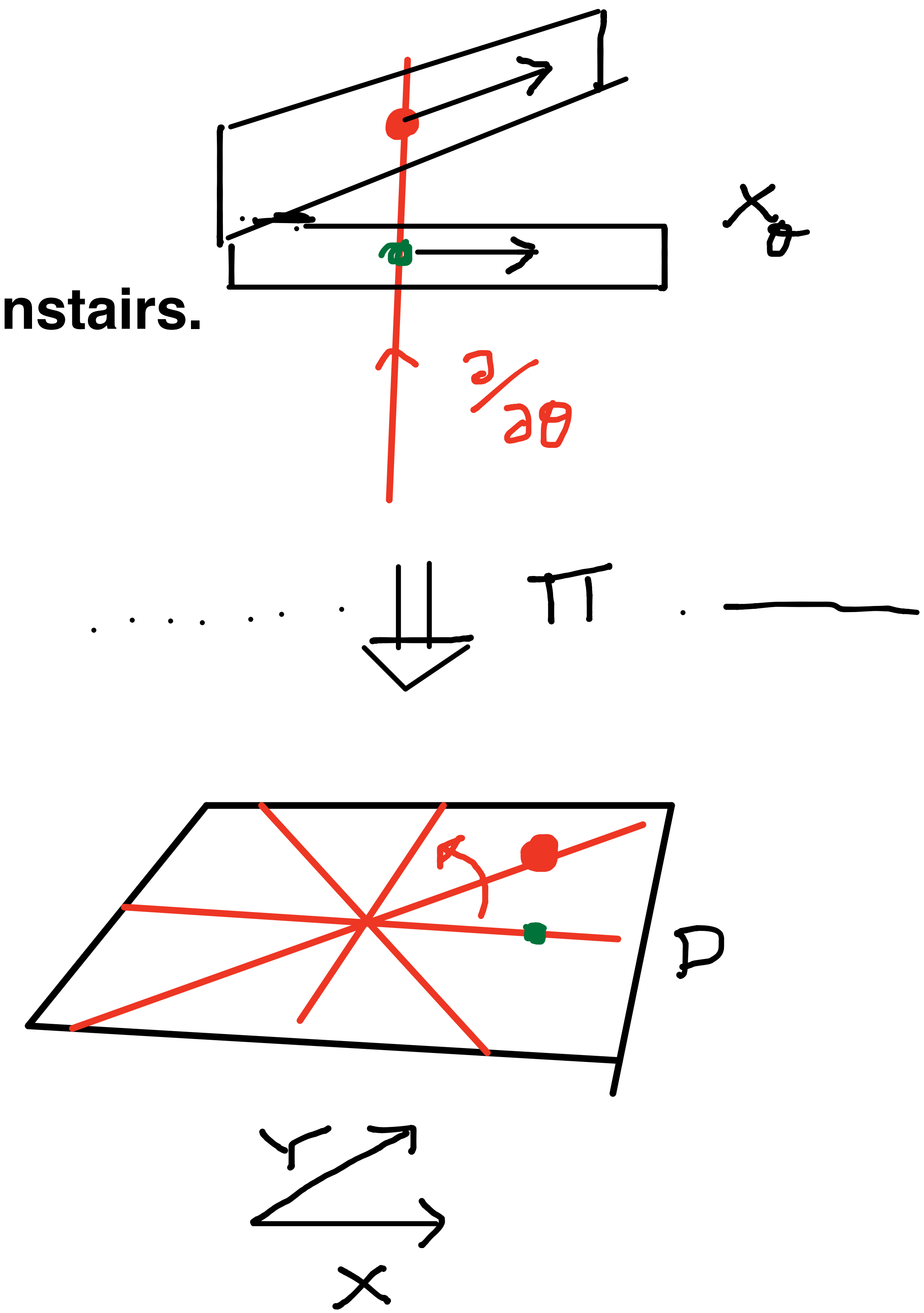
old space: sR manifold Q,
w distribution D, inner prod on…

new space: points are rays in D **downstairs.**

$$\tilde{Q} = \mathbb{P}D$$

so: points in new space:
(pt, line): $(q, \ell), \ell \subset D(q)$, $q \in Q$

new distribution:
def a) : $\tilde{D}(q,\ell) = d\pi_q^{-1}(\ell)$
def b): horiz. curves are curves s.t.
$(q(t), \ell(t))$: $dq/dt \in \ell(t)$



Prolonging horizontal curves

$$q(t) \rightsquigarrow (q(t), \ell(t)) = (q(t), span(dq/dt))$$

Tacnode case: $x = t, y = \pm t^2, z = 0;$

θ for fiber

`Prolong: introduce fiber variable u`

$$[dx, dy] = [1, u] = [\cos(\theta), \sin(\theta)]$$

$$u = \tan(\theta) = dy/dx$$

$$x = t; u = \begin{cases} 2t, t < 0 \\ -2t, t > 0 \end{cases}$$

NOW A CORNER!

want to invoke Hakavouri-LeDonne…

Prolonging sR struc.:

use that fiber inherits metric from inner prod on D . Eg: rank 2 case: identify proj line w unit circle [doubled]

X, Y o.n. for D,

o.n. for prolongation of D $$X_\theta = \cos(\theta) X + \sin(\theta) Y; \frac{\partial}{\partial \theta}$$

Exer: The proj. map $\pi : \tilde{Q} \to Q$ is distance decreasing (actually ``non-increasing'')

Exer: The prolongation of a geodesic is a geodesic.

Proof of theorem: $k = 2$:
Say a geodesic is p.w. C^2.
Prolong it, and its sR struc. :
Result: a C^1 geod in the prolongation
WITH corners!

Contradicts `no corners' theorem of H-LeD.

So the original curve must have been C^2

General k. Similar. Use induction on k.

fini. Thank you all , esp. Enrico,
for the opportunity to talk and
inspiration to prove a theorem

in base
circle radius r = 1/\lambda

in space:
roughly, helices:
lie on cylinder of radius,

in sR case: they climb
area of circle/ revolution

in Riem case: increase height
from sR case by \eps*\lambda
per rev.