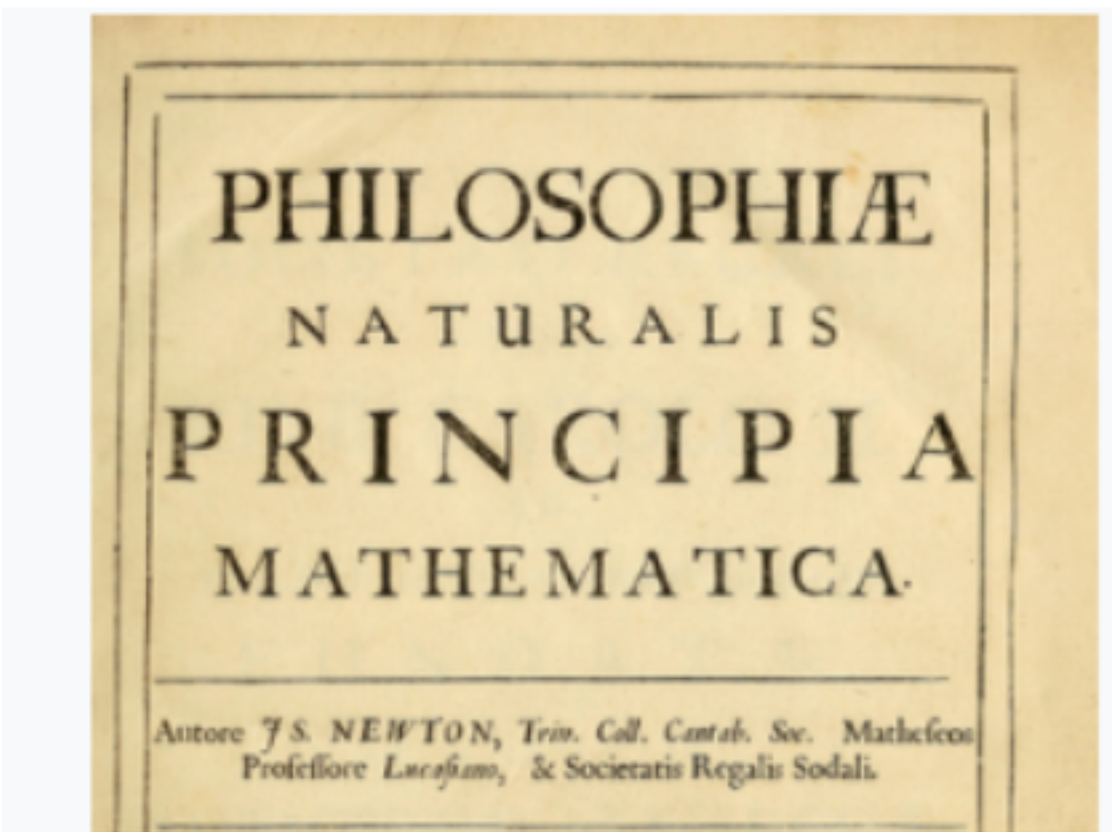
Two Open Questions in the N-body problem

-Richard Montgomery,
prof. emeritus, UC Santa Cruz

NEWTON. 1667.

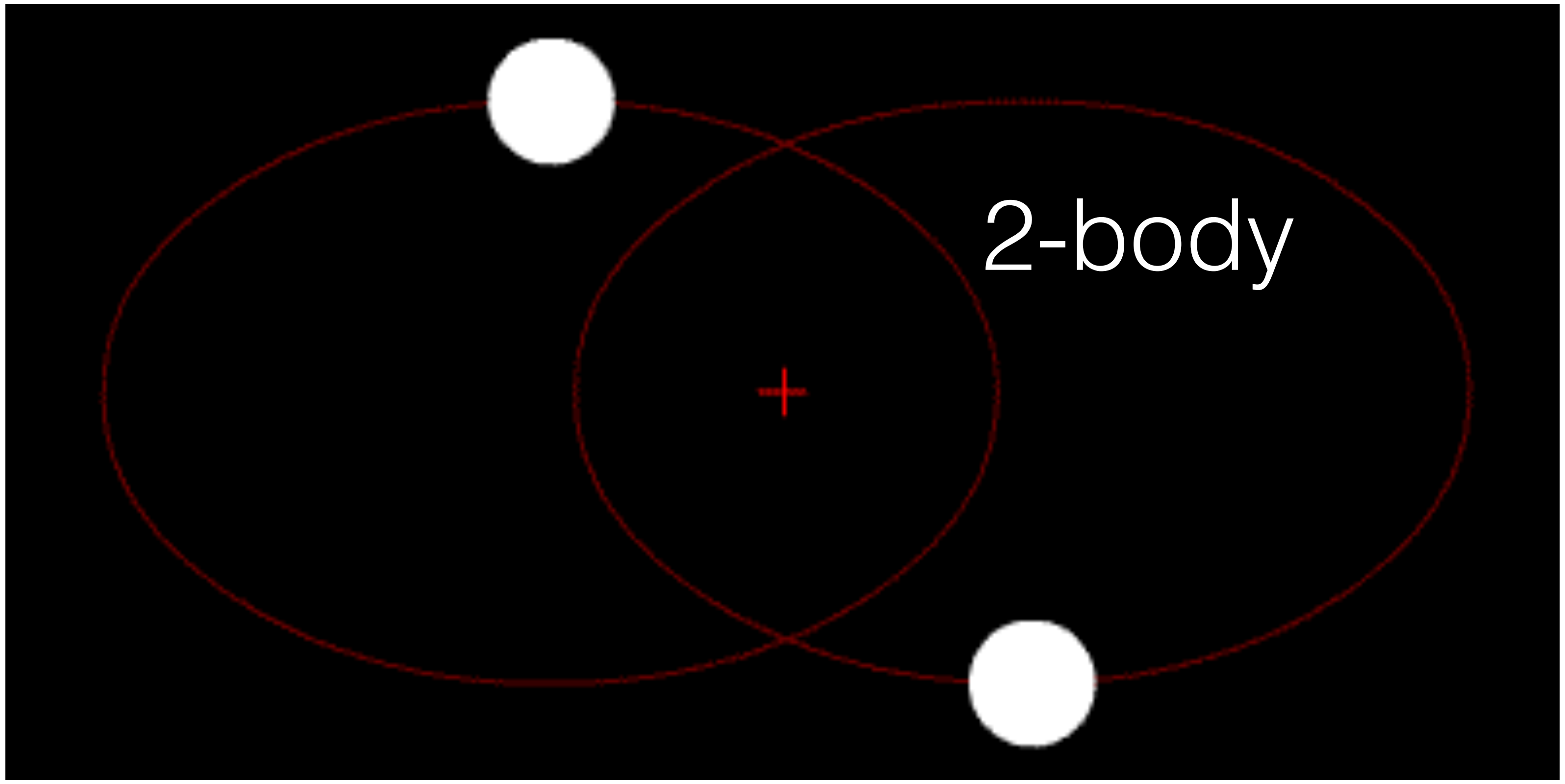
Poses the N-body problem


PHILOSOPHIÆ
NATURALIS
PRINCIPIA
MATHEMATICA.

Autore J S. NEWTON, *Trin. Coll. Cantab. Soc.* Matheseos Professore *Lucasiano*, & Societatis Regalis Sodali.

Solves the 2-body problem

deriving Kepler's laws of planet motion



NEWTON'S EQNS

$$m_1\ddot{q}_1 = F_{21},$$

$$F_{21} = -F_{12} = \frac{Gm_1m_2(q_2 - q_1)}{r_{12}^2}$$

$$m_2\ddot{q}_2 = F_{12}$$

$$\Longrightarrow \qquad \ddot{q} = -\mu\frac{q}{|q|^3}$$ where $$q = q_1 - q_2$$

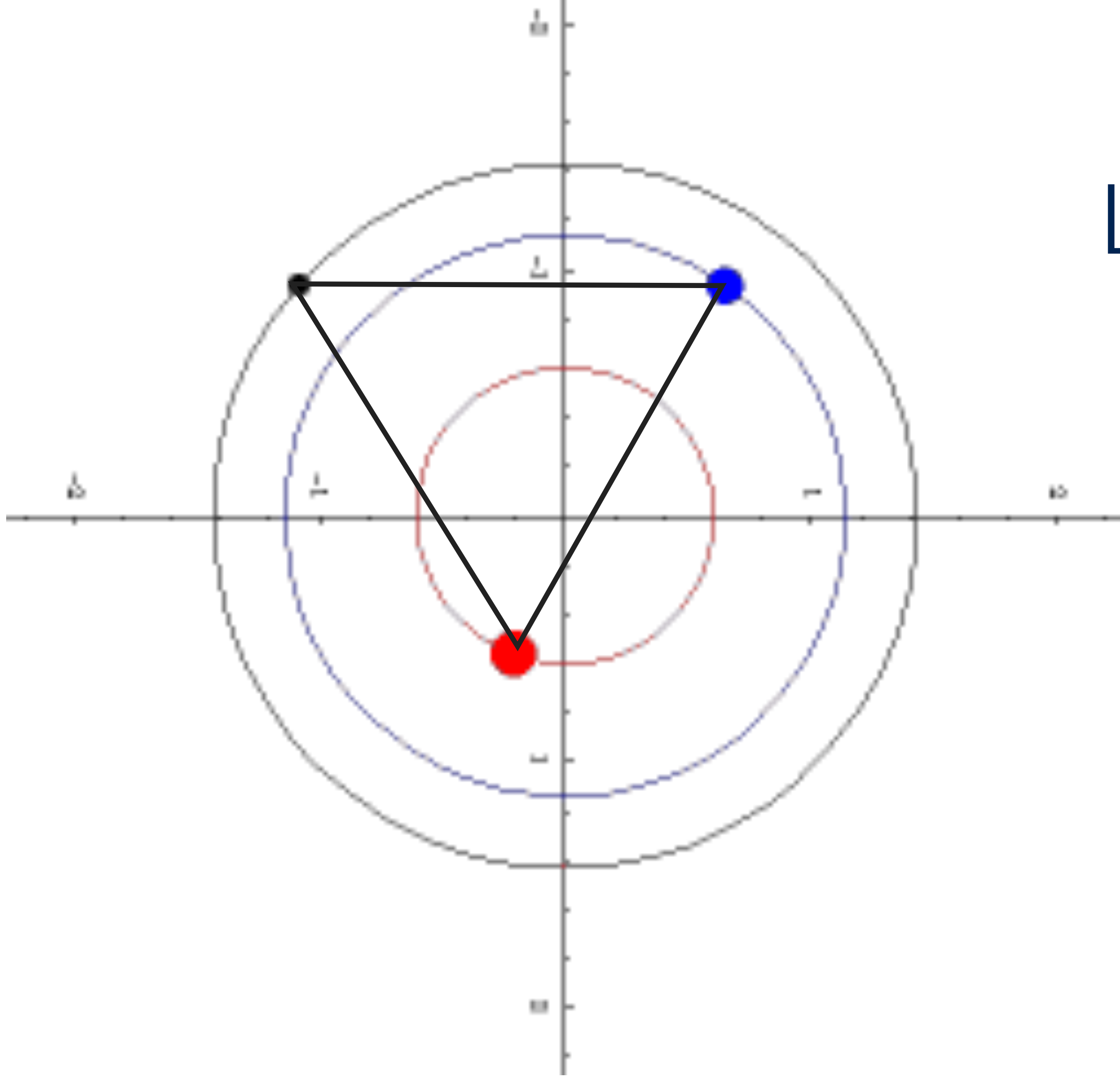
The center of mass $\dfrac{m_1 q_1 + m_2 q_2}{m_1 + m_2}$ moves in straight line

Viewed from a moving frame with this center of mass as origin,
the individual masses move on *homothetic* ellipses with origin as center

Kepler's 3 laws!

each of the two planets moves in a conic about their shared center of mass

3-body.
Lagrange's solution(s)
Equilateral triangles.

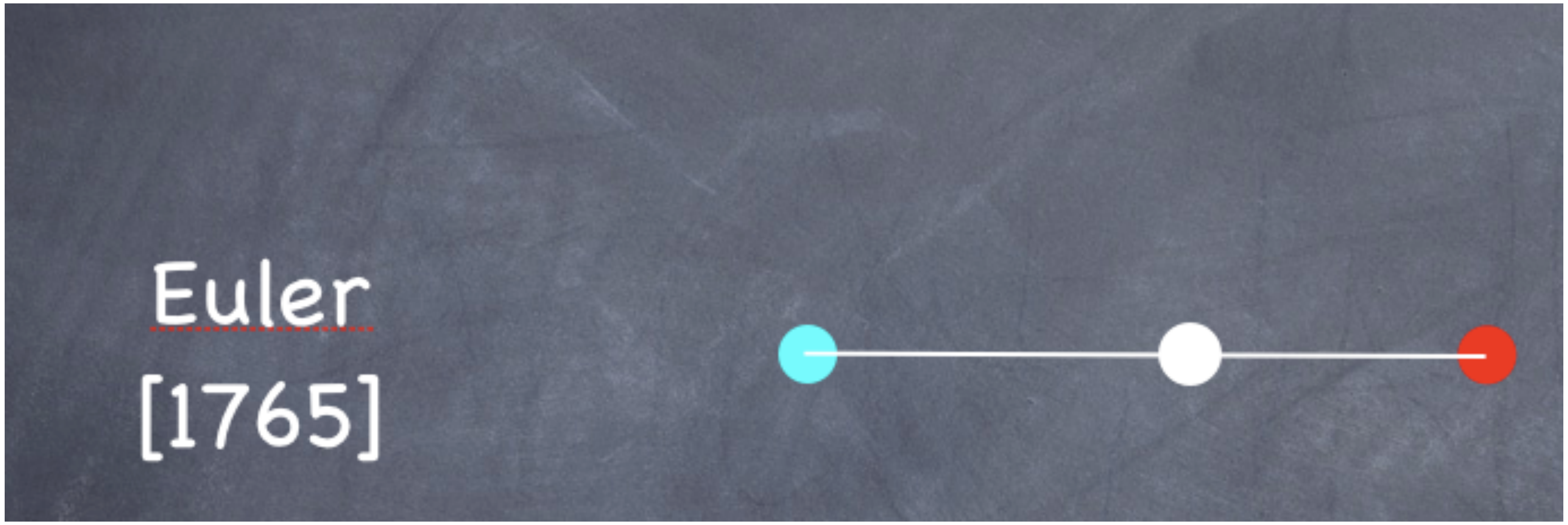


Lagrange 1772

$$m_1\ddot{q}_1 = F_{21} + F_{31}$$

$$m_2\ddot{q}_2 = F_{12} + F_{32}$$

$$m_3\ddot{q}_3 = F_{23} + F_{13}$$

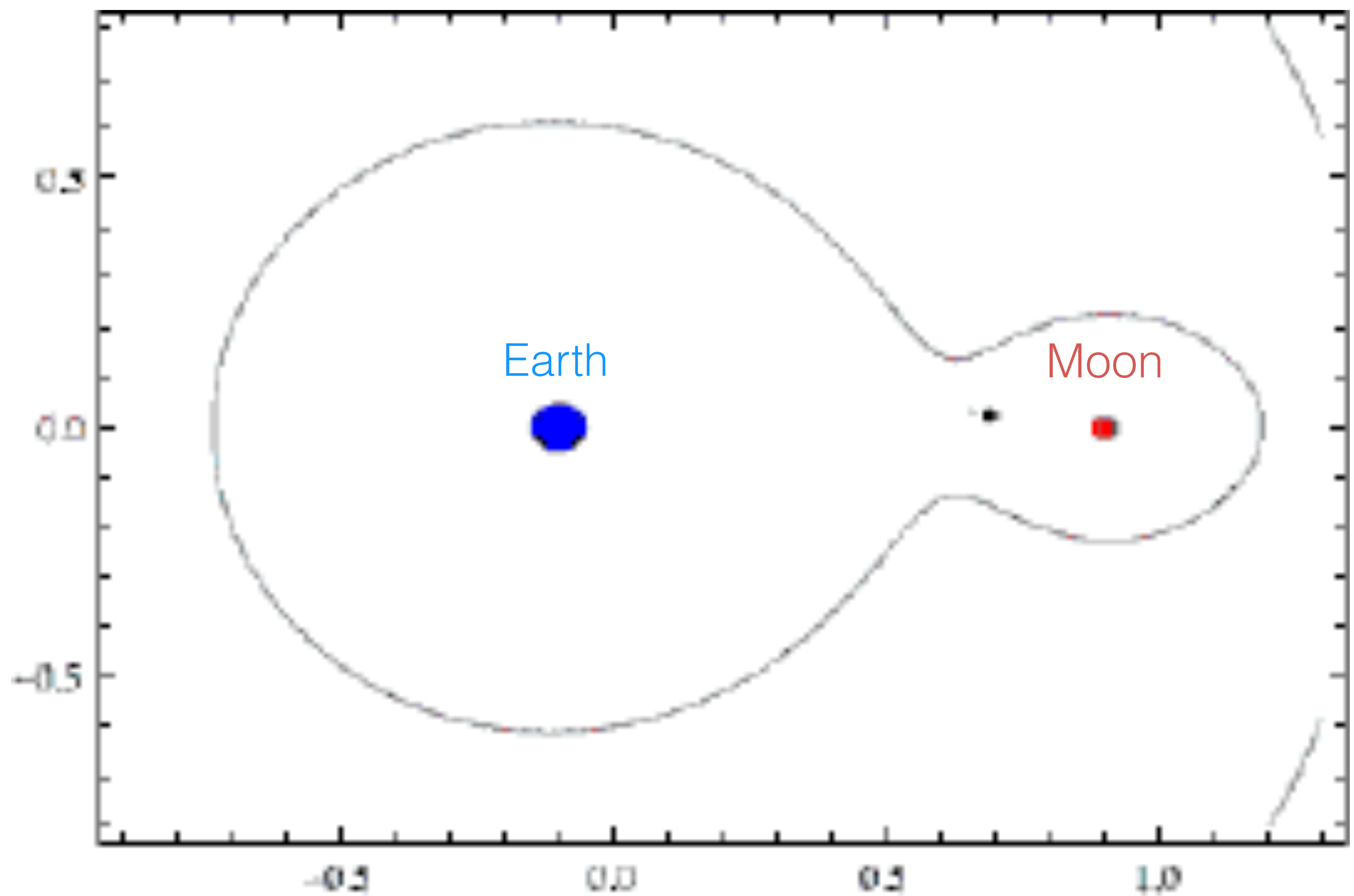


for each mass distribution
for each Kepler conic…

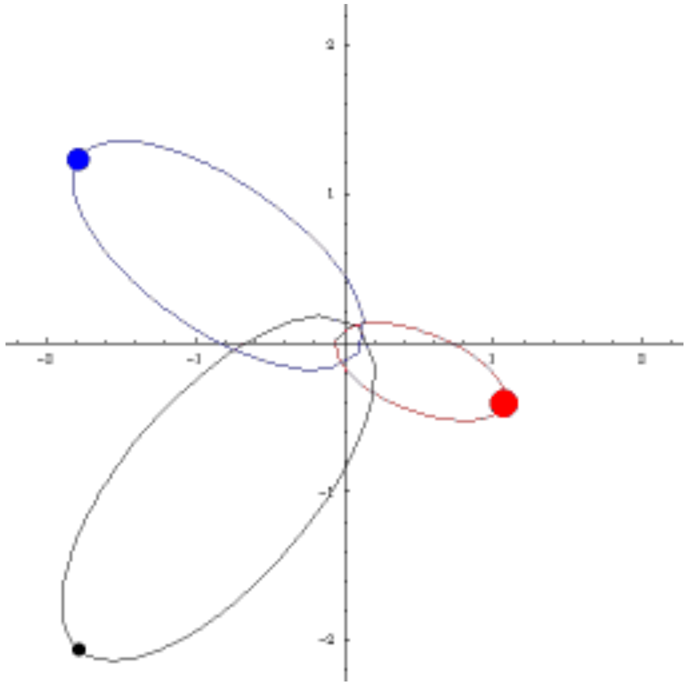
Euler 1767

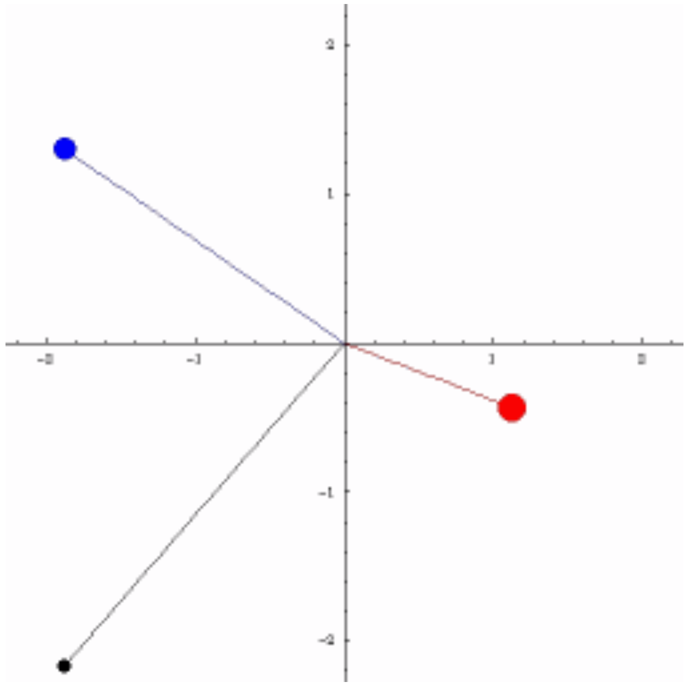
Poincare. 1892: **Chaos** in Restricted three-body problem. Shown here: a transit orbit, in a rotating frame

first example of `deterministic chaos'; discovered ``homoclinic tangles' led to Smale horseshoe



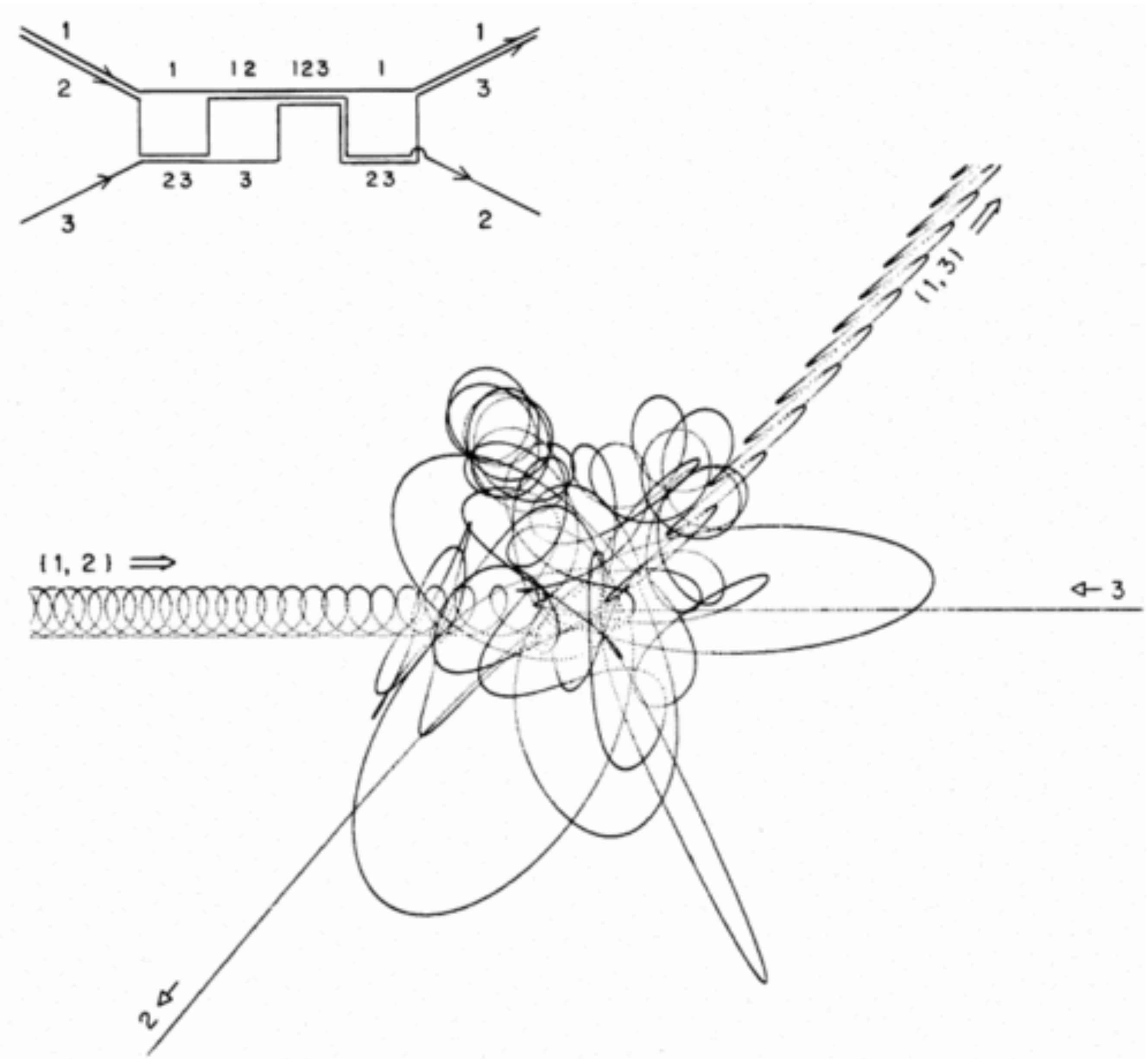
co-rotating frame, so earth, moon fixed



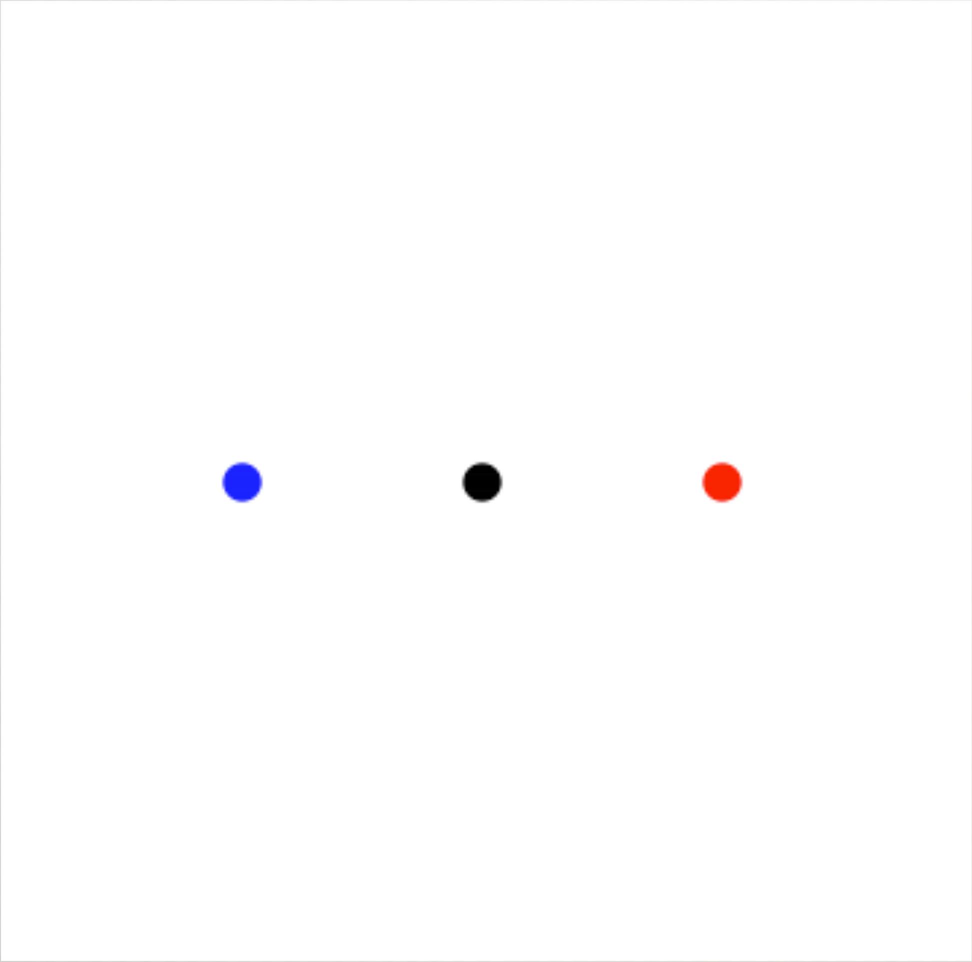


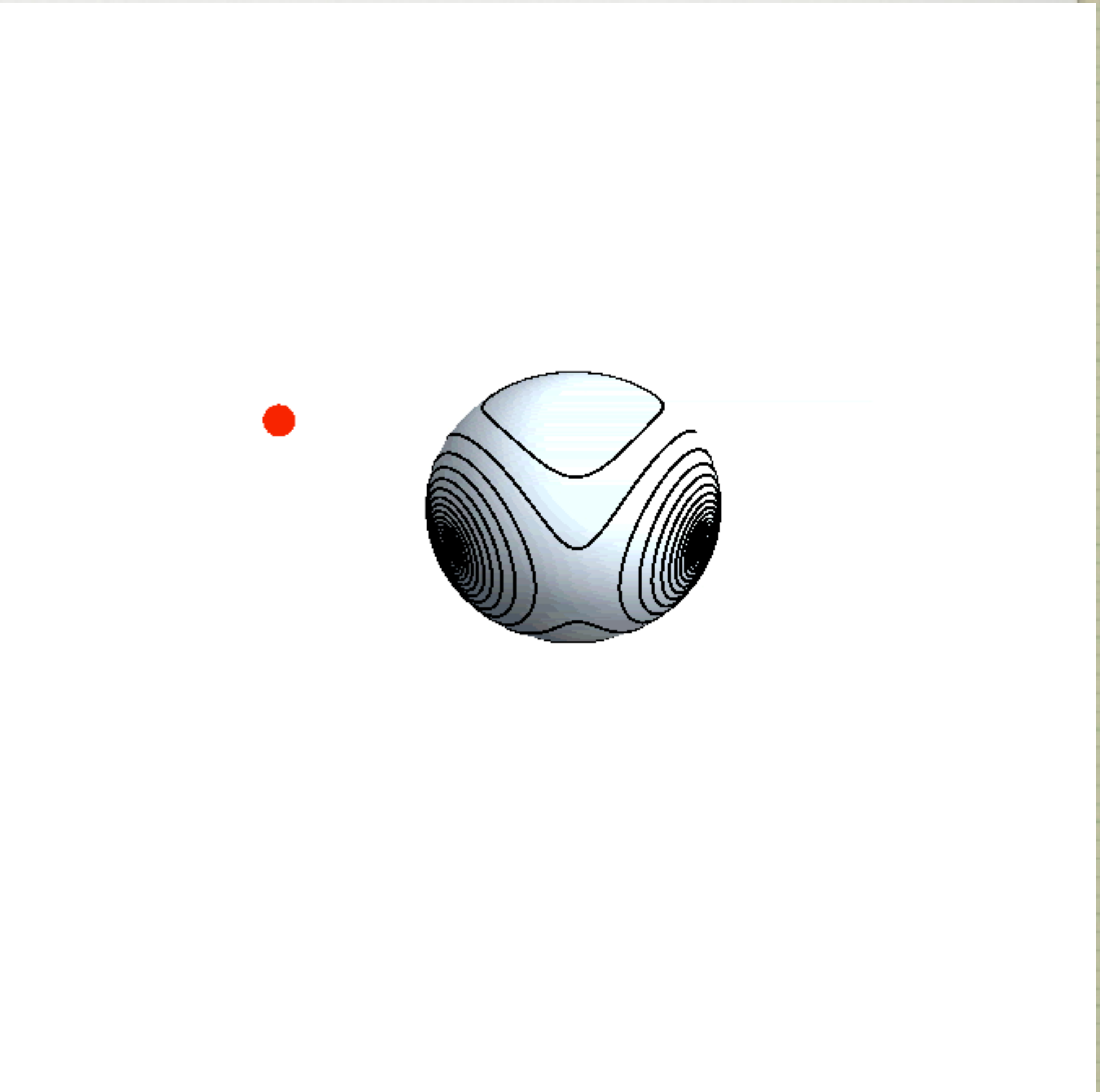
for each mass distribution
for each Kepler conic…

Hut. 1970s



Cris Moore 1994; Chenciner-Montgomery 2000





pictures, animations

thanks to:
Rick Moeckel, Gil Bor,
Carles Simo

Galileo:

1600

The laws of physics are invariant under the \`Galilean group' of isometries of space, or time, and of space-time \`boosts'

$$q_a(t) \text{ solves}$$

$$q_a(t) \text{ solves}$$

$$\iff q_a(t) + d \text{ solves}$$ (translations)

$$\iff Rq_a(t), R \in O(d), \text{ solves}$$

$$\iff q_a(t - t_0) \text{ solves}$$

$$\iff q_a(-t) \text{ solves}$$

$$\iff q_a(t) + tv \text{ solves}$$

Newton: “ $$m_a \ddot{q}_a = F_a$$

$$q_a \in \mathbb{R}^d, a = 1, \ldots, N$$ where $$m_a > 0$$

$$F_a = \Sigma_{b \neq a} F_{ba}$$

$$F_{ba} = \frac{G m_b m_a (q_b - q_a)}{r_{ba}^3}$$ where $$r_{ba} := |q_b - q_a|$$ ”

A system of dN non-linear 2nd order analytic ODEs having singularities at collisions.
It has a large Lie group of symmetries (Galileo) and associated conservation laws (energy, ang. mom, linear momentum)

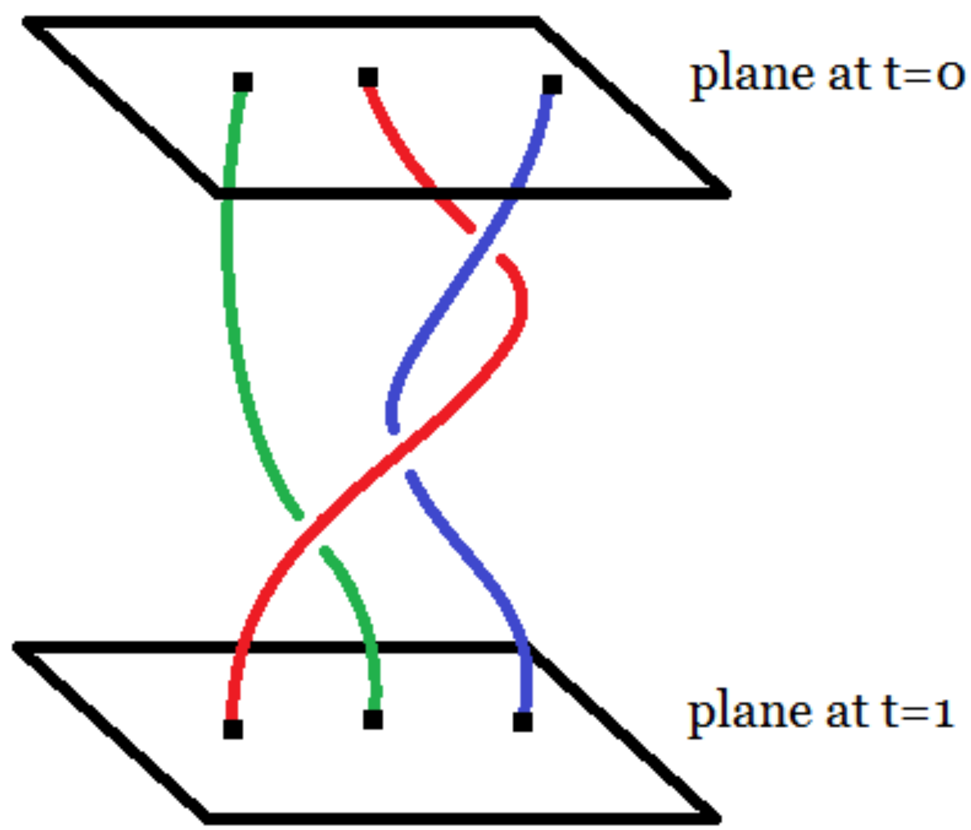
Some open questions
within the N-body problem:

1. Is the number of central configurations finite ?
2. Are there *any Lyapunov* stable periodic solutions ?

* 3. Is every braid on N strands realized ? (eight, choreos)
* 4. Is the scattering image open and dense?

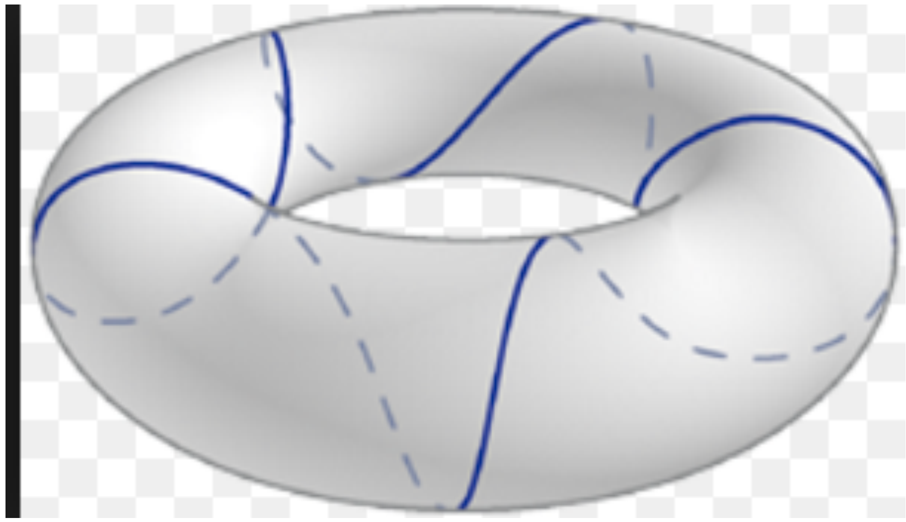
1st Question.

¿Is every braid on 3 strands realized by some periodic solution?



Inspiration:

Thm.
In a *compact* Riemannian geometry
every **free homotopy class of loops**
is realized by a periodic *geodesic*.



Pf. Direct method of the calculus of var'ns. Minimize length of loops over all loops which represent the given class

3-body. A conjugacy class in the pure braid group on 3 strands
=
a free homotopy class of loops in the ***collision-free*** planar 3-body config . space

3-body problem:
a free homotopy class of loops for the collision-free planar 3-body config . space
=
a conjugacy class in the pure braid group on 3 strands

Answer: `Yes',
if I cheat.

THE N-BODY PROBLEM, THE BRAID GROUP,

AND ACTION-MINIMIZING PERIODIC

SOLUTIONS.

1998, -R.Mont.

and if I don’t cheat ??

Annals of Mathematics, **152** (2000), 881–901

A remarkable periodic solution of the three-body problem in the case of equal masses

By Alain Chenciner and Richard Montgomery

Dedicated to Don Saari for his (censored) birthday

Abstract

Using a variational method, we exhibit a surprisingly simple periodic orbit for the newtonian problem of three equal masses in the plane. The orbit has zero angular momentum and a very rich symmetry pattern. Its most surprising feature is that the three bodies chase each other around a fixed eight-shaped curve. Setting aside collinear motions, the only other known motion along a fixed curve in the inertial plane is the “Lagrange relative equilibrium” in which the three bodies form a rigid equilateral triangle which rotates at constant angular velocity within its circumscribing circle. Our orbit visits in turns every “Euler configuration” in which one of the bodies sits at the midpoint of the segment defined by the other two (Figure 1). Numerical computations

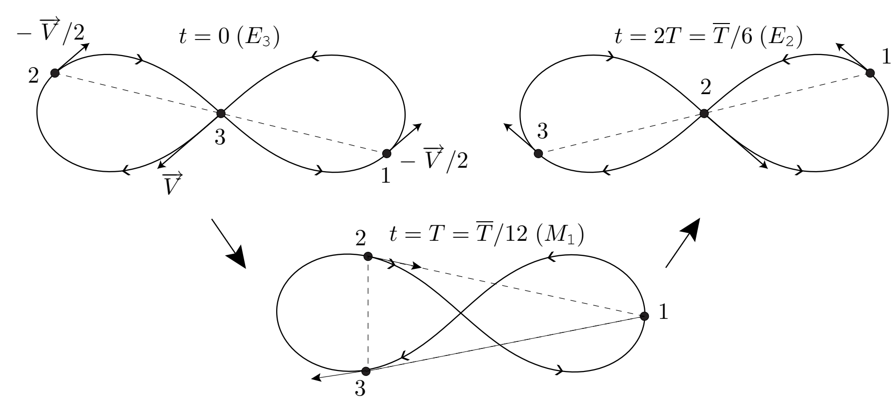


Figure 1 (Initial conditions computed by Carles Simó)

$x_1=-x_2=0.97000436-0.24308753i, x_3=0$; $\vec{V}=\dot{x}_3=-2\dot{x}_1=-2\dot{x}_2=-0.93240737-0.86473146i$

$\overline{T}=12T=6.32591398$, $I(0)=2$, $m_1=m_2=m_3=1$

2000

But… (priorities !)

PHYSICAL REVIEW
LETTERS

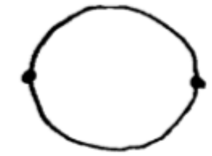
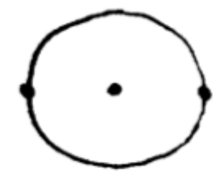
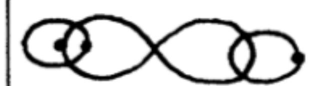
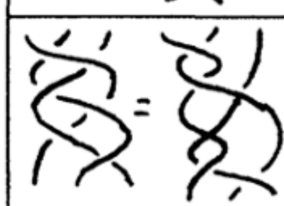
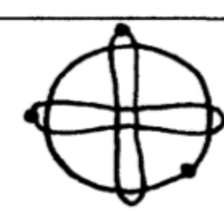
OLUME 70 14 JUNE 1993

Braids in Classical Dynamics

Cristopher Moore
Santa Fe Institute, Santa Fe, New Mexico 87501
(Received 19 October 1992)

Point masses moving in 2+1 dimensions draw out braids in space-time. If they move under influence of some pairwise potential, what braid types are possible? By starting with fictional path the desired topology and "relaxing" them by minimizing the action, we explore the braid types of po tials of the form $V \propto r^a$ from $a \leq -2$, where all braid types occur, to $a = 2$, where the system is int able. We also discuss issues of symmetry and stability. We propose this kind of topological classifica as a tool for extending the "symbolic dynamics" approach to many-body dynamics.

1993

braid	b_i	orbit	existence
	b_1^2		exists for all α
	—		exists for all α
	$b_1^2 b_2^2$		$\alpha < -1.1 \pm 0.05$
	$b_1^2 b_2^{-2}$		$\alpha < -1.4 \pm 0.05$
	$(b_1 b_2)^3$		exists for all α
	$(b_1 b_2^{-1})^3$		$\alpha < 2$
	$(b_1^2 b_2)^2$		$\alpha < -1.0 \pm 0.05$
	$(b_1^2 b_2^{-1})^2$		$\alpha < -1.7 \pm 0.05$
	$b_1 b_2 b_1^{-1} b_2 b_1 b_2^{-1}$ $= b_1^2 b_2 b_1^{-2} b_2$		at least $\alpha \leq 2$

and even earlier:

SUR LES SOLUTIONS PÉRIODIQUES ET LE PRINCIPE DE MOINDRE ACTION

Comptes rendus de l'Académie des Sciences, t. 123, p. 915-918 (30 novembre 1896).

La théorie des solutions périodiques peut, dans certains cas, se rattacher au principe de moindre action.

Supposons trois corps se mouvant dans un plan et s'attirant en raison inverse du cube des distances ou d'une puissance plus élevée de ces distances; j'appelle a, b, c ces trois corps

1896

Methods:

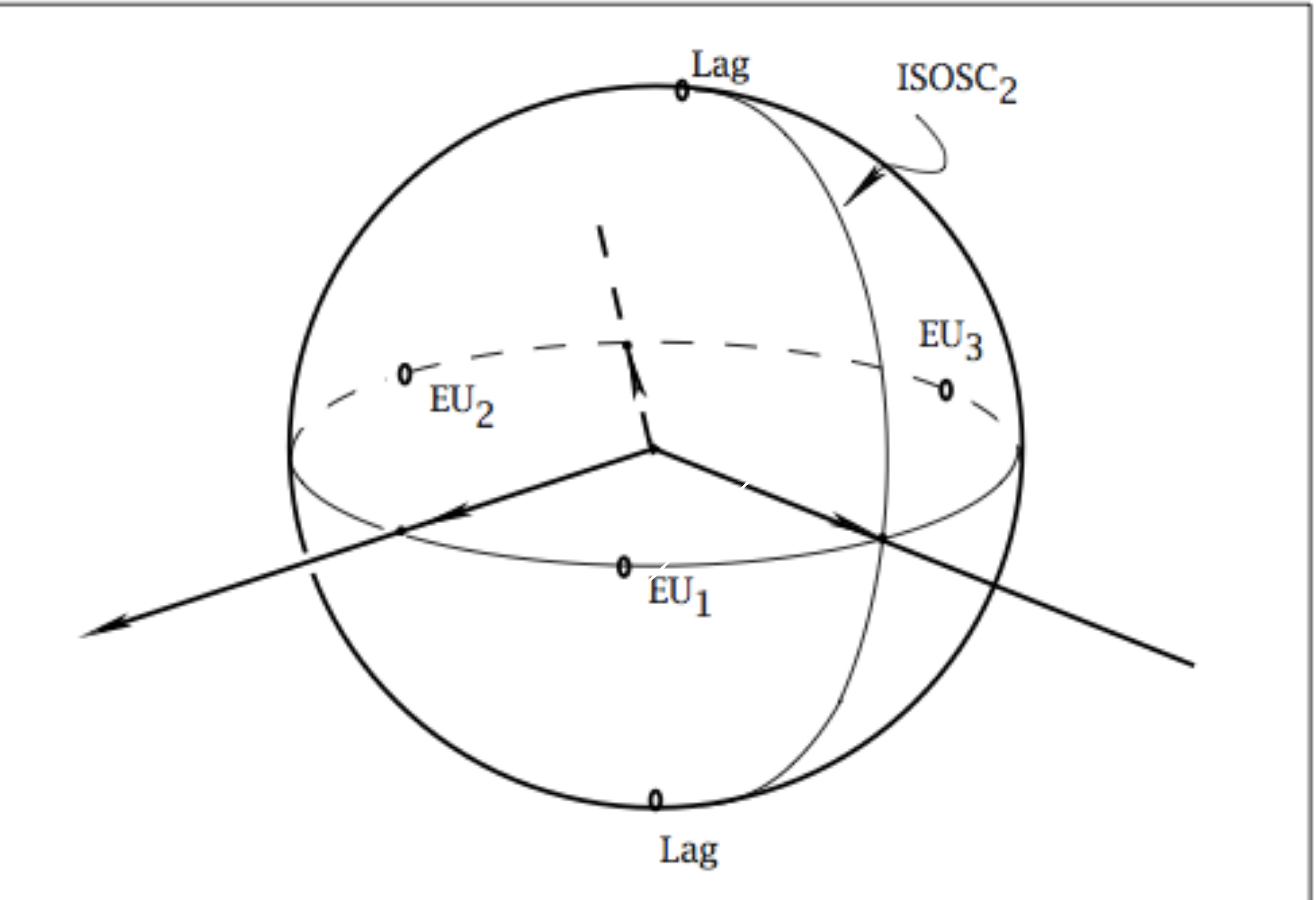


Figure 4. The shape sphere.

Eclipse sequences.

..Every free homotopy class…

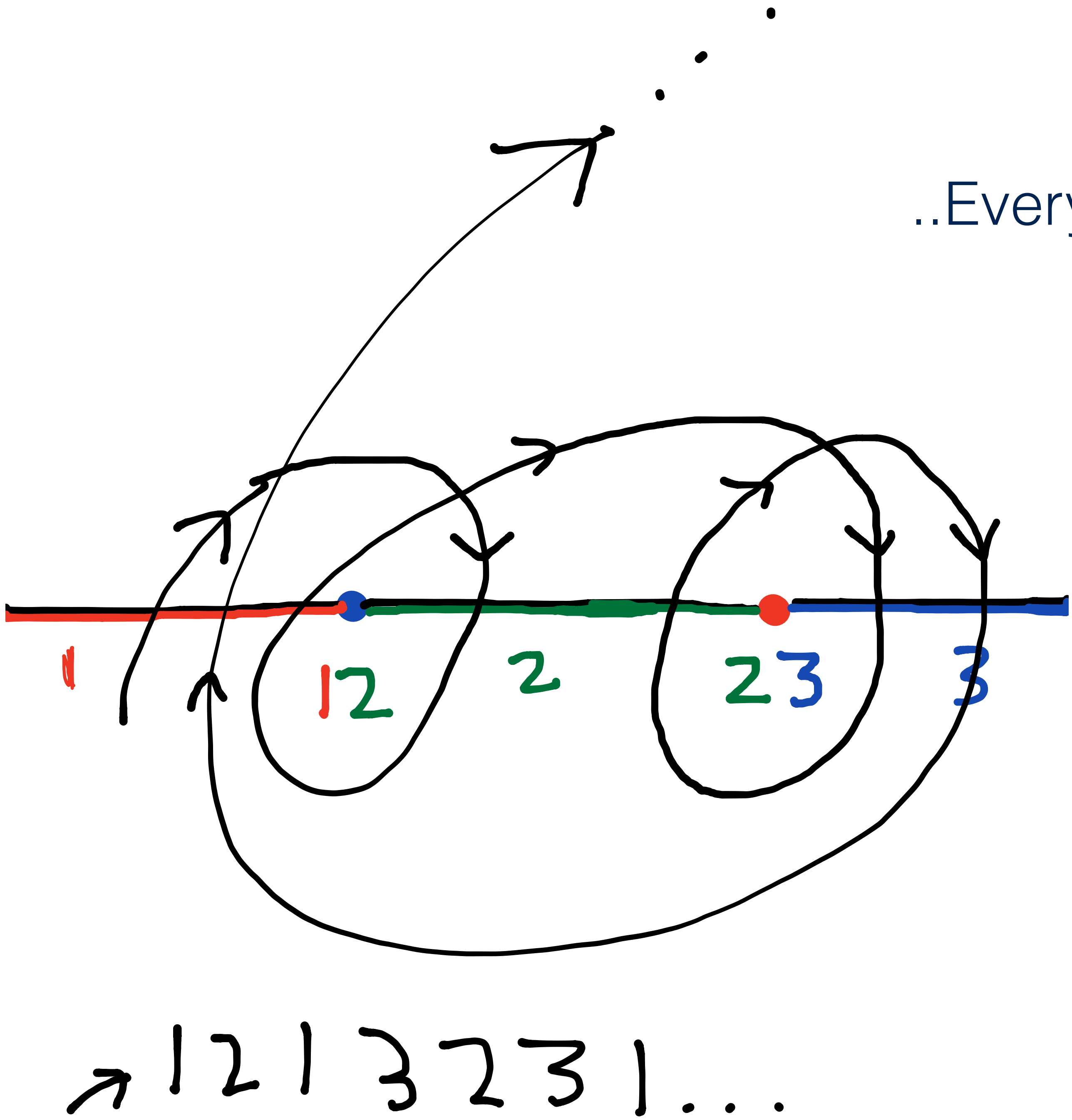


Figure 8: 123123

N= 3, the eclipse sequence of a periodic collision-free curve represents its free homotopy class (braid) *mod* rotation.

The eclipse sequence of the eight is `123123'

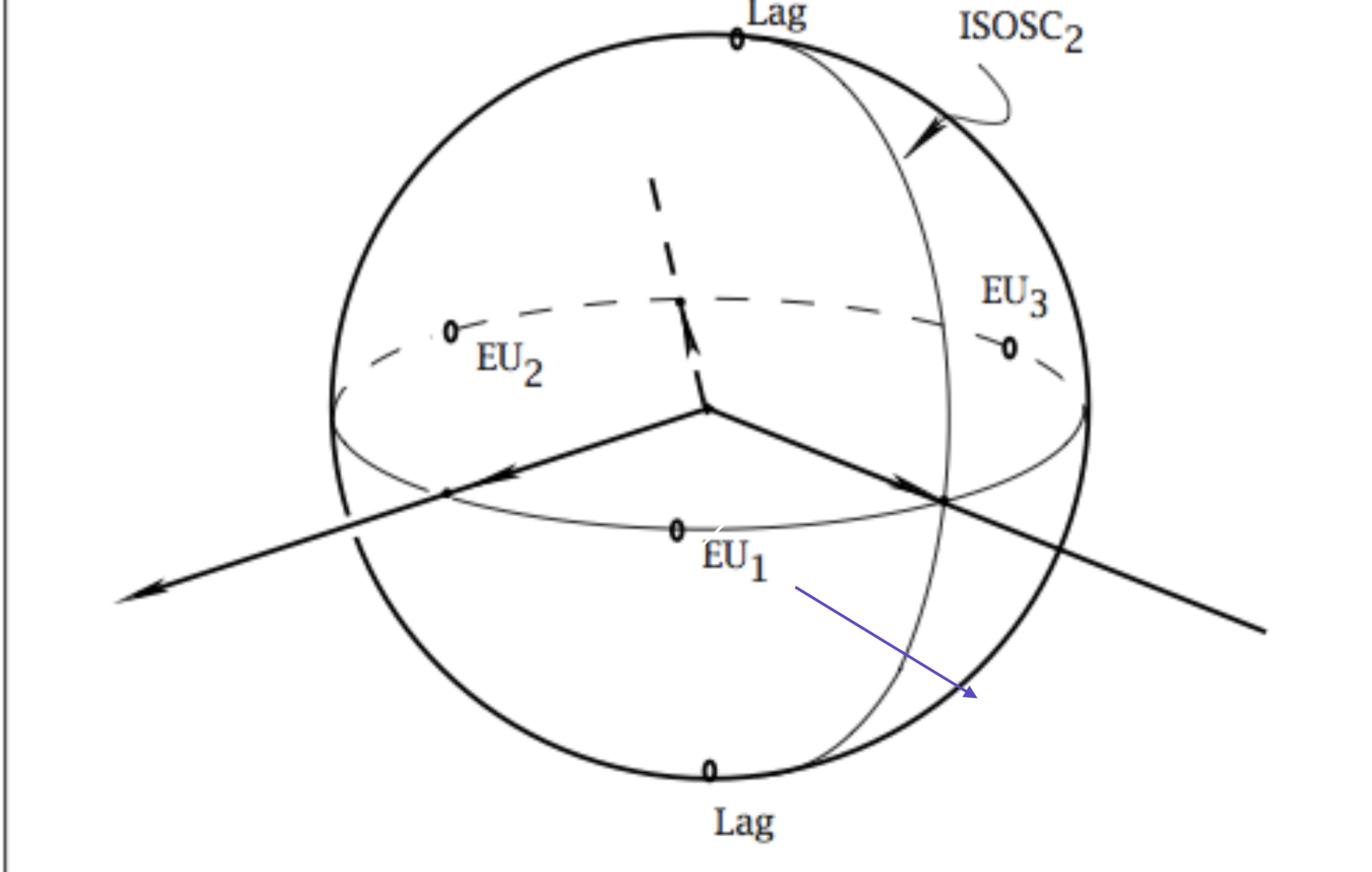
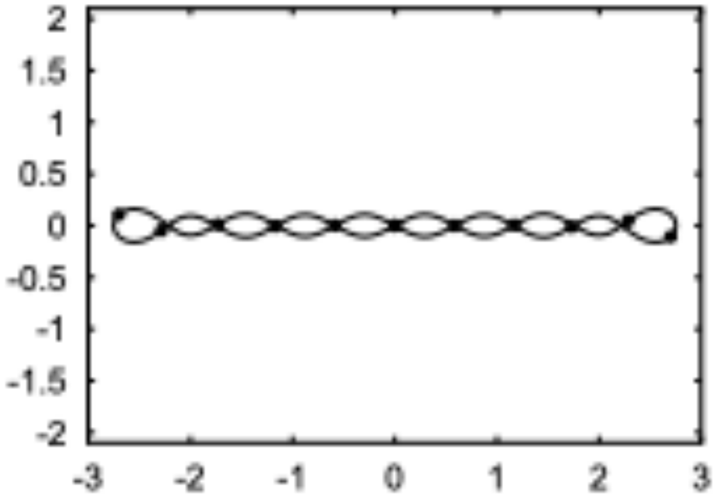


Figure 4. The shape sphere.

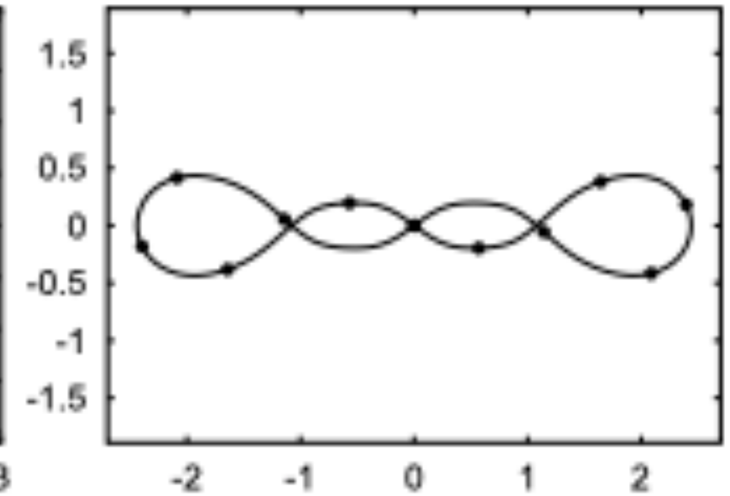
Minimize the Classical action among all paths joining EU_1 to ISOSC_2

Technical heart: existence of a **collision-free** minimizer

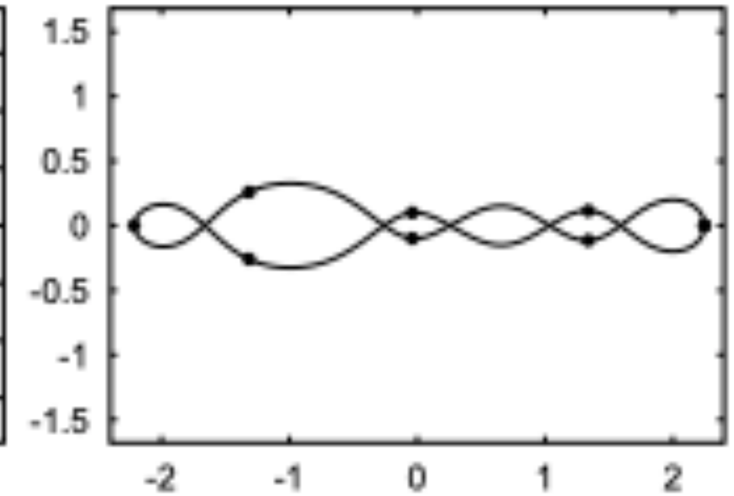
Choreographies, etc. : a 15 year detour from question



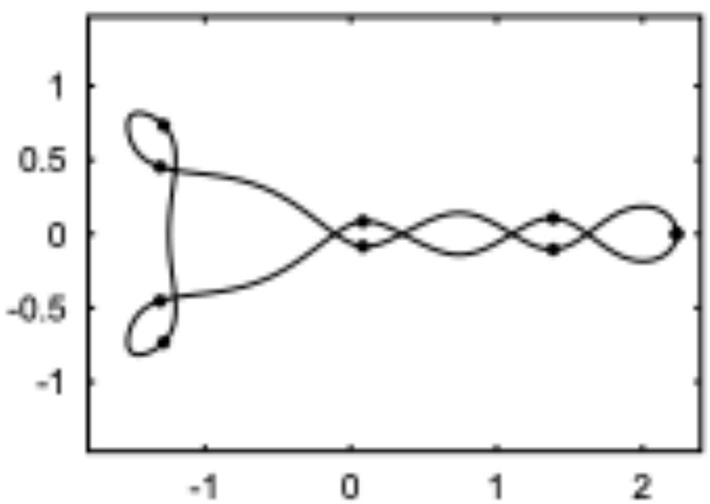
a) Chain with 11 bodies in 10 loops.



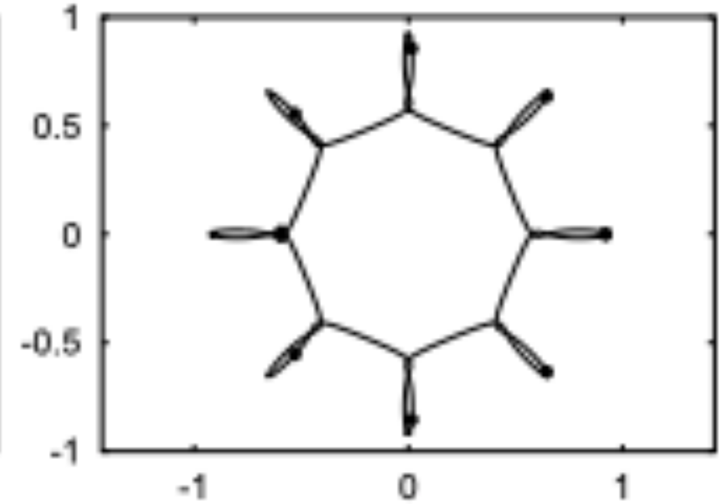
b) Chain with 11 bodies in 4 loops.



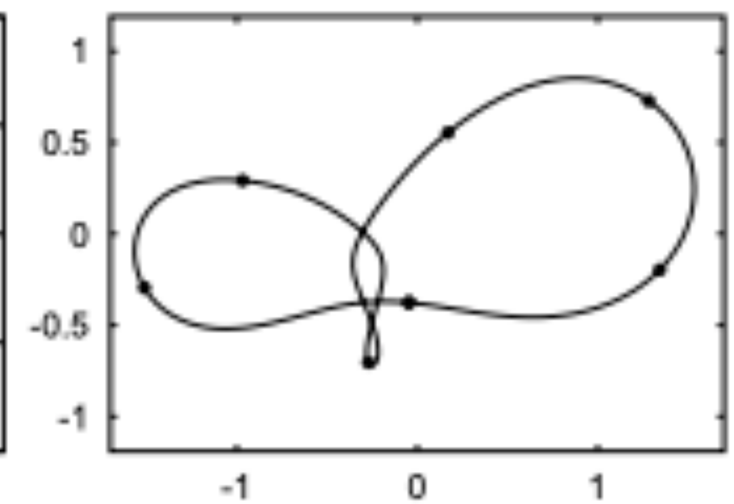
c) Chain with 8 bodies in 6 loops.



d) Bifurcated chain with 9 bodies

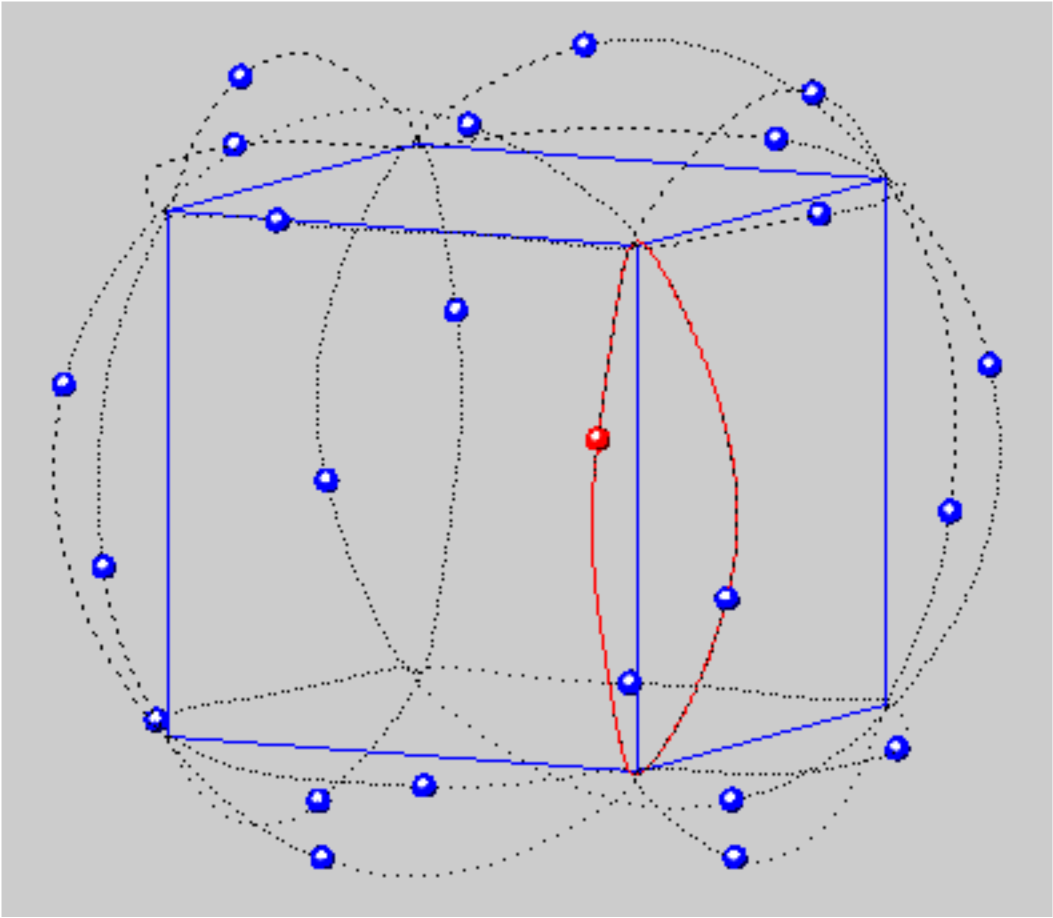


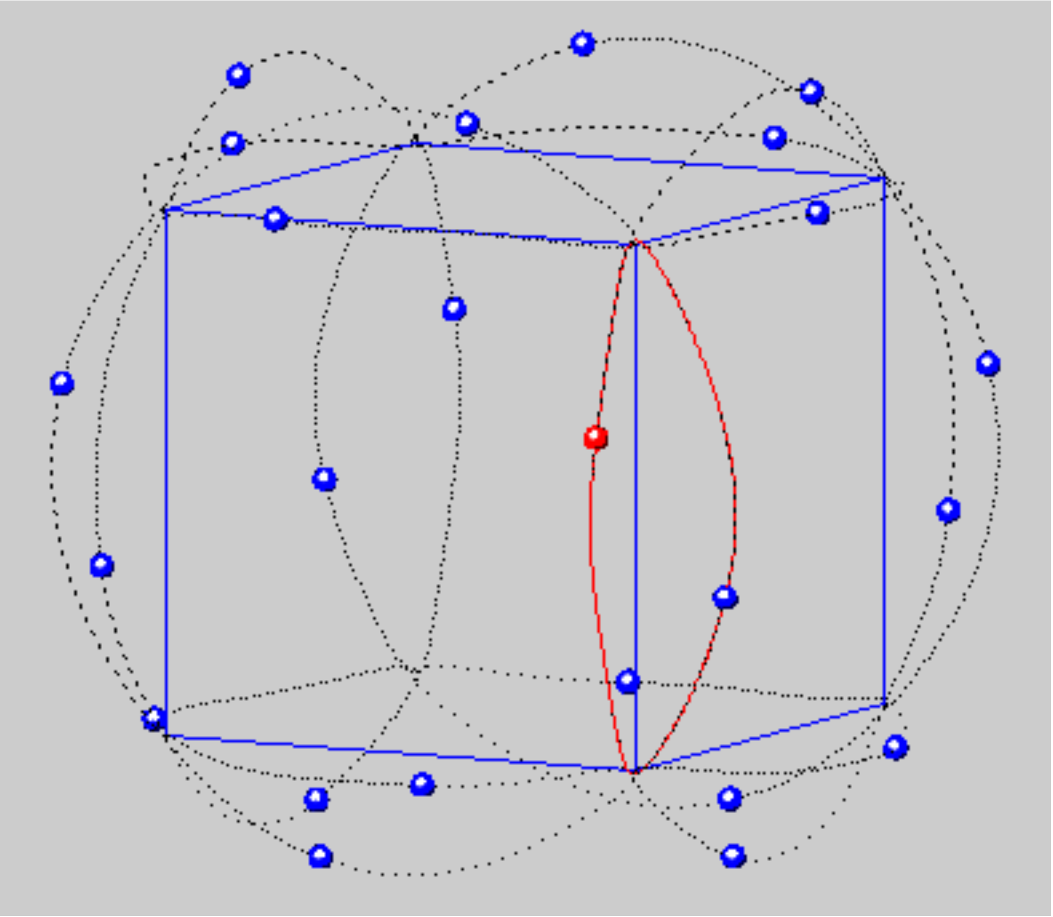
e) 8 petal daisy with 9 bodies.



f) An asymmetric case with 7 bodies.

N=24, d= 3:





and about 100 more for d=3; roughly six per Platonic solid

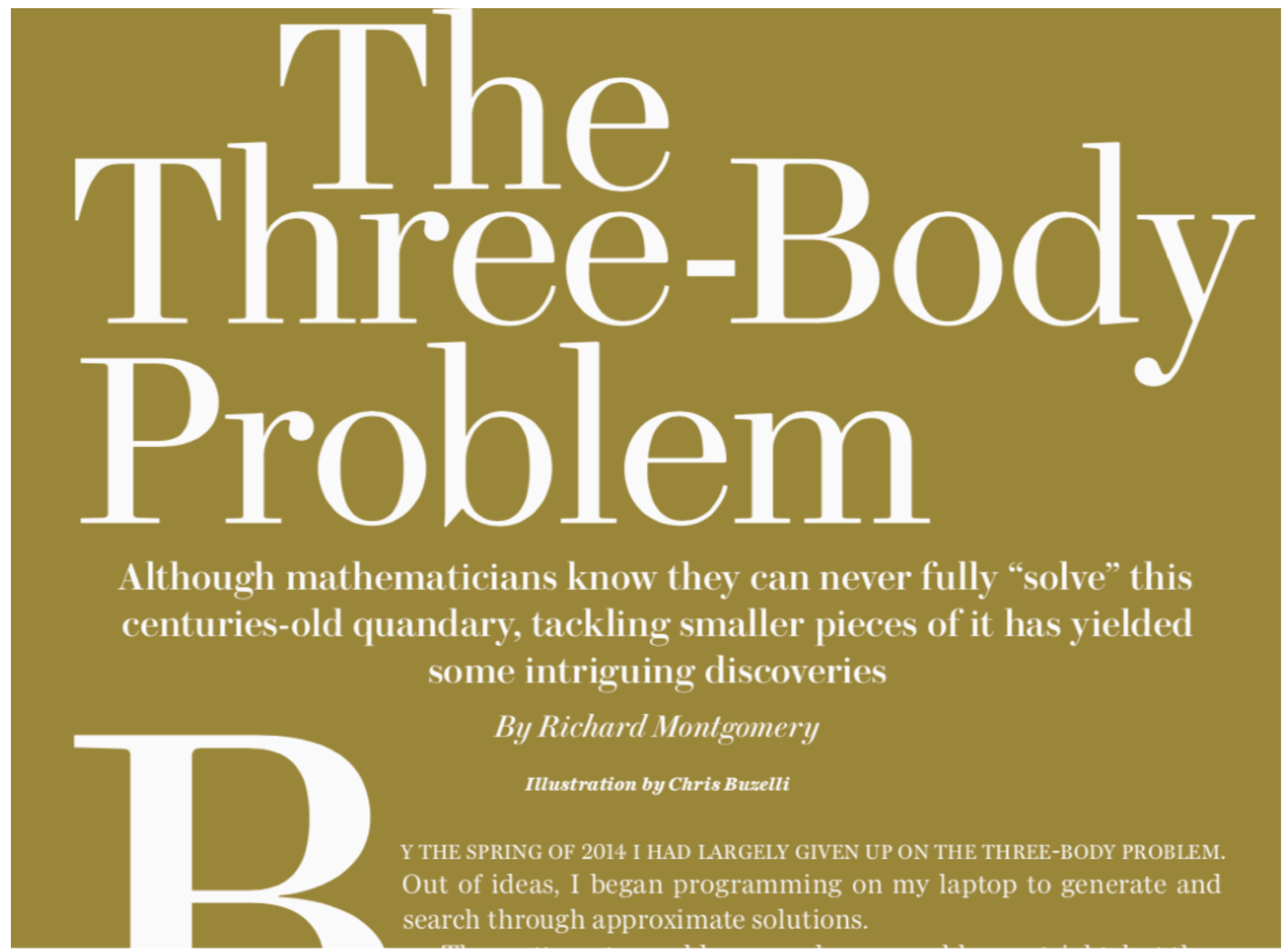
Fusco,Gronchi, Negrini.
Platonic polyhedra, topological constraints and periodic orbits of the classical N-body problem' ,
Invent. Math., Vol.285/2, 283-332. 2011

The original question was specific to **zero angular momentum** and *for realizing all eclipse* sequences. **It is still open.**

I worked on it off and on for 17 years before a 2014 theorem with Rick Moeckel, valid for angular momentum epsilon that I will describe

Scientific American,

Aug 2019


The Three-Body Problem

Although mathematicians know they can never fully "solve" this centuries-old quandary, tackling smaller pieces of it has yielded some intriguing discoveries

By Richard Montgomery

Illustration by Chris Buzelli

BY THE SPRING OF 2014 I HAD LARGELY GIVEN UP ON THE THREE-BODY PROBLEM. Out of ideas, I began programming on my laptop to generate and search through approximate solutions.

cancelling stutters.

Notation: $1^5 = 11111$ etc

thus $1^5 2^4 3^3 = 13$

periodic **reduced** eclipse sequences.

<—> free homotopy classes of loops (mod rotation) .

Taking $a_i \in \{1, 2, 3\}$

Thm: [RM & RM 2014] There exists an M such that for all sequences of integers

$$n_i \geq M$$

every infinite eclipse sequence of the form

$$\ldots a_1^{n_1} a_2^{n_2} a_3^{n_3} \ldots$$

is realized by a sol'n having this sequence. If the eclipse sequence is periodic, then so is the solution.

We need masses equal or close to equal.
Need **some** angular momentum.

Cor.: [Moeckel & M-] every free homotopy class mod rotations is realized for the planar 3 -body problem !

Pf: take all the $n_i \geq M$ to be odd. Then

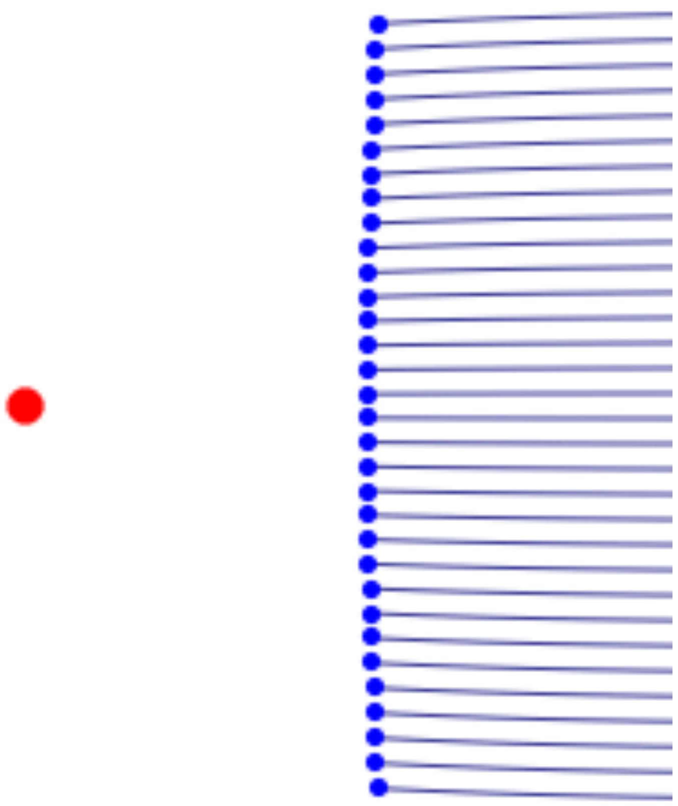
$$\ldots a_1^{n_1} a_2^{n_2} a_3^{n_3} \ldots = \ldots a_1 a_2 a_3 \ldots \quad \text{(homotopically)}$$

Caveats: Need masses equal or `close to equal'.
Need some angular momentum.

2nd Question:
¿Is the scattering image open and dense?

Rutherford scattering, 1917:

essential to his discovery that nuclei were very tiny and dense



3-body scattering?

is **anisotropic :** different **directions** of the incoming ``beam'' lead to different outgoing scattering maps

2-body scattering is **isotropic:** scattering map the same regardless of direction of incoming beam

What is a `direction' for a beam/solution q(t) to the positive energy N-body problem ?

Answer: $$\lim_{t \to -\infty} \frac{q(t)}{t} = \lim_{t \to -\infty} \dot{q}(t) := a$$

$$a \in (\mathbb{R}^d)^N \setminus \text{ collisions}$$

The energy E of such a solution is positive and equal to E = $\frac{1}{2}\|a\|^2$

where

$$\|a\|^2 := \Sigma_i m_i |a_i|^2$$

The space of all such solutions for fixed **a** sweeps out a Lagrangian submanifold lying in the energy E hypersurface of phase space.

Call it $\mathcal{L}_{\mathbf{a}}^{-}$

Similarly:

$$\lim_{t\to+\infty}\frac{q(t)}{t} = \lim_{t\to+\infty}\dot{q}(t) := b$$

and `forward' Lagrange submanifold

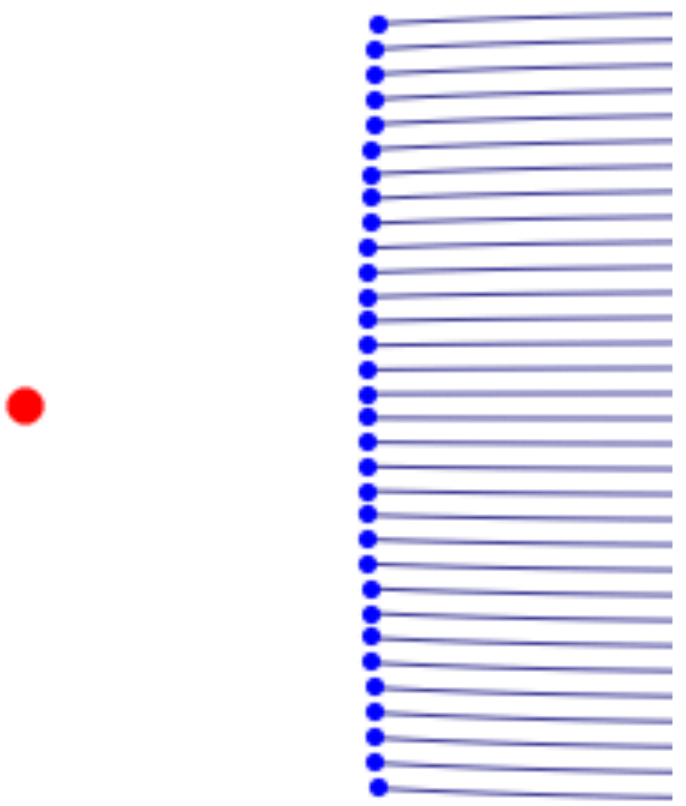
$$\mathcal{L}_{\mathbf{b}}^{+}$$

Question: if $\|a\| = \|b\|$ (energies equal)

and $a \neq \pm b$

does $\mathcal{L}_{\mathbf{a}}^{-} \bigcap \mathcal{L}_{\mathbf{b}}^{+} \neq \emptyset$ **?**

True for N = 2, i. e ``Rutherford scattering''



Original Question: Fix **a** in the N-body configuration space. Is it true that for an open and dense set of **b** lying in the sphere of radius $\|a\|$ we have that

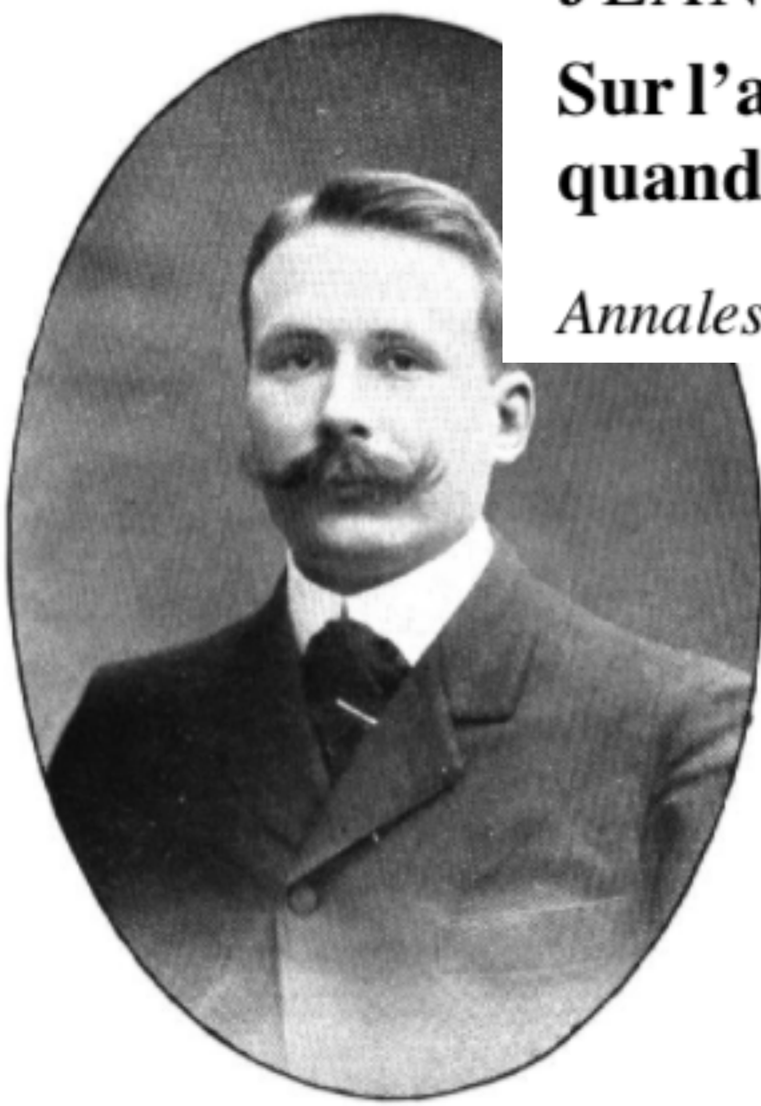
$$\mathcal{L}^{-}_{\mathbf{a}} \bigcap \mathcal{L}^{+}_{\mathbf{b}} \neq \emptyset \quad \textbf{?}$$

Inspiration

JEAN CHAZY

Sur l'allure du mouvement dans le problème des trois corps quand le temps croît indéfiniment

Annales scientifiques de l'É.N.S. 3e série, tome 39 (1922), p. 29-130.

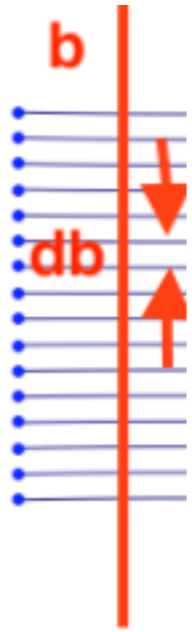


JEAN CHAZY

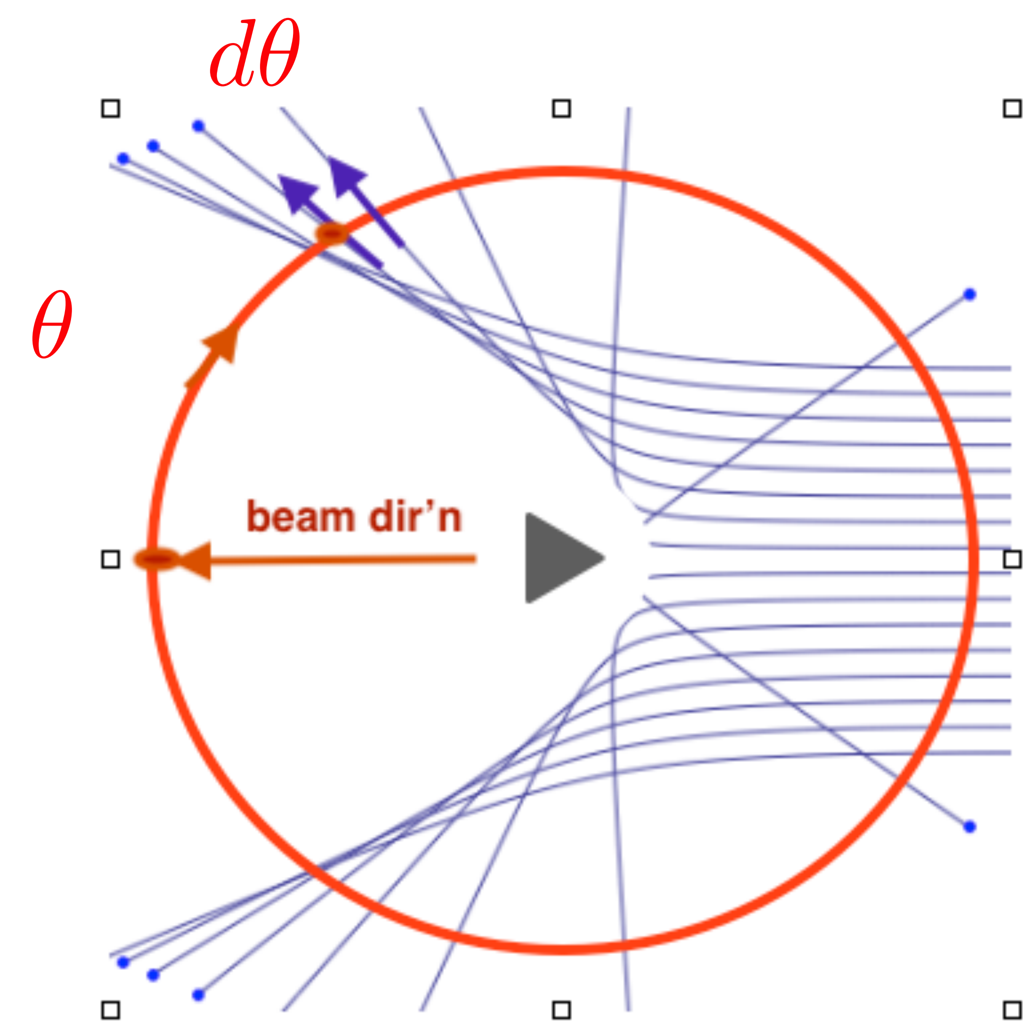
$$q(t) = at - F(a)\log(t) + c + o(1)$$

asymptotic shape,
(or velocity)

impact
parameter



$$b \in \mathbb{R} \mapsto \theta \in S^1$$



b = Chazy'ys c
= impact parameter

Call this map the
``scattering map"

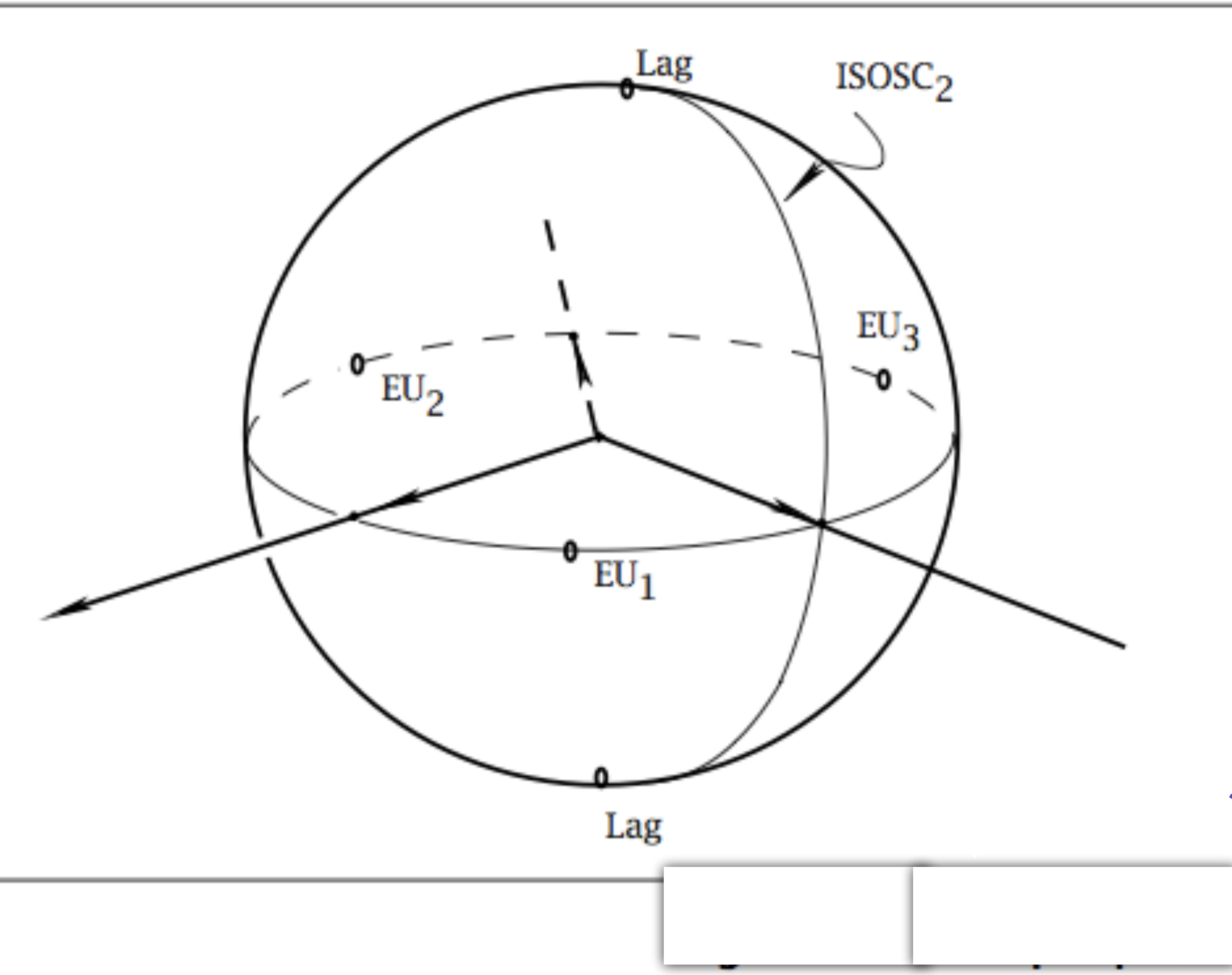
denote it by : $\pi : \mathbb{R} \to S^1$

$$(\quad \pi(b) = 2Arctan(\frac{Z}{2Eb})$$

$$\pi(\pm\infty) = 0 \qquad)$$

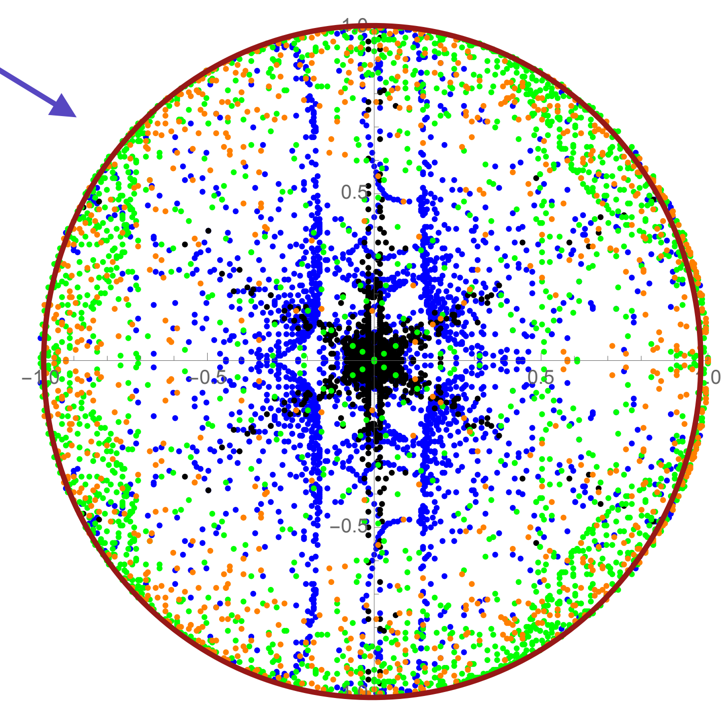
Numerical Experiments
(Rick Moeckel, the other `RM')

4. Is the scattering image open and dense?



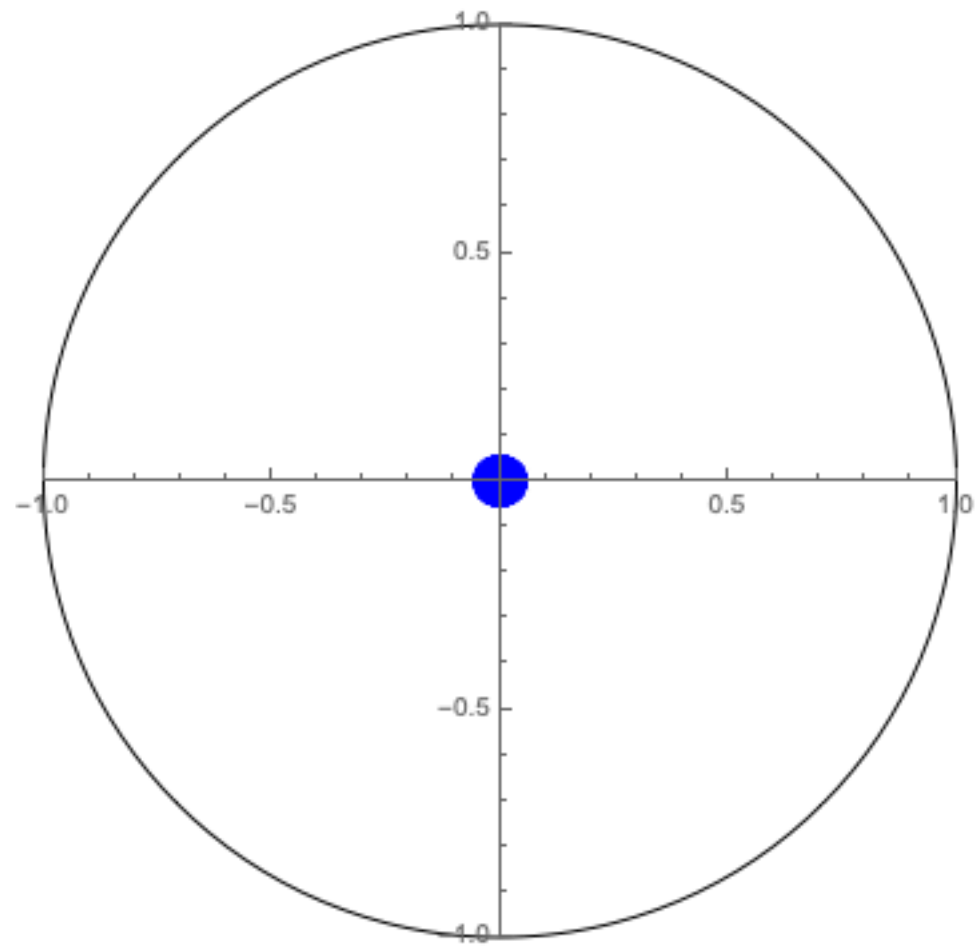
The **shape sphere**: the quotient of 3-body configuration space (minus triple collision) by the group of (orient. pres. isometries) x (scaling)

mod reflection



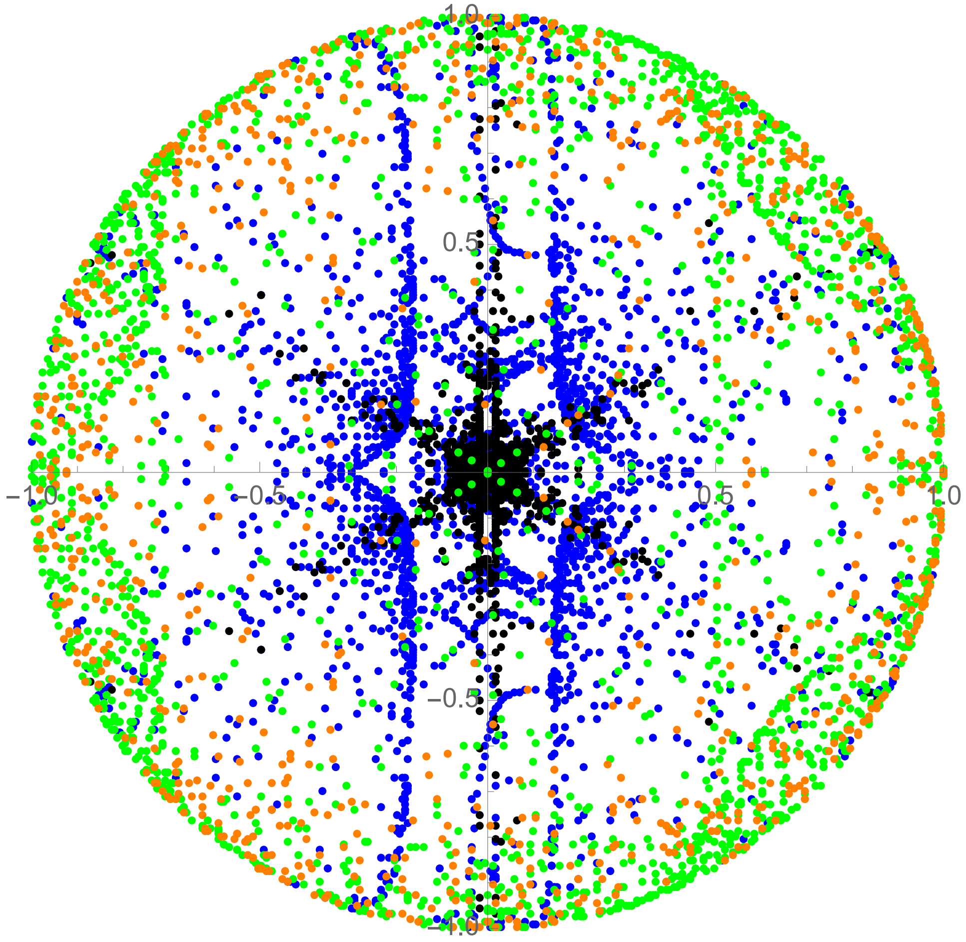
shape disc: quotient of 3-body configuration space (minus triple collision) by (all isometries) x (scaling). Equals shape sphere modulo reflection about the collinear equator.

4. *Is the scattering image open and dense?*

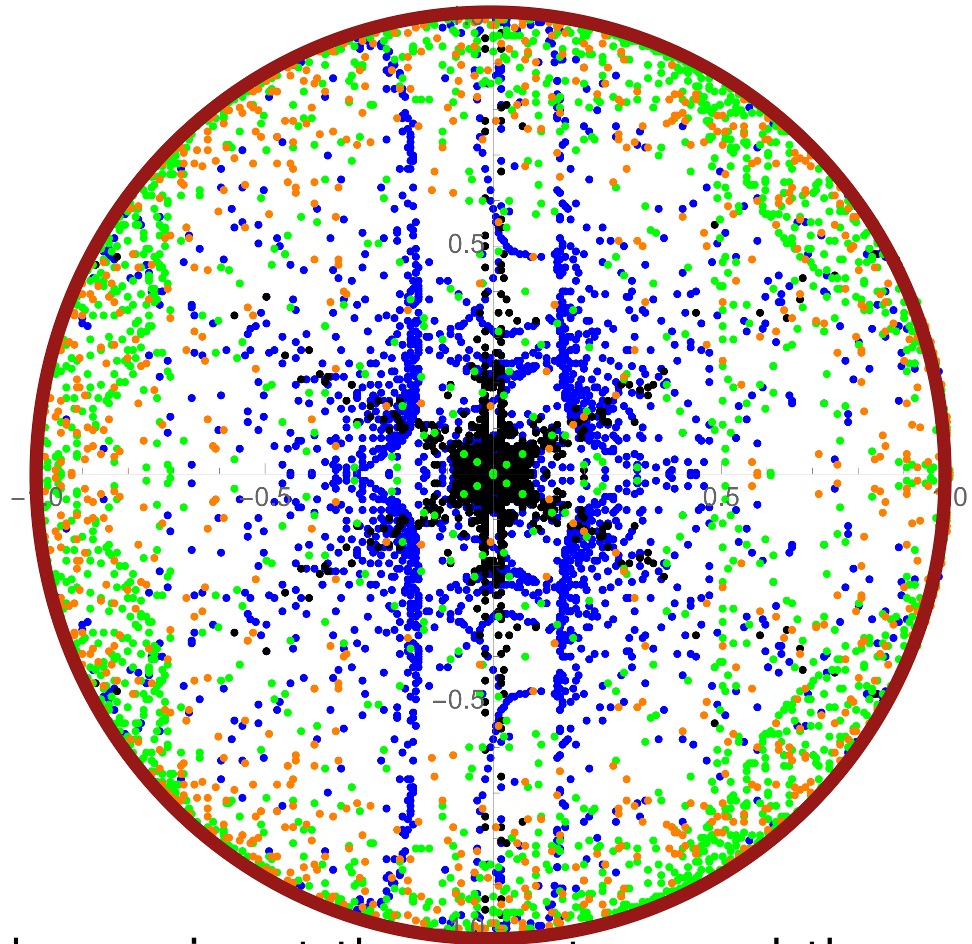


4. Is the scattering image open and dense?

A picture Rick Moeckel made of the **image** of the scattering map for an incoming equilateral triangle (Lagrange) beam projected onto the **shape disc**



colors indicates how close the trajectories stays to infinity

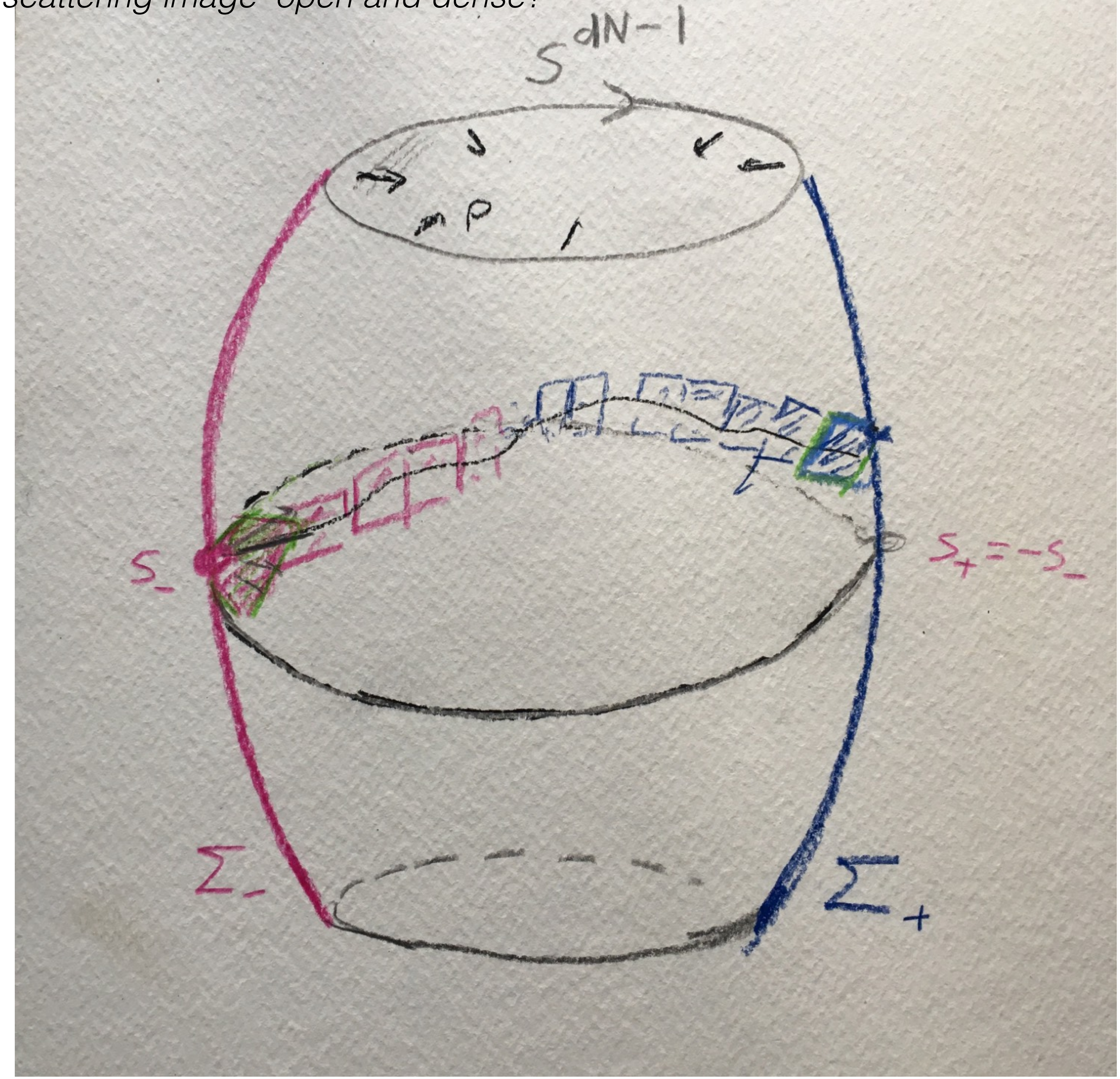


``The equilateral shape is at the center and the collinear shapes are at the outer edge. The isosceles shapes form three diameters of the disk. The collision shapes are at the third roots of unity on the diameter.

The **unstable manifold is a 3D disk whose boundary is a 2 sphere** in the infinity manifold. The points to follow are chosen from other 2D spheres in this disk. **Black points are near the infinity manifold** and blue, green orange farther from infinity. Very crude experiment so far, but encouraging. How to prove ? ''

-email, Rick Moeckel , .. 2020 (?)

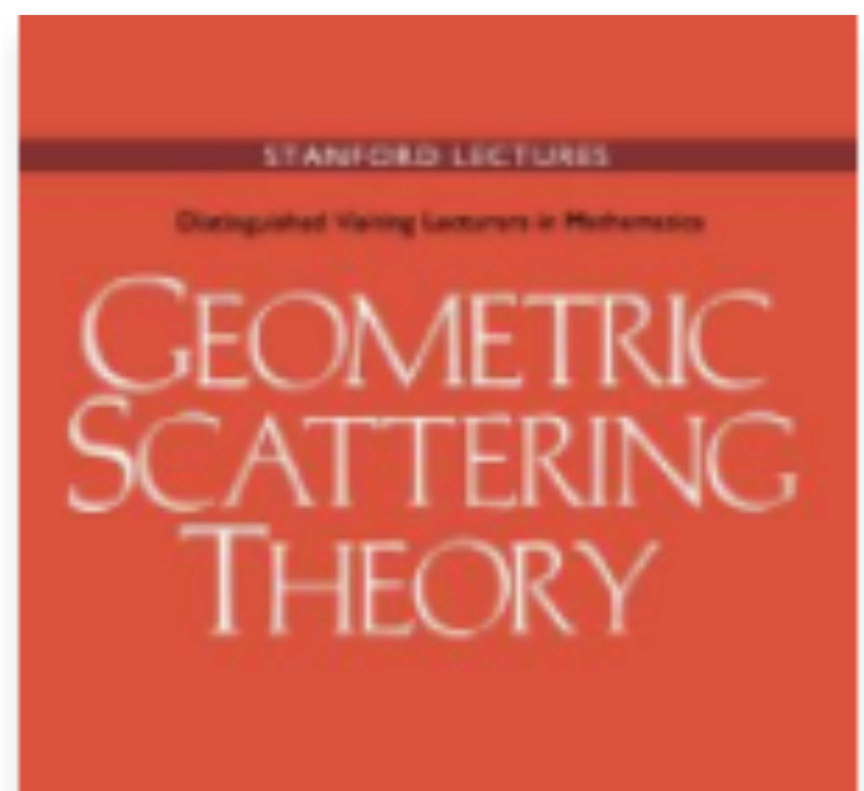
4. *Is the scattering image open and dense?*



4. *Is the scattering image open and dense?*

McGehee's blow-up

Melrose's view of:



Set-up and eqns N bodies in d-dimensional Euc. space:

Newton's eqns: $\iff \ddot{q} = \nabla_m U(q)$

$$q = (q_1, \ldots, q_N) \in \mathbb{E} := \mathbb{R}^{Nd} \qquad q_a \in \mathbb{R}^d, a = 1, \ldots, N$$

Conserved energy

$$E(q, \dot{q}) = \frac{1}{2}\langle \dot{q}, \dot{q} \rangle_m - G \sum \frac{m_a m_b}{r_{ab}}$$
$$= h.$$
$$= K(\dot{q}) - U(q)$$

where

$$2K(\dot{q}) = \langle \dot{q}, \dot{q} \rangle_m = \sum m_i \|\dot{q}_i\|^2 =$$

and

$$U(q) = G \sum \frac{m_a m_b}{r_{ab}}$$

$\nabla_m = \nabla =$ gradient relative to mass metric.

4. Is the scattering image open and dense?

\`Spherical ' change of var's :

$$\mathbf{q} = r\mathbf{s}$$

$$r = \|\mathbf{q}\|_m$$

$$\dot{\mathbf{q}} = v\mathbf{s} + \mathbf{w}, \mathbf{s} \perp \mathbf{w}$$

$$s \in \mathbb{S} \cong S^{Nd-1}$$

$$\rho = \frac{1}{r}$$

$$d\tau = rdt$$

ENERGY: $\frac{1}{2}v^2 + \frac{1}{2}\|w\|^2 - \rho U(s) = h.$

Newton's eqns $\iff$

$$\begin{aligned} \rho' &= -v\rho \\ s' &= w \\ v' &= |w|^2 - \rho U(s) \\ w' &= \rho\tilde{\nabla} U(s) - vw - |w|^2 s \end{aligned}$$

($\tilde{\nabla} U(s) = \nabla U(s) + U(s)s =$ tangential proj of $\nabla U(s)$ by Euler's ident.)

Spatial Infinity : $\rho = 0$, an invariant submanifold

the **infinity manifold.**

4. Is the scattering image open and dense?

Flow at infinity. Set $\rho = 0.$

$$s' = w$$
$$w' = -vw - \|w\|^2 s$$
$$v' = \|w\|^2$$

$$s \in \mathbb{S} \cong S^{dN-1}$$

$$v \in \mathbb{R}, v \neq 0$$

Energy at infinity: $\frac{1}{2}v^2 + \frac{1}{2}\|w\|^2 = h.$

Flow *at infinity is* ***independent*** *of U !*

Equilibria! $$(\rho, s, v, w) = (0, s, v, 0)$$

form a normally hyperbolic manifold of equilibria within the full phase space.

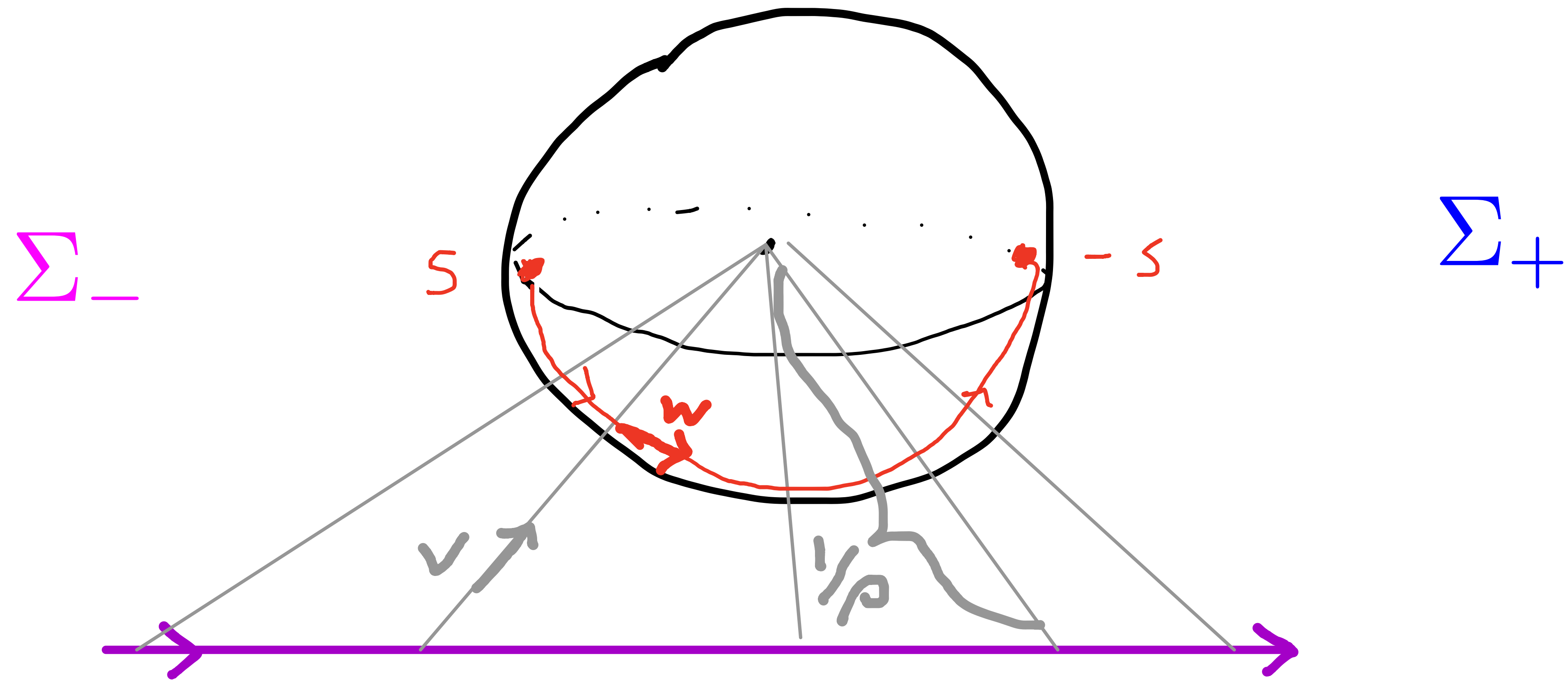
$$\Sigma = \Sigma_- \cup \Sigma_+$$

disjoint union of unstable (v > 0) and stable (v < 0) equilibria
representing past and future end shapes

4. *Is the scattering image open and dense?*

Flow *at infinity is* ***independent*** *of U.*

Set $U = 0$ to understand the dynamics at infinity.
Flow = reparam. of free motion -
projected onto the sphere !:



v <0

rays outgoing
to infinity

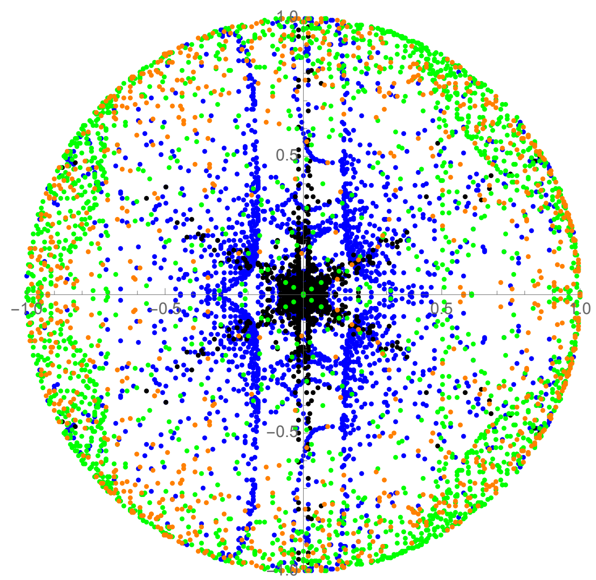
v >0
rays incoming
from infinity

s, -s become equilibria! ; flow is gradient like between them…

Q. **Unstable manifold?** Of what?

Answer. A beam is the family of solutions making up the unstable manifold of an equilibrium point (**a,** |a|) lying on the **infinity manifold**

Q. **Why those black diameters of `near infinity points"?**



Answer. The image of `scattering orbits" that `stay near infinity' converge to ``broken geodesics" on the sphere -*`linear point billiards' or `train tracks' come in.*

What we can prove:

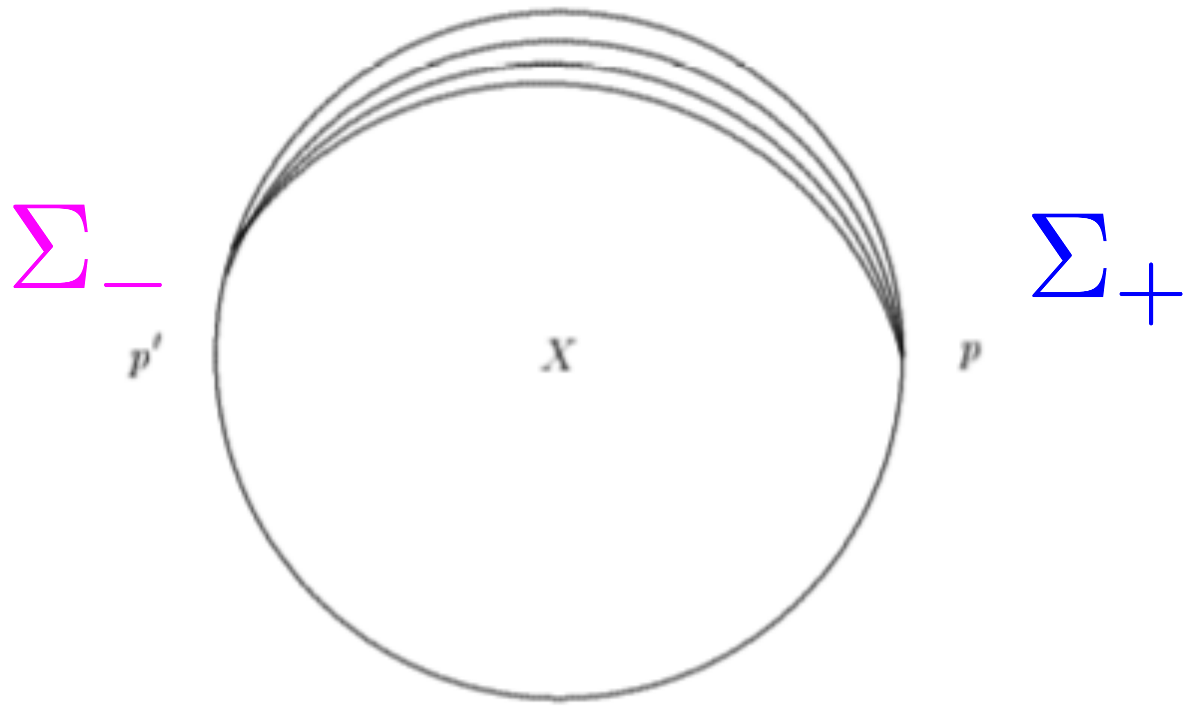
Thm[Nathan Duignan, RM, RM, and Guowei Yu]
The image of the scattering map has non-empty interior.

What E.M. and A.V. can prove:

Thm[Ezequiel Maderna and Andrea Venturelli]
$\mathcal{L}_{\mathbf{a}}^{-}$ projects **onto** configuration space (including collisions.

some words on broken geodesic flow
and the mystery of the
black diameters

a picture from Melrose



p. 80. Geometric Scattering Theory -Melrose.

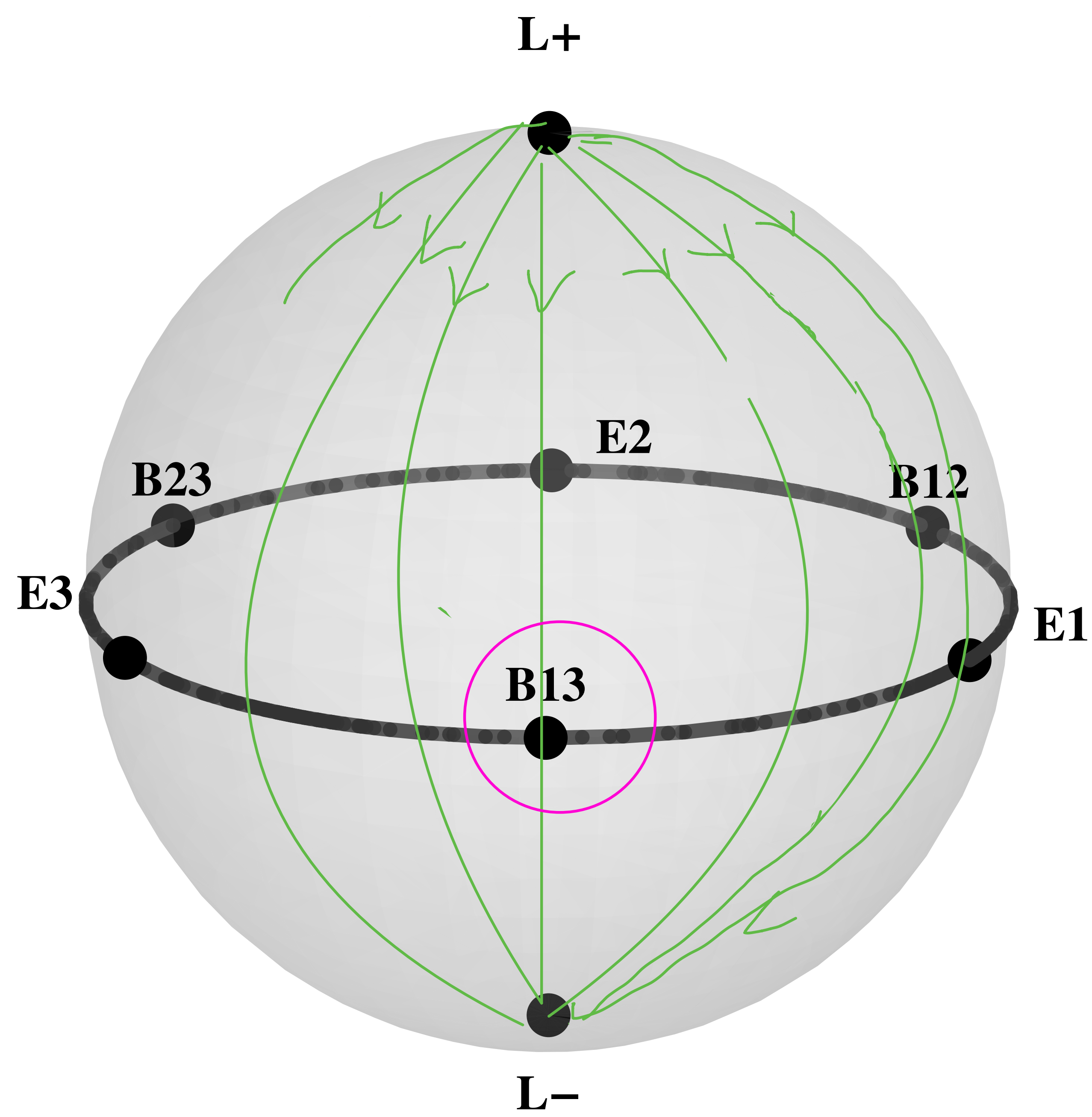
Fig. 11. Geodesic of a scattering metric.

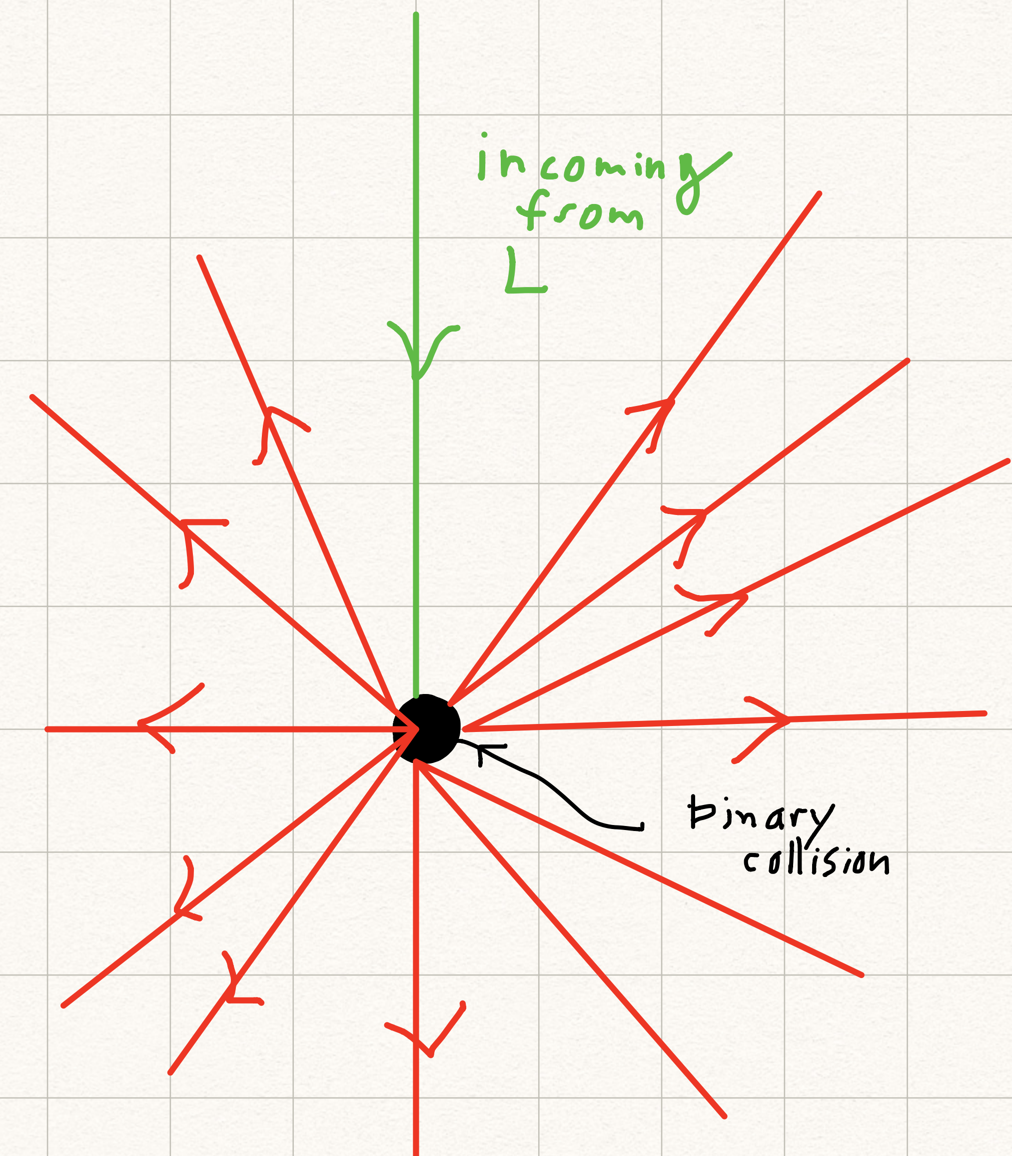
Vasy. Knauf.

… Mazzeo. Zworski. Also:

A COMPARISONS OF THE GEORGESCU AND VASY SPACES ASSOCIATED TO THE N-BODY PROBLEMS AND APPLICATIONS

BERND AMMANN, JÉRÉMY MOUGEL, AND VICTOR NISTOR

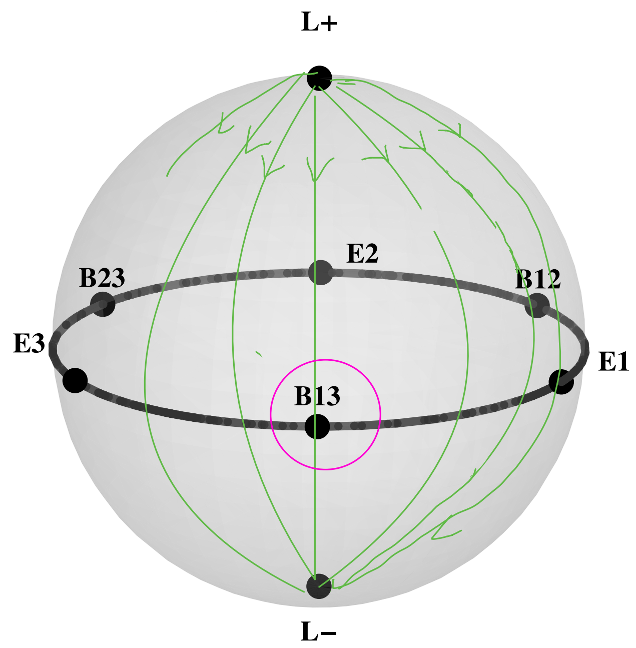

L+
E2
B23
B12
E3
E1
B13
L−



`Broken' geodesic flow: the collision loci on the sphere act as `perfect reflectors'

Non-deterministic!

If a geodesic hits a point on the collision locus it bounces off in a random direction, continuing until either it hits another , continuing in this manner `flowing' for a total time = spherical arclength of π



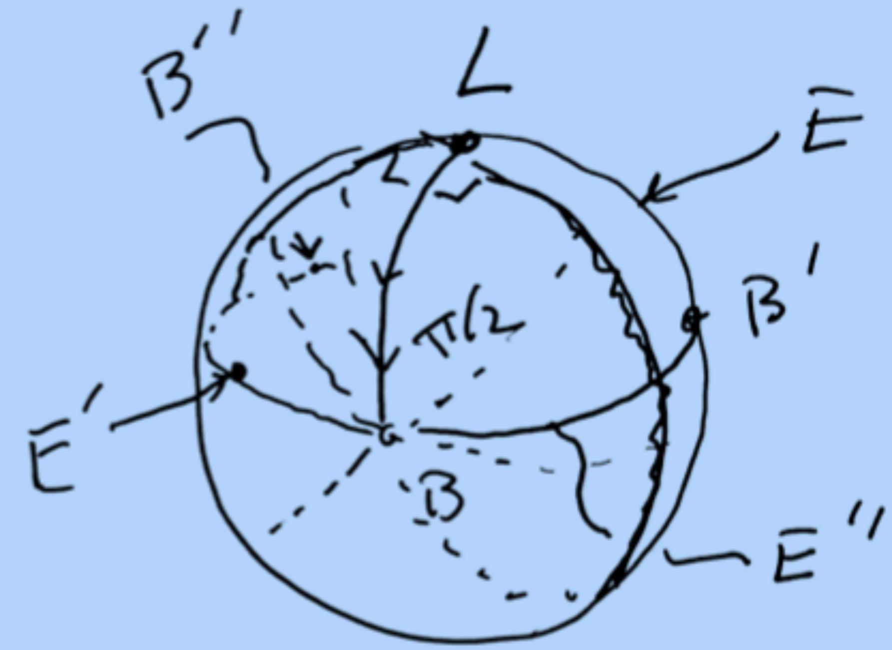
Lag-> Collision = Lag -> Collinear Equator = $\frac{1}{2}\frac{\pi}{2} = \frac{\pi}{4}$

$$\frac{\pi}{4} + x = \pi \implies x = \frac{3\pi}{4} = \frac{1}{2}\frac{3\pi}{2}$$

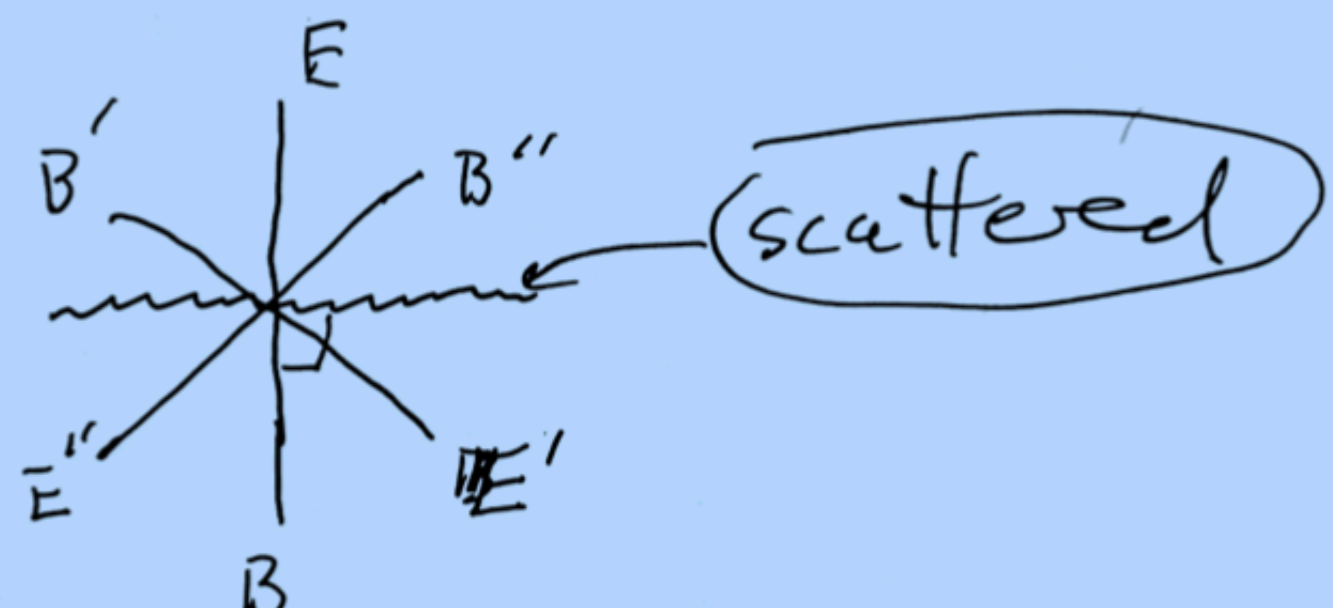
Scenario: Leave binary. Hit collision locus at a point B. Go 3/2 away around the sphere in any direction and mark the resulting points:

Circle of radius $\frac{3\pi}{2}$ about B on standard unit sphere ircle of radius

= circle of radius $\frac{\pi}{2}$ = great circle midway between B and -B.



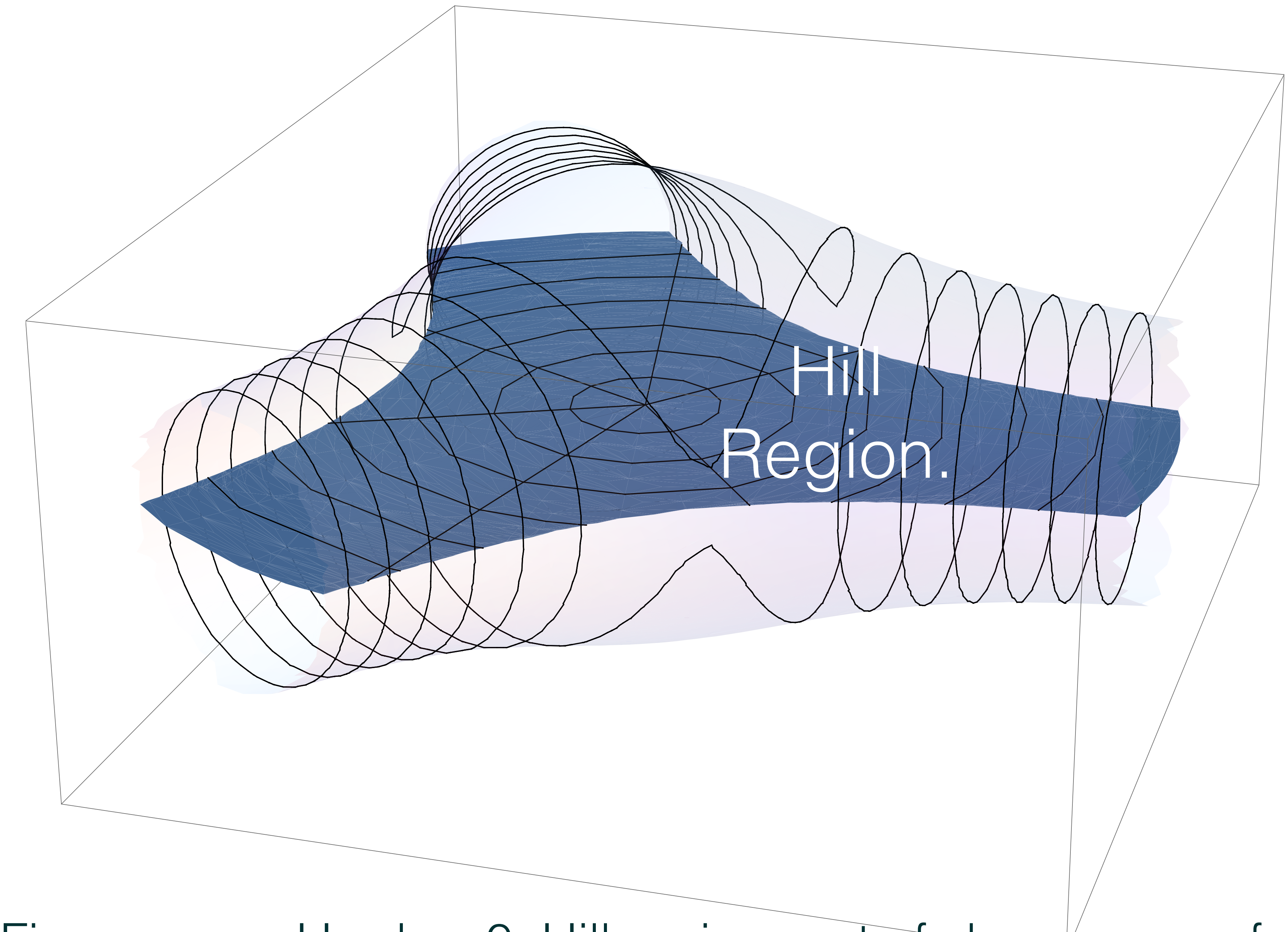
B''
L
E
B'
$\pi/2$
E'
B
E''



E
B'
B''
scattered
E''
E'
B

FINI

FINI



Fix energy = H =-h < 0. Hill region:part of shape space for which there is a v and H(q,v) = -h. Domain where motion occurs. Identical to region with U(q) > +h

..Every free homotopy class…

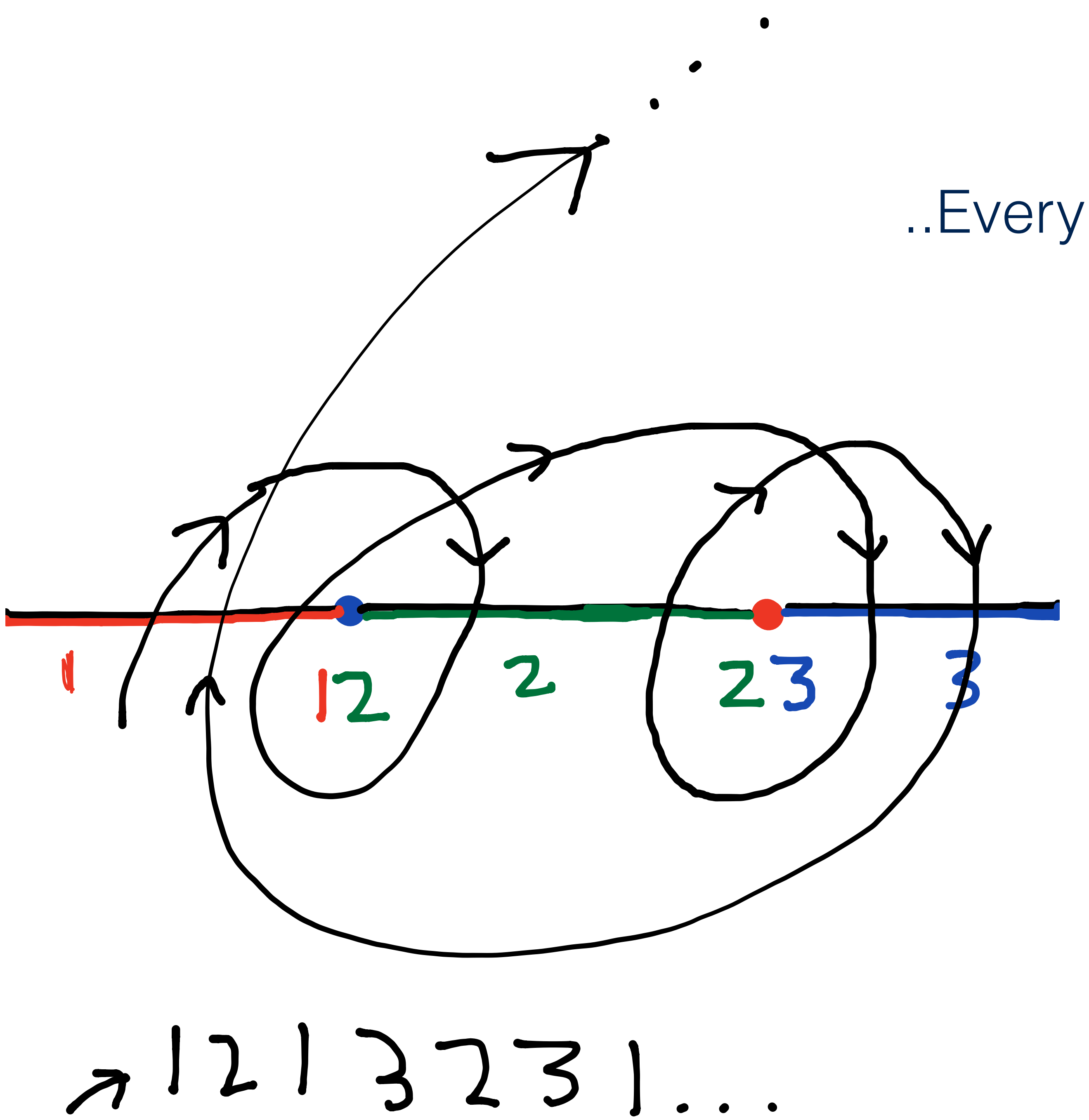


Figure 8: 123123