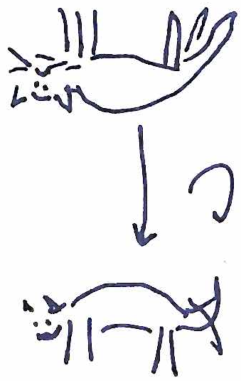
lecture 1 A

~~$F_* U_1 = U_1$ vector . . .~~
~~$F_* U_2 = -U_2$~~

1. A



story : Jair → Wilc. Shapere.

$J = 0$

set up. $q = (q_1 \ldots q_N) \in (\mathbb{R}^3)^N$

for example.

or continuum model.

'what is a "shape?"'

$Q \subset (\mathbb{R}^3)^N$.

$K.E. = \sum m_a |dq_a|^2$.

explain

: etc. $J = 0 \Leftrightarrow v \perp G \cdot q$.

$\sum m_a q_a \times v_a$.

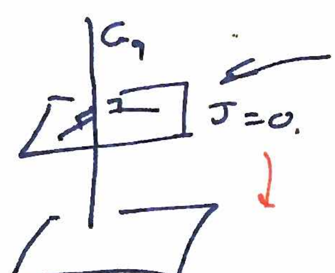
1.B. general : $G \circlearrowright Q$ freely transitively.

*assume G cpt
or action proper
so metric, ∃, quotient
good

$$Q \xrightarrow[G]{\pi} S = Q/G$$

S for shape

G-inv. metric



a connec.

G_q

$J = 0$.

$d\pi_q|\mathcal{H}_q \simeq T_{\pi(q)} S$

$\mathcal{H}_q$

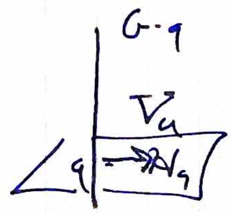
$\sigma_q : \mathfrak{g} \to T_q Q$

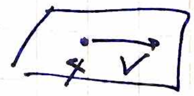
$J(q,p) = \sigma_q^* p \in \mathfrak{g}^*$: "momentum map"

$J = 0 \Leftrightarrow v \perp G \cdot q$.

assuming $p \leftrightarrow v$ metric dual generalities ctd.

$$Q \xrightarrow{G} S = Q/G.$$





$x = \pi(q)$

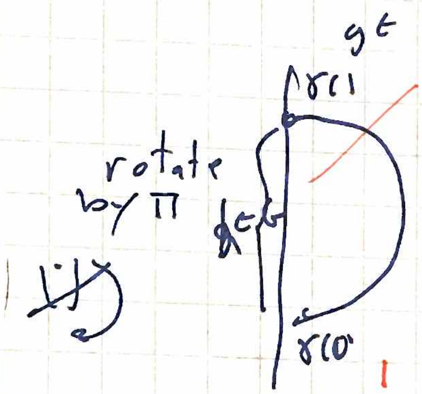
$\exists!\ V_q \in H_q$ s.t. $d\pi_q V_q = v$

H's.

so can lift paths: c.

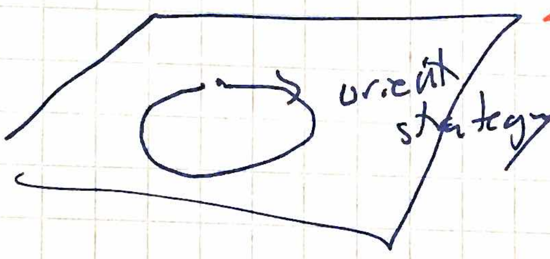
$h\gamma$ by $\pi\, h\gamma = \gamma$.

& $\frac{d}{dt} h\gamma(t) \in H_{h\gamma}$.



$G = SO(3)$ or $SE(3)$ or ...

Metric on the $\pi^* ds^2_S$



or walking or swimming.

holonomy.

$h\gamma(0) = \gamma(1)$

$\uparrow$ is a conjugacy invariant

Different lifts $\gamma_2 = g\gamma$

$g\ h \mapsto g h g^{-1}$

the lift is unique once a pt on single pt over $\gamma(0)$ chosen.

If γ_1, γ_2 the lifts then $\exists\ g \in h$ $g\gamma_1(t) = \gamma_2(t)$

Simplest model cases:

$S = \mathbb{R}^2$.

$G = S^1$ or $\mathbb{R}$

$Q =$

$$\mathcal{H}_{(x,y,z)} = \{dz \underbrace{- A_1 dx - A_2 dy}_{} = 0\}.$$

$a = A_1 dx + A_2 dy$

$A_i = A_i(x,y)$.

That A_i indep of z

$\iff \mathcal{H}$ invariant under G.

Metric on base: $dx^2 + dy^2$.

Then: Horiz v-flds

$$\frac{\partial}{\partial x} + A_1 \frac{\partial}{\partial z}, \quad \frac{\partial}{\partial y} + A_2 \frac{\partial}{\partial z}.$$

are o.n. for $\mathcal{H}$.

$da = B\, dx \wedge dy$

~~Recall~~ &

$$\Delta z = \int dz = \oint_\gamma a = \iint da := \iint_\Omega B\, dx \wedge dy.$$

c classe

Eg: Heis:

$$a = \frac{1}{2}(x\,dy - y\,dx)$$

so

$$da = dx \wedge dy = \text{area fm.}$$

<u>Recall</u>:

generalities

Here next p. $4\frac{1}{2}$ ☆
insert

$$\Rightarrow H = \frac{1}{2}\Big[\underbrace{(p_x - a_1 p_z)}_{v_x}{}^2 + \underbrace{(p_y - a_2 p_z)}_{v_y}{}^2\Big]$$

Ham eqns

$$\dot{x} = \{x, H\} = \{x, \tfrac{1}{2}v_x^2\} = v_x.$$

$$\dot{y} = v_y \quad \ldots$$

$$\dot{z} = v_x\{z, v_x\} + v_y\{z, v_y\} = -a_1 v_x - a_2 v_y$$

re states horizontality

$$\dot{v}_x = \{v_x, H\} = v_y\{v_x, v_y\} = -\Omega v_y$$

$$\dot{v}_y = \{v_y, H\} = v_x\{v_y, v_x\} = +\Omega v_x$$

$$\dot{p}_z = \{p_z, H\} = 0.$$

No z's

$4\frac{1}{2}$*

How to compute? (sR geod)

Generalities

$(Q, \mathcal{H}, \langle , \rangle)$

1) Choose loc. o.n. frame $X_a, a=1, \dots, r$ for $\mathcal{H}$.

2) Turn the X_a into fiber linear fns on T^*Q:

Mom fns; Sachkov notation? often $H_1, H_2 \dots$

$P_a := P_{X_a}$

where if X is a v-fld on Q

$(P_X)(q,p) = p(X(q))$

3) Sum their squares:

$H = \sum P_a^2 : T^*Q \to \mathbb{R}$.

H is the Ham'n generating the normal geod flow.

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*Works for Riem case, too: $\mathcal{H} = TQ$

Exer.: verify.

H is the "principal symbol" of the (sub) Laplacian in PDE terms.

Now

$\{p_x - a_1 p_z, p_y - a_2 p_z\}$

$=$ well $\{f(x,y), p_x\} = \dfrac{\partial f}{\partial x}$

$\{f(x,y), p_y\} = \dfrac{\partial f}{\partial y}.$

$$\Rightarrow \Omega = \left(\frac{\partial a_1}{\partial y} - \frac{\partial a_2}{\partial x}\right) p_z \quad \cancel{a} \quad da = B\, dx \wedge dy$$

$$B = \frac{\partial a_1}{\partial x} - \frac{\partial a_2}{\partial y}.$$

So:

$\dot x = v_x$

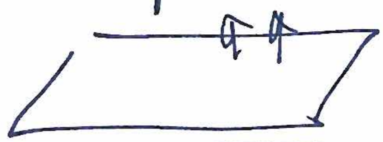
$\dot y = v_y$

$$\frac{d}{dt}\begin{pmatrix} \dot v_x \\ \dot v_y \end{pmatrix} = p_z B \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} v_x \\ v_y \end{pmatrix}.$$

$\dot p_z = 0$ & horiz lift z.

ie solve for $x(t), y(t)$ & s.

eqns of a planar particle in a mag field of strength B

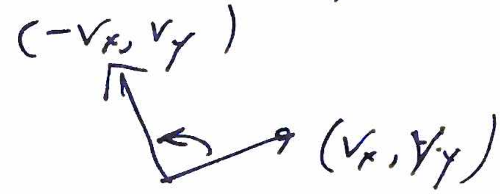


$\perp$ plane $\Leftrightarrow$ $k = p_z B$.

Indeed may assume $H \equiv \frac{1}{2}$ w/ $v_x^2 + v_y^2 = 1$ as $T = (\dot{x}, \dot{y})$ a unit vector

Then
$$\frac{dT}{ds} = -k N$$

$$N = \begin{pmatrix} -v_x \\ v_y \end{pmatrix}$$



$\Rightarrow$ Anything we know about motions of particles in mag. fields. (over 100 yrs of research ...

applies to these sR geometries
