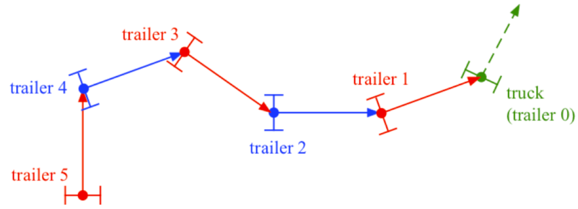
Find & describe interesting integrable sR geodesic flows.

Does the Monster tower admit an integrable sR geodesic flow?

Do ~~the~~ K-trailer systems?



warm-up:

Geodesic flows

on

Surfaces:

Integrable:

plane

sphere

surface of revolution.

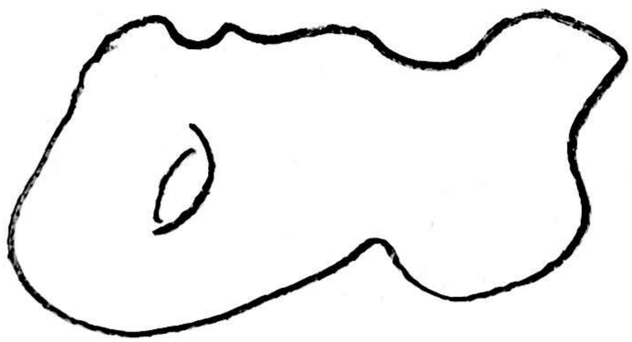
(Jacobi's ellipsoids.

Non-integrable

Σ_g $g>1$

compact.

· on any curvature $K<0$

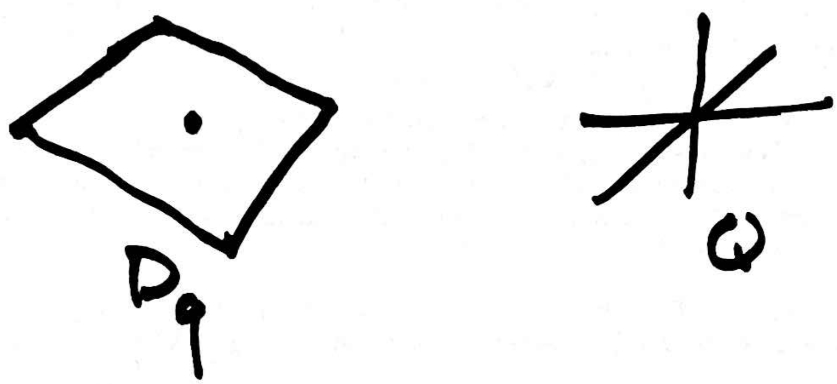


Typical generic metric

sub Riemannian geometry.

- $D \subset TQ$

  linear sub-bundle



- $\langle \, , \, \rangle$ on D.

- Horizontal path

$$\gamma : I \subset \mathbb{R} \to Q$$

$$\dot{\gamma}(t) \in D_{\gamma(t)}$$

- $\ell(\gamma) = \int_I \sqrt{\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle}\, dt$

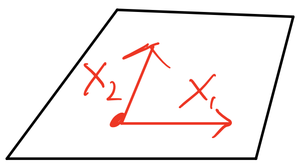
→ Allows us to define

- distance
- geodesics

Geodesic equns

suppose, for simplicity

D has rank 2



X_1, X_2 o.n. frame

by duality

$P_1, P_2 : T^*Q \to \mathbb{R}$

fiber linear

$$P_a(q,p) = p(X_a(q))$$

$a = 1, 2$

$$= \sum p_\mu X_a^\mu (q^1, q^2, \dots, q^n)$$

Define

$$H = \frac{1}{2}(P_1^2 + P_2^2).$$

the sR "Kinetic energy"

H's Hamilton's eqns

generate the "sR geodesic flow"

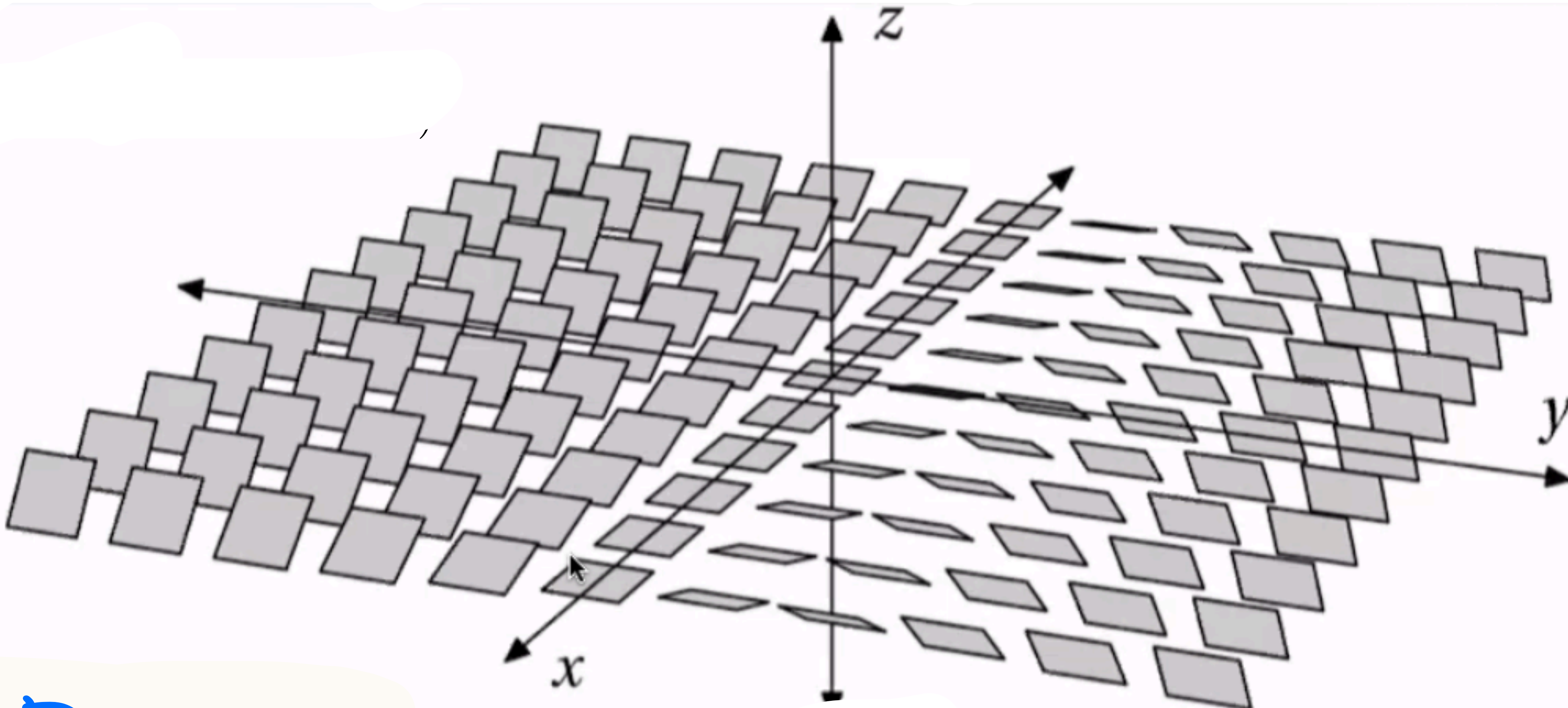
$$\Phi_t : T^*Q \to T^*Q$$

$$\frac{df}{dt} = \{f, H\} \quad \text{where } f_t = \Phi_t^* f$$

Poisson brackets

Prop. The projections to Q of solutions to these Hamilton's equations are sR geodesics.

Heisenberg group



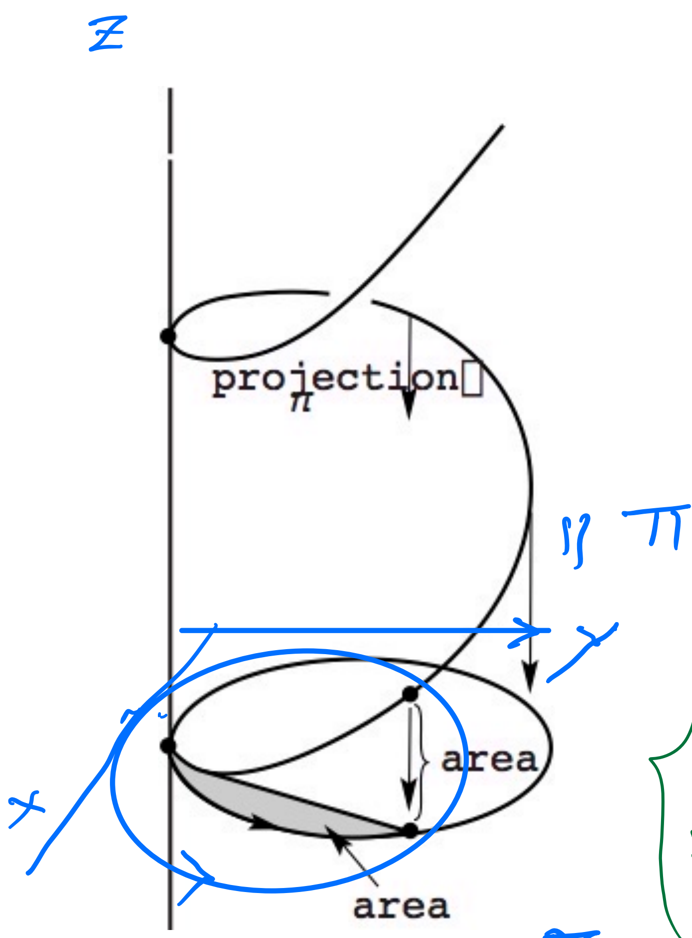
D:

$= \{ dz - y\,dx = 0 \}$

metric:

$dx^2 + dy^2 \,\big|_D$

Heis geodesics...



Heisenberg geodesic problem $\Longleftrightarrow$ isoperimetric problem in plane.

Heis. geodesics given in terms of trig. functions; cosine, sine, & linear:

Project to

 or 

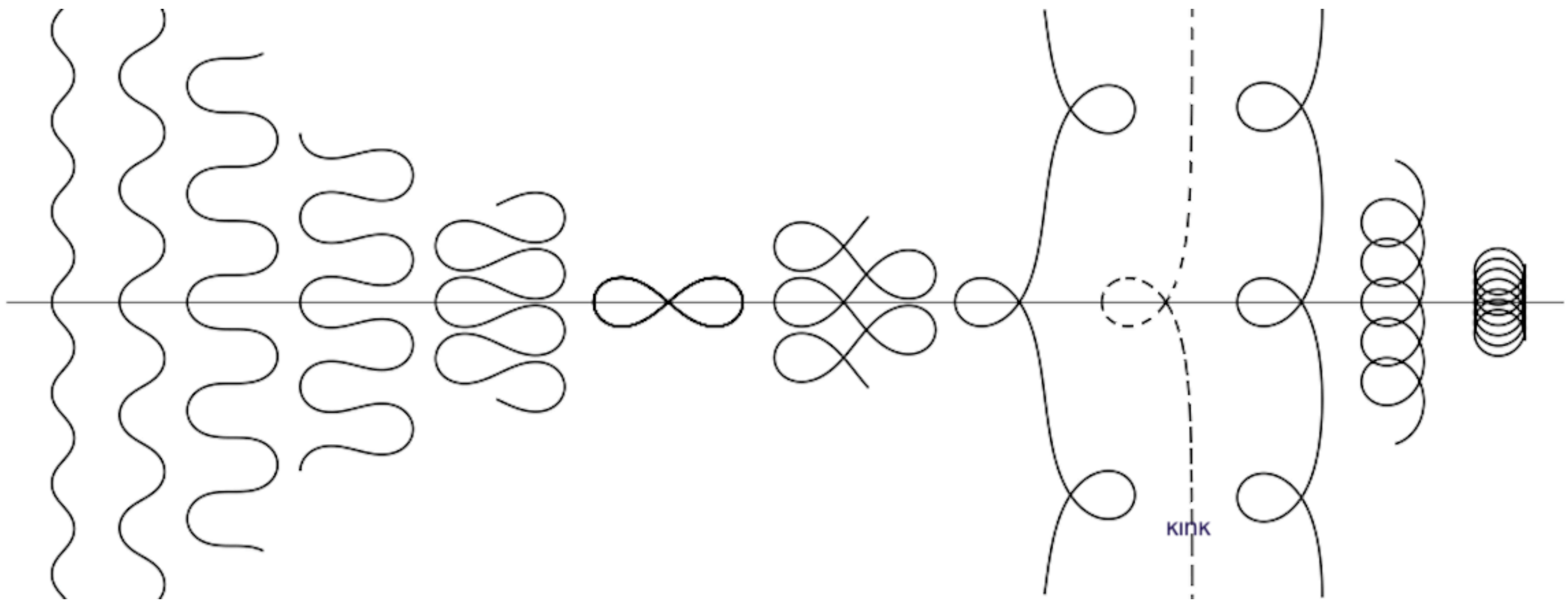
circles or lines

via $\mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$$(x,y,z) \longrightarrow (x,y)$$

After the trig/exponential fns, the next simplest transcendental fns are the elliptic fns

→ Euler's elastica.



Modulo isometries & scaling
a 1-parameter family.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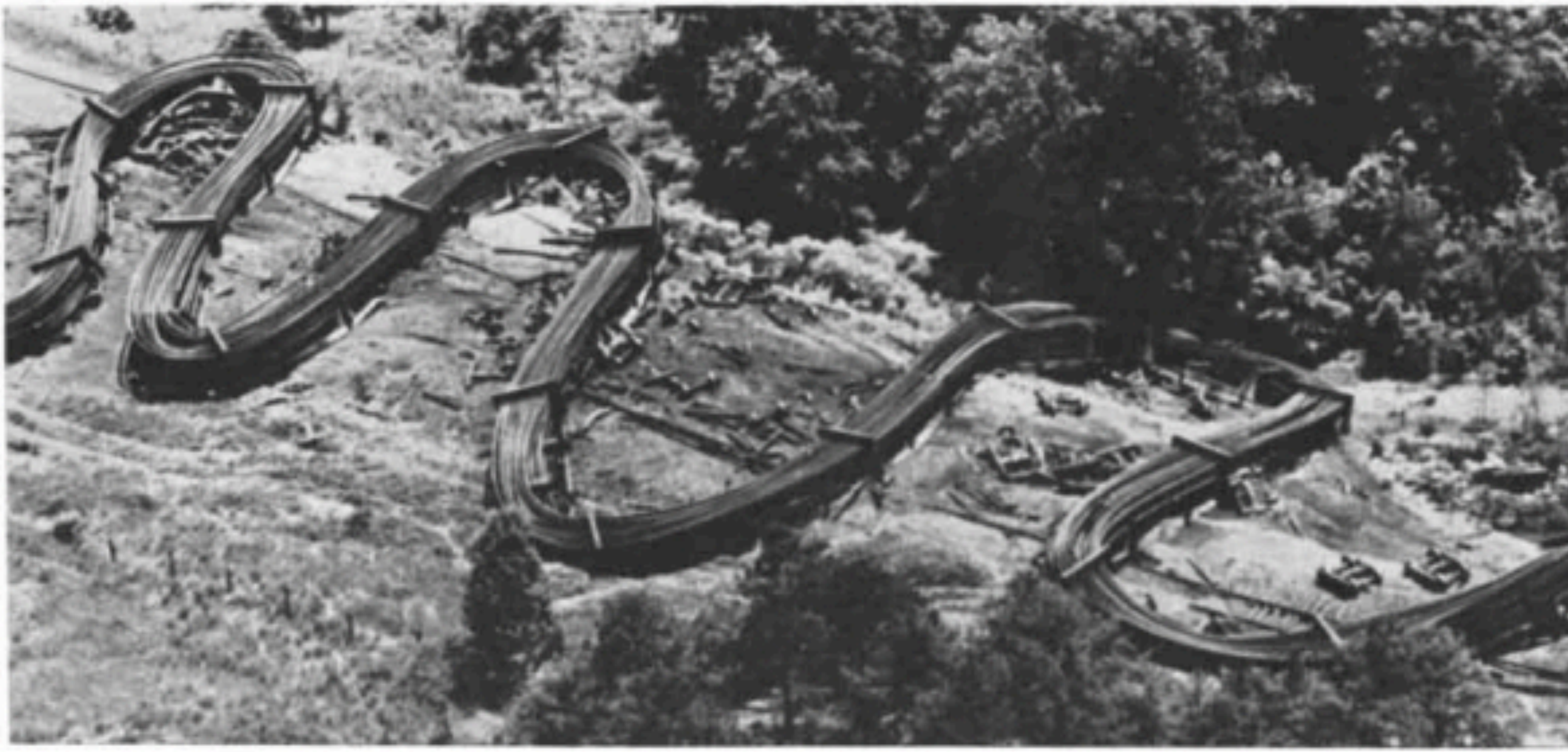


CATASTROPHIC EXAMPLE of a sine-generated curve on a much ... compressive strain folded the trainload of rails into

SubRiemannian geometries whose solutions are lifts of elastica

	manifold	name(s)	discover (?) / ref
#1.	$\mathbb{R}^3$	Flat Martinet	Agrachev-Chyba-.. -Gauthier- Kupka (1997)
#2.	$\mathbb{R}^4$	Engel group or $J^2(\mathbb{R},\mathbb{R})$	Ardentov-Sachkov (2011)
#3.	$\mathbb{R}^5$	Cartan group or Free 3-step Carnot gp or Generalized Dido problem	Sachkov (2003)
#4.	$\mathbb{R}^2 \times \mathbb{S}^1 = SE(2)$	bicycle rolling space or roto-translational space or 1st level of Monster tower	Citti-Sarti (2006) Moiseev-Sachkov (2010) Ardentov-Bor-LeDonne-M.-Sachkov (2020)
#5.	$\mathbb{R}^2 \times SO(3)$	rolling space for sphere on plane	Arthur and Walsh (1986) Jurdjevic (1995) and final sections of book
#6.	$\mathbb{R}^2 \times SL(2,\mathbb{R})$	rolling space for hyperbolic plane on plane	Jurjevic; Bor ..

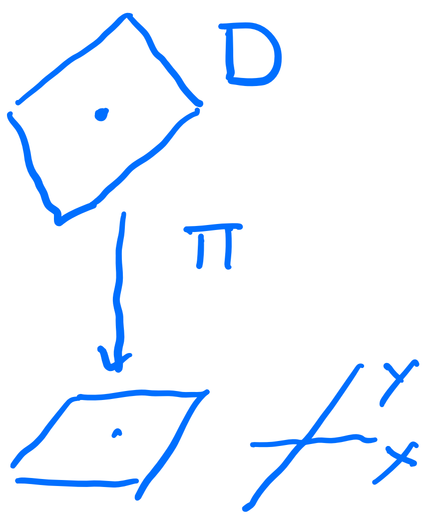
so all these have (sR) geodesic equns which are integrable in terms of elliptic functions.

#1. space: $\mathbb{R}^3$

Martinet:

distribution: $dz - y^2\,dx = 0$ D

metric: $dx^2 + dy^2$ restricted to D



(x,y,z)

$\downarrow$

(x,y)

$$d\pi\big|_{D_{(x,y,z)}} : D_{(x,y,z)} \overset{\text{isom.}}{\simeq} \mathbb{R}^2$$

1.=

FIGURE 15

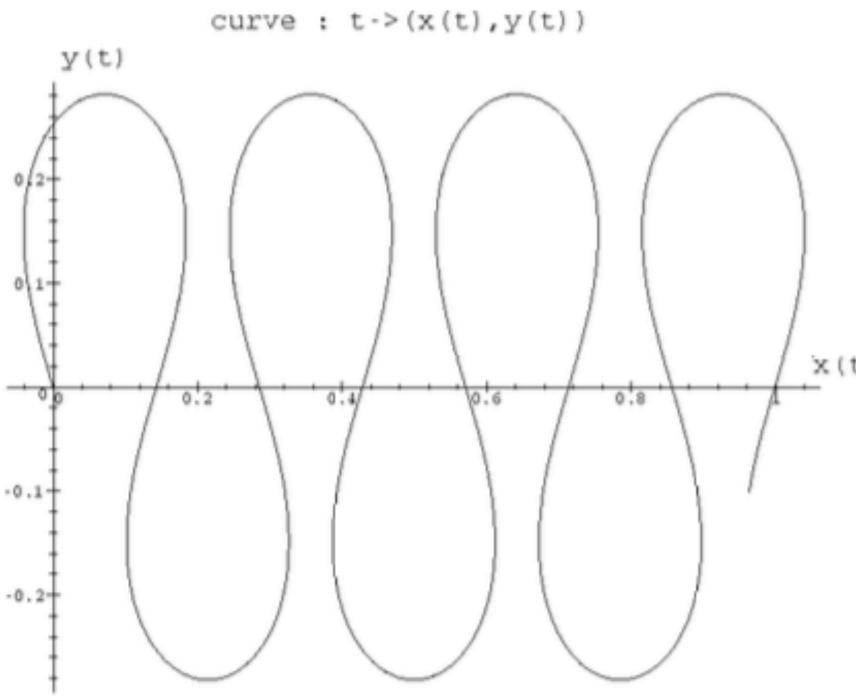
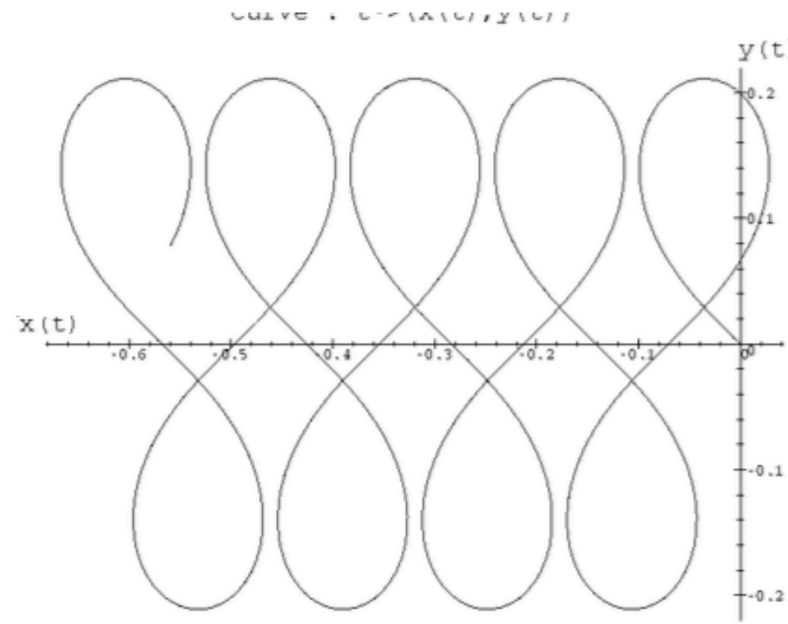


FIGURE 16



SUB-RIEMANNIAN SPHERE IN MARTINET FLAT CASE

A. AGRACHEV, B. BONNARD, M. CHYBA, AND I. KUPKA

(1997)

#1 $\boxed{\text{(Martinet)}}$

curve : $(x(s), y(s))$ ← arc length

$$\kappa(s) = \frac{d\theta}{ds} = \pm \frac{1}{r(s)}$$

ELASTICA ODE :

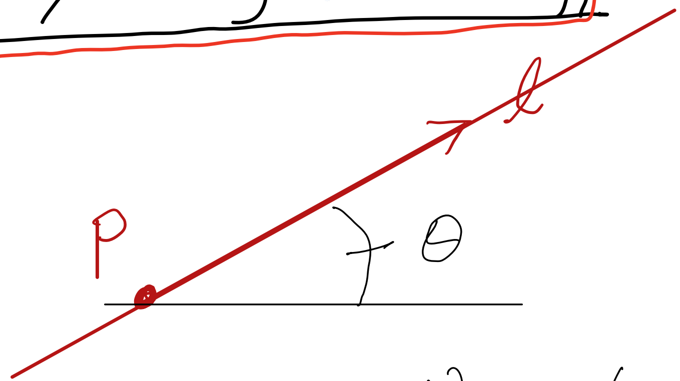
$$\boxed{\kappa(s) = a + by(s)}$$

(More generally

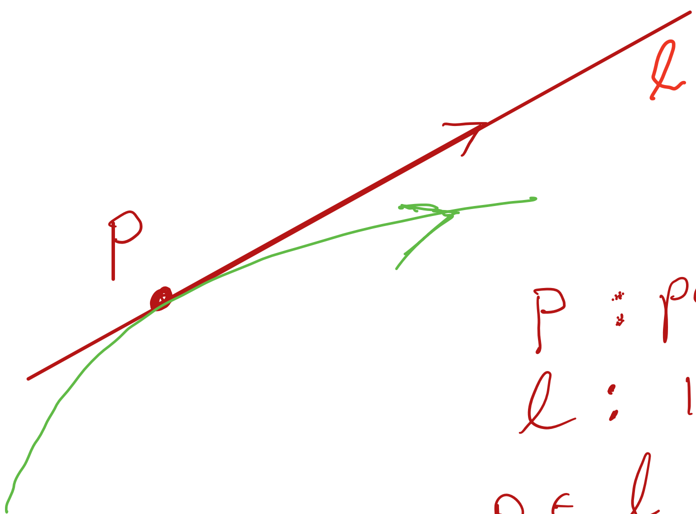
$= \text{affine fn of } x(s), y(s).$)

bicycling (#4)

(geometry 4. of list)



$$(p, \ell) \simeq (p, \theta) \in \mathbb{R}^2 \times S^1$$



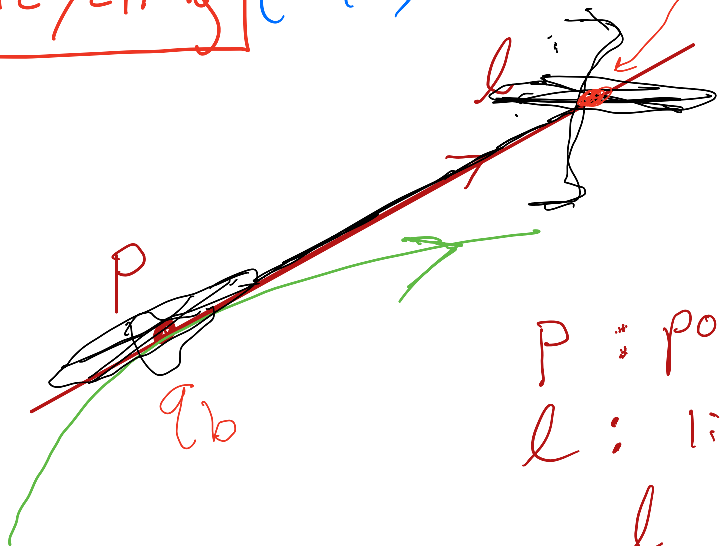
p : point in plane

ℓ : line.

$p \in \ell.$

$$\frac{dp}{dt} \parallel \ell$$

bicycling (#4)



p : point

ℓ : line

ℓ

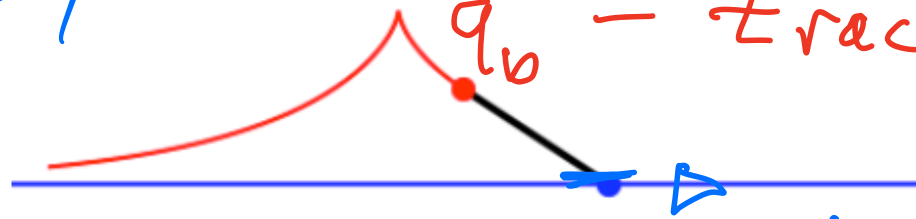
$$\frac{dq_b}{dt} \in \overleftrightarrow{q_f - q_b}$$

SR length = length of front path q_f

$\Rightarrow$

Bicycling along a straight line

#4



q_b – tractrix

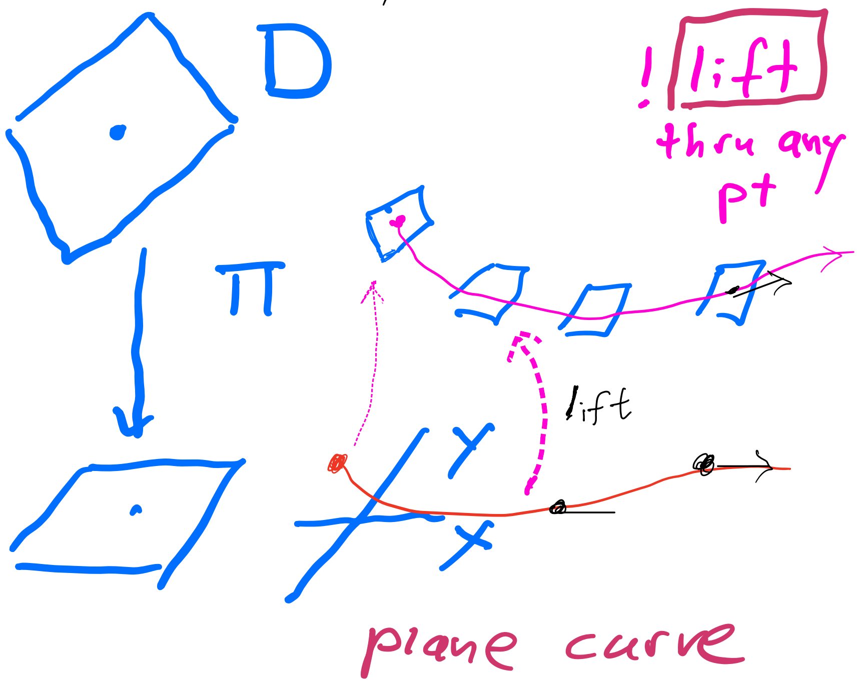
q_f : along line

All admit a sR submersion

$\pi : Q \to \mathbb{R}^2$ ..

onto the Euclidean plane .

$d\pi|_{D_z} : D_z \xrightarrow[\text{isometry}]{\sim} T_{\pi(z)}\mathbb{R}^2 = \mathbb{R}^2$



Corollaries

- a horizontal curve & its plane projection have the same length

- the horizontal lifts "h" of a straight line are sR geodesics

- $X_1 = h\frac{\partial}{\partial x}$, $X_2 = h\frac{\partial}{\partial y}$ frame D.

- If $(x(s), y(s))$ is the plane projection of a sR geod. then $\dot{x} = P_1(s)$
$\dot{y} = P_2(s)$
& the curvature $\kappa(s)$ of this plane curve sat. $\kappa = \{P_1, P_2\}$.

"Integrable Geodesic flows"

- "Flows you can "solve""

Formal def: Arnold-Liouville

$n = \dim Q. = \frac{1}{2} \dim T^*Q$

$\exists\, f_1, \ldots, f_{n-1} \in \mathcal{F}(T^*Q)$.

s.t.

1) $\{f_i, f_j\} = 0 = \{f_i, H\}$

2) $df_1 \wedge df_2 \wedge \cdots \wedge dH \neq 0$ a.e.

$\mathcal{F}$ = Algebraic, or Meromorphic or Analytic or C^∞.
on T^*Q

Some

Integrable sR geometries whose space of projected geodesics is <u>larger</u> than the space of elastica integrable via hypergeometric fns

$\downarrow$

$J^k(\mathbb{R},\mathbb{R}) := J^k$

Doddoli: 2021-2 ★

Perez-Monroy 1997-8

& Anzaldo-Meneses

integrable via 2nd level of the filament hierarchy

$\downarrow$

$\mathbb{R}^2 \times \mathbb{T}^2$

'tricycling'

noslip @ midpoints

★ Perline-Tabachnikov 2023

$J^1 \simeq$ Heisenberg

$J^2 \simeq$ Engel , on list

J^K : integrable by hyperelliptic functions : $\left(\frac{dx}{dt}\right)^2 = P_{2K}(x)$ *

deg $2K$

Felipe Perez-Monroy & Alfonso Anzaldo-Meneses , 2003

see also Alejandro Bravo-Doddoli : 2021

* From

$$\frac{1}{2}\dot{x}^2 + \frac{1}{2}F_K(x)^2 = \frac{1}{2}$$

deg K

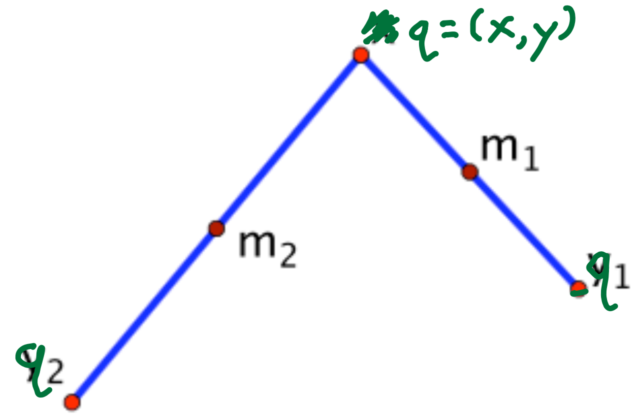
Ham'n 1-deg-of freedom.

Geometry of Integrable Linkages

Ron Perline * Serge Tabachnikov†

July 27, 2023

“ ... we have observed is that the presence of elastica in a geometrical context often indicates that there is an integrable system floating around in the background. ”



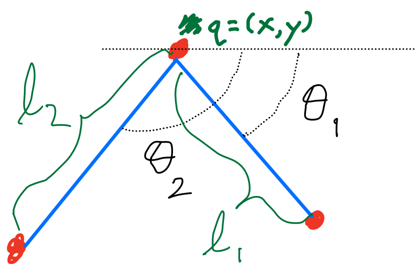
No-slip imposed at midpoints m_i

$$Q = \mathbb{R}^2 \times S^1 \times S^1$$

$$(x,y) \quad \theta_1, \; \theta_2$$

$$X_1 = \frac{\partial}{\partial x} - \frac{\sin\theta_1}{\ell_1}\frac{\partial}{\partial \theta_1} - \frac{\sin\theta_2}{\ell_2}\frac{\partial}{\partial \theta_2}$$

$$X_2 = \frac{\partial}{\partial y} + \frac{\cos\theta_1}{\ell_1}\frac{\partial}{\partial \theta_1} + \frac{\cos\theta_2}{\ell_2}\frac{\partial}{\partial \theta_2}$$



No-slip imposed at midpoints $\underline{m_i}$!

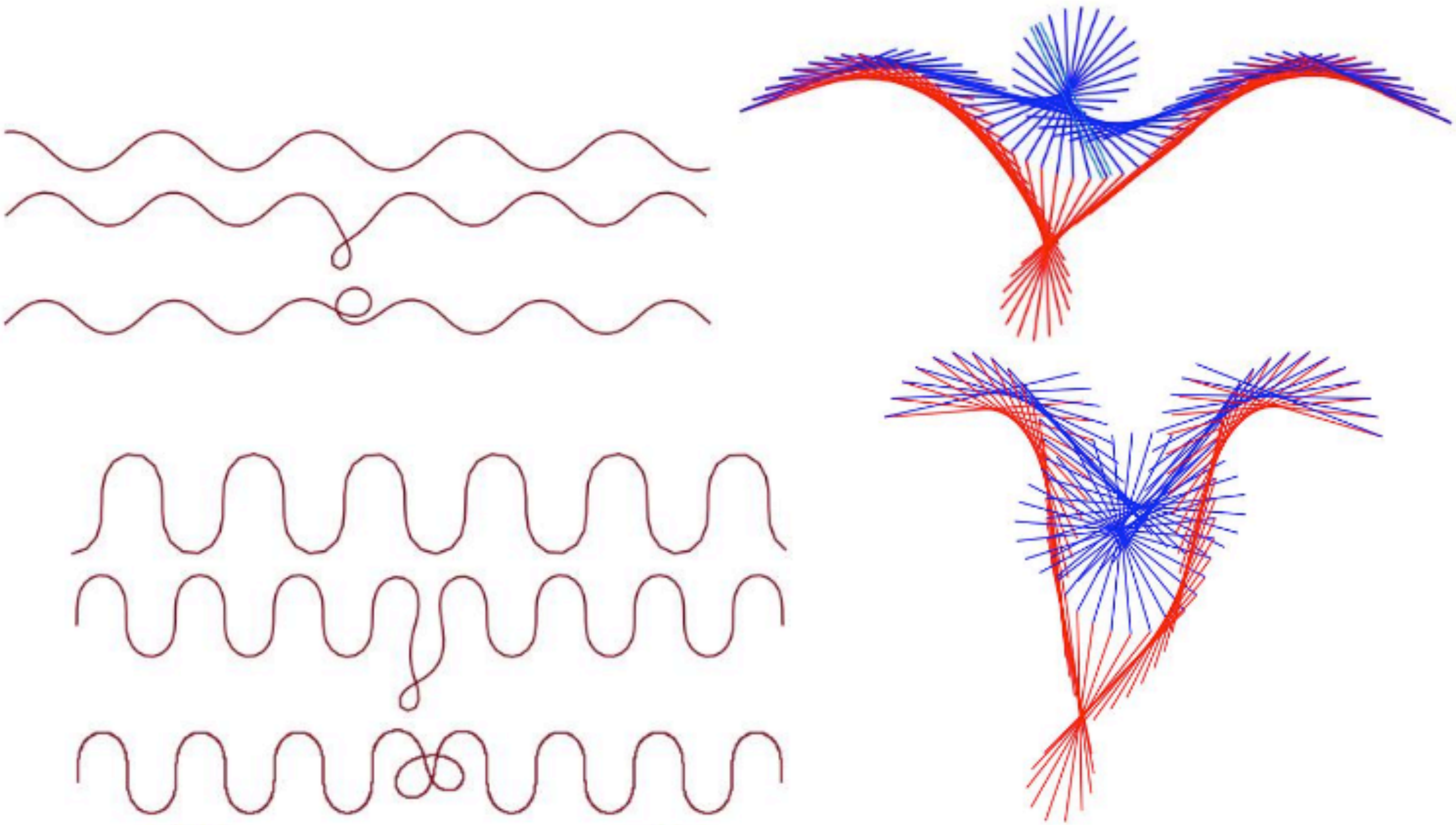


Figure 4. On the left, the trajectories of points x, y_1, and y_2 are shown, and on the right, the respective motion of the 2-linkage whose two links are colored red and blue. The values of k are 0.1 and 0.707, respectively.

$$\gamma_t = \frac{\kappa^2}{2} T + \kappa_s N,$$

$\gamma = \gamma(t, s)$

$$\kappa_t = \kappa_{sss} + \frac{3}{2}\kappa^2 \kappa_s.$$

$' = \frac{d}{ds}$

$$\begin{aligned} I_0 &= \int_\gamma 1\, ds, \quad E_0 = \kappa(s), \\ I_2 &= \int_\gamma \kappa^2\, ds, \quad E_2 = \kappa'' + \frac{\kappa^3}{2}, \\ I_4 &= \int_\gamma (\kappa')^2 - \frac{1}{4}\kappa^4\, ds, \quad E_4 = \frac{5}{2}\kappa^2\kappa'' + \frac{5}{2}\kappa\,\kappa'^2 + \frac{3}{8}\kappa^5 + \kappa''''. \end{aligned}$$

vill refer to critical points of linear combinations of these functionals
on *curves*; for example, a curve satisfying $aE_0 + E_2 = 0$ is a 1-soliton, a
ve satisfying $aE_0 + bE_2 + E_4 = 0$ is a 2-soliton.

$$a E_0(\gamma) = 0 \iff \text{lines, circles}$$

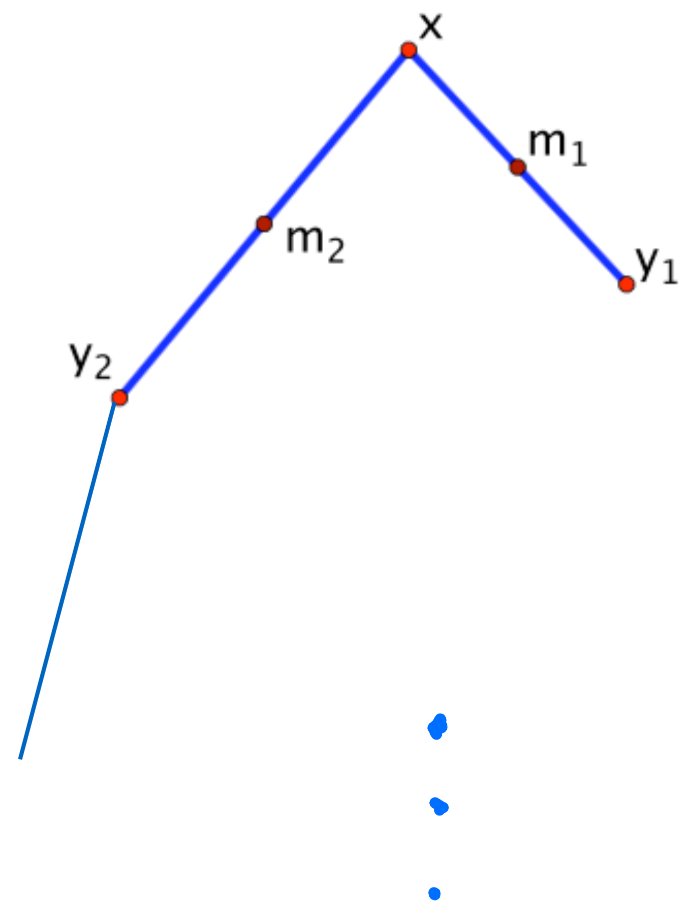
$$a E_0(\gamma) + b E_2(\gamma) = 0 \quad (b \neq 0)$$

$\iff$ elastica (1-solitons

$$a E_0(\gamma) + b E_2(\gamma) + c E_4(\gamma) = 0$$

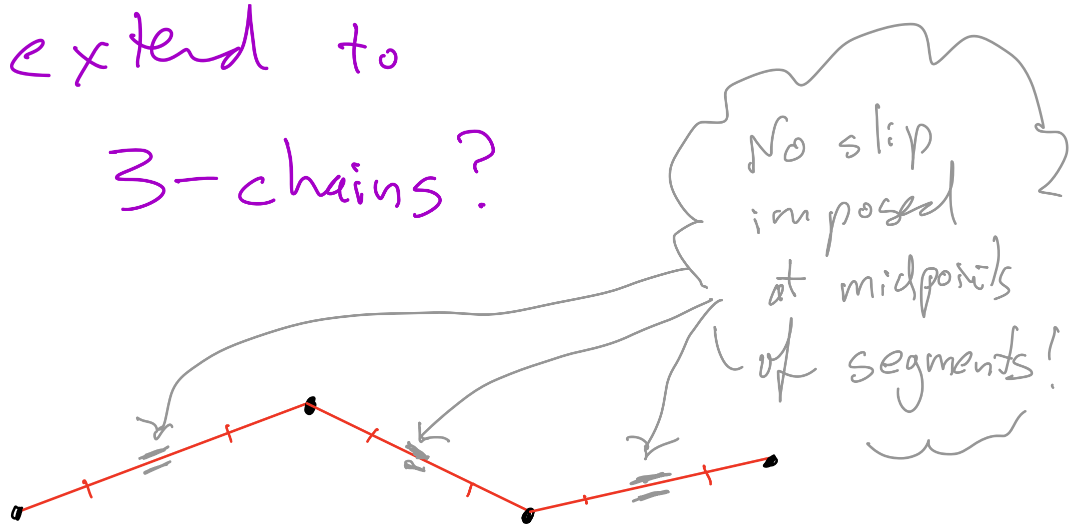
$(c \neq 0)$

$\iff$ tricycle 2-solition curves.



x
m1
m2
y1
y2

Does the Perline-Tab. 'tricycle' integrability (a 2-chain) extend to 3-chains?



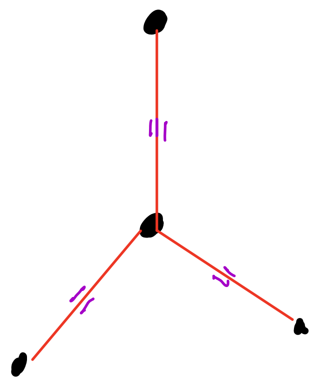
almost certain.

E_0, E_2, E_4, E_6.

Perline; numerical evidence

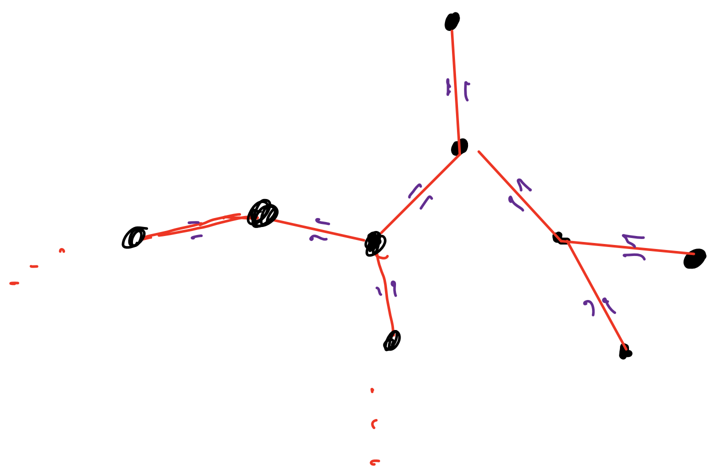
K-chains, any K. ?

or



seems to be integrable √√

& next ... ??



Note!

placement of wheels @ midpoints vs ends of segments

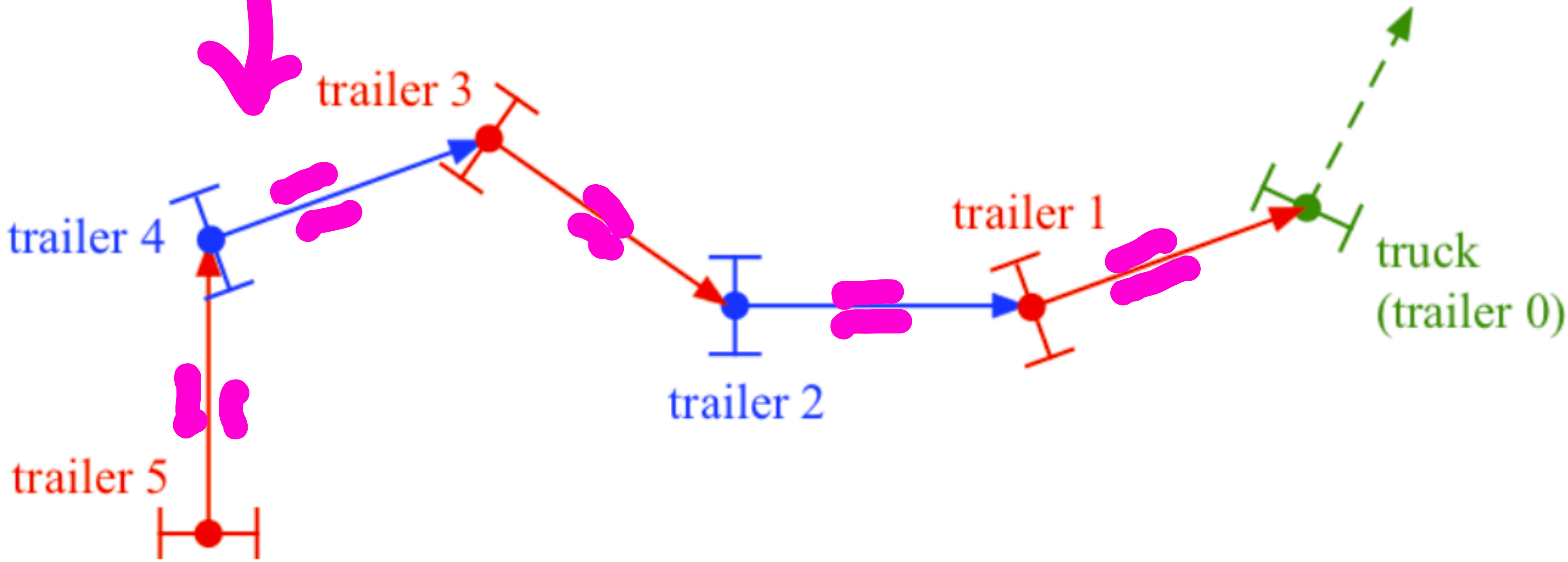


Figure 8: A train, made up of a truck pulling 5 trailers.

Jet space | Monster-Semple tower | k-trailer spaces

$$
\begin{tikzcd}
J^k \arrow[d, "\mathbb{R}"] & \subseteq & M_k \arrow[d, "\mathbb{P}^1"] & T_k \arrow[l, "2:1"'] \arrow[d, "S^1"] \\
J^{k-1} \arrow[d, "\mathbb{R}"] & \subseteq & M_{k-1} \arrow[d, dotted] & T_{k-1} \arrow[l, "2:1"'] \arrow[d, dotted] \\
\mathbb{R}^2 \times \mathbb{R} \arrow[d] & \subseteq & \mathbb{R}^2 \times \mathbb{P}^1 \arrow[d] & \mathbb{R}^2 \times S^1 \arrow[l, "2:1"'] \arrow[d] \\
\mathbb{R}^2 & & \mathbb{R}^2 & \simeq \quad \mathbb{R}^2
\end{tikzcd}
$$

Not sR ~~sal~~ sub-mersions!

"Jean trailer"

no slip @ ends not midpts

Different distributions?

almost certainly
but not checked

Q. Is there an inner product on the K-trailer distribution for which the corresponding sR geodesic flow is integrable? (on T^*T_K)

- same Q, but on M_K? (so flow on T^*M_K)

Overflow

Classify rank 2 integrable ~~C~~ Carnot flows

$\mathfrak{g} = V_1 \oplus V_2 \oplus \cdots \oplus V_k$

$[V_i, V_j] \subseteq V_{i+j}$

$V_s = 0 \quad , s > k$

$V_1 \simeq \mathbb{R}^2$ Lie generates

curve : $(x(s), y(s))$

arc length

$$\kappa(s) = \frac{d\theta}{ds} = \pm \frac{1}{r(s)}$$

ELASTICA ODE :

$$\boxed{\kappa(s) = a + b\,y(s)}$$

(More generally $=$ affine fn of $x(s), y(s)$.)

$$\begin{tikzcd} Q \simeq \mathfrak{g} \simeq \mathbb{R}^N = G, \ \text{Nilp.} \arrow[d, "sR \ \text{submn.}"] \\ V_2 = \mathbb{R}^2 \simeq G/[G,G] \end{tikzcd}$$

$$N = 2 + d_2 + d_3 + \cdots + d_k$$

$$d_j = \dim V_j$$

$(2, d_1, \dots, d_k)$ graded dim'n

$(2,1)$: Heisenberg

$(2,1,1)$ Engel.

$(2,1,2)$ Free 3-step.

$(2, \underbrace{1, 1, \dots, 1}_{k})$ $J^k(\mathbb{R}, \mathbb{R})$

all integrable

$(2,1,2,3)$ Free 4-step.

[growth 2,3,5,8] Non-integrabl

others...

Classify the
Carnot groups
admitting ~~on~~ algebraically*
integrable
subRiemannian structures.

- - - - - - - .

$^*f_i \in \mathrm{Polyn}(T^*G)$
$\cong \mathrm{Polyn}(\mathbb{R}^n \times \mathbb{R}^n)$