Geodesics in Jet Space.

— Alejandro Doddoli,
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and R. Montgomery
who is sorry he is not there in Paris with you

with big thanks to Felipe Monroy-Perez

$J^k = J^k(\mathbb{R}, \mathbb{R}) =$ space of k-jets of functions y = f(x)

coordinates $x, y, u_1, u_2, \ldots, u_k$

u_j represents $\frac{d^j y}{dx^j}$

rank 2 distribution

defined by vanishing of the system of k one-forms

$$\begin{aligned} dy - u_1 dx &= 0 \\ du_1 - u_2 dx &= 0 \\ \vdots &= \vdots \\ du_{k-1} - u_k dx &= 0 \end{aligned}$$

so again:

$$\begin{aligned} u_1 &= \frac{dy}{dx} \\ u_2 &= \frac{du_1}{dx} \\ \vdots &= \vdots \\ u_k &= \frac{du_{k-1}}{dx} \end{aligned}$$

admitting the global frame:

$$X_1 = \frac{\partial}{\partial x} + u_1 \frac{\partial}{\partial y} + u_2 \frac{\partial}{\partial u_1} + \ldots + u_k \frac{\partial}{\partial u_{k-1}}$$

$$X_2 = \frac{\partial}{\partial u_k}$$

which generates a nilpotent Lie algebra:

$$[X_2, X_1] = \frac{\partial}{\partial u_{k-1}}$$

$$(ad_{-X_1})^j X_2 = \frac{\partial}{\partial u_{k-j}}, i = 1, \ldots, k \qquad (\; u_0 = y \;)$$

$\Longrightarrow$ J^k has the structure of a Carnot group of `Goursat' type: growth (2,3,4, …, k+1, k+2)

Declare X_1, X_2 to be an **orthonormal** frame for Δ_k

to get a subRiemannian structure on this Carnot group

Q1: ¿What are its geodesics?

Q2: ¿What are its *globally minimizing* geodesics
$\gamma : \mathbb{R} \to J^k$ **?**

Q1 fully answered in 2002-3 by

Alfonso Anzaldo-Meneses and Felipe Monroy-Perez

available through Research Gate:

Goursat distribution and sub-Riemannian structures,
December 2003 Journal of Mathematical Physics

Integrability of nilpotent sub-Riemannian structures
(preprint; INRIA, 2003; inria-00071749]

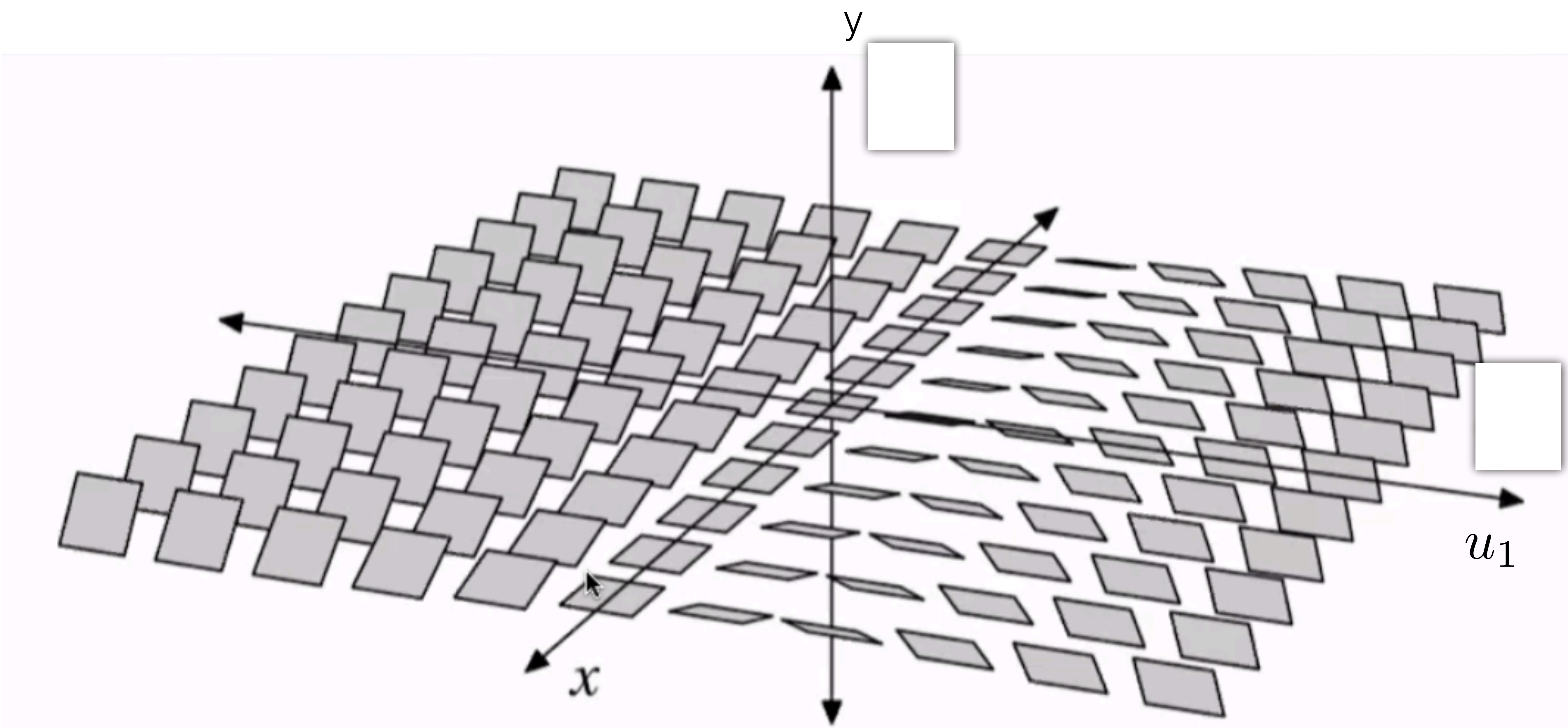
Optimal Control on Nilpotent Lie Groups
October 2002 Journal of Dynamical and Control Systems

and their results form ~ 75% of my talk

k=1: Heisenberg:

`distribution', D

$$dy - u_1 dx = 0$$



o.n. frame for

$$X_1 = \frac{\partial}{\partial x} + u_1 \frac{\partial}{\partial y} \qquad X_2 = \frac{\partial}{\partial u_1}$$

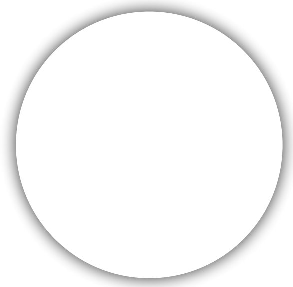
(k=1)

Geodesics: consider the projection

$$J^1 \to \mathbb{R}^2; \pi(x, y, u_1) = (x, u_1)$$

geodesics project to lines

or circles



in the (x, u_1)-plane

i.e. those plane curves with curvature

$$\kappa = \kappa(s) = \text{constant}$$

Among these geodesics , only those corresponding to *lines* *are* global minimizers

Remark on geodesics for general Carnot groups G.

exp: $\mathfrak{g} \simeq G$

$$\mathfrak{g} = V_1 \oplus V_1 \oplus \ldots V_s$$

V_1 Lie generates, corresponds to distribution

$$\pi : G \to G/[G,G] \simeq V_1 \simeq \mathbb{R}^r$$

Lines in the Euclidean space V_1 horizontally lift to global minimizers (=metric lines) in G.

¿Are there any other global minimizers?

For s=2 (two-step, like Heis.) **no.** [Thm: Eero Hakavouri]

but, for s = 3…

(k=2) k=2: Engel:

$$dy - u_1 dx = 0$$
$$du_1 - u_2 dx = 0$$

coords: (x, y, u_1, u_2)

o.n. frame: $X_1 = \frac{\partial}{\partial x} + u_1 \frac{\partial}{\partial y} + u_2 \frac{\partial}{\partial u_1}$ $X_2 = \frac{\partial}{\partial u_2}$

$$\pi : J^2 \to \mathbb{R}^2_{x,u_2} \qquad \pi(x, y, u_1, u_2) = (x, u_2)$$

¿ What plane curves arise as projections of geodesics?

(Any horizontal lift of such a plane curve is then a geodesic)

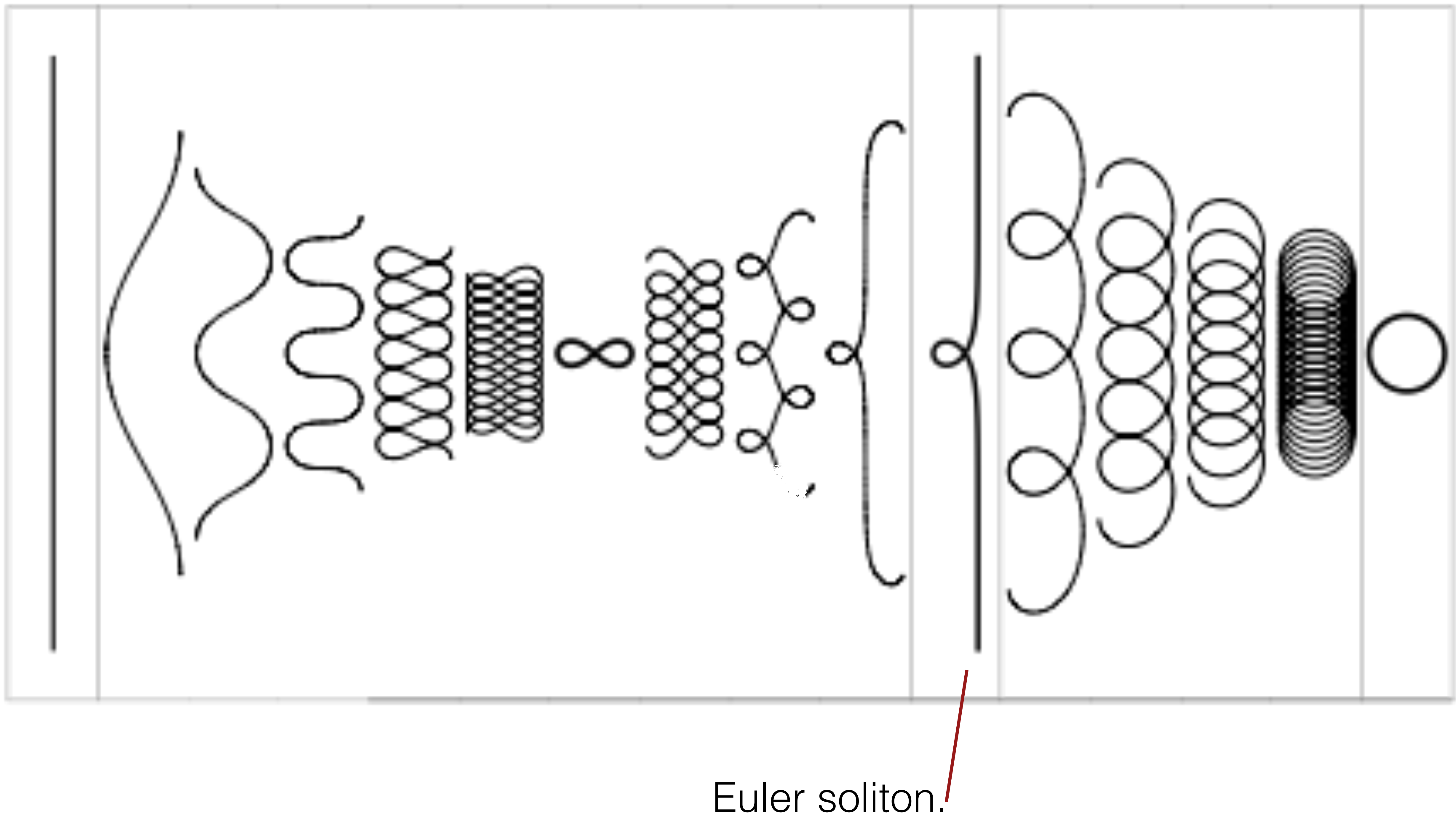
Thm[Ardentov-Sachkov] Besides the lines and circles, we get **Euler elastica** whose directrices are `vertical' (= parallel to u_2 - axis)

These elastica are the plane curves param. by arclength s whose curvature satisfies:

$$\kappa(s) = a + bx(s)$$

i.e. the curvature is a linear polynomial in the coordinate x

(k=2)



Thm, ct'd [Ardentov-Sachkov] among these,
only the lines and the Euler solitons correspond to globally minimizing geodesics.

General k

$$\pi : J^k \to \mathbb{R}^2_{x,u_k}; \pi(x, y, u_1, \ldots, u_k) = (x, u_k)$$

is $\pi : G \to V_1$ and, as such, is a **subRiemannian submersion.**

Def. a *subRiemannian submersion* between two sR mfds whose distributions have the same rank r is a submersion whose differential, upon restriction to each distribution r-plane, is a linear isometry

Us: $d\pi : \Delta_k \to \mathbb{R}^2$

is a linear isometry since $ds^2 = dx^2 + du_k^2$ restricted to Δ_k

here Δ_k denotes the distribution on J^k

Q1' : ¿ What are the *planar projections* of geodesics in J^k?

Write c(s) = $(x(s), u_k(s))$ for such a planar projection, with s = arc-length.

Answer :

Thm A. [Anzaldo-Meneses & Monroy-Peréz; 2002] The curvature $\kappa(s)$ of c(s) is given by a degree k-1 polynomial K in x:

$$\kappa(s) = K(x(s)) \qquad (*)$$

Conversely, for any degree k-1 polynomial K(x), any *horizontal lift* to J^k of *any plane curve* whose curvature satisfies (*) is a geodesic.

Moreover, $$\dot{u}_k(s) = F(x(s))$$

for some anti-derivative F(x) of K(x),
i.e. F(x) is a degree k polynomial such that
dF(x)/dx = K(x),

To understand x(s) , use F to form the (1-deg-of-freedom) Ham. sys:

$$H(x,p) = \frac{1}{2}p^2 + \frac{1}{2}(F(x))^2$$ so potential is $$V(x) = \frac{1}{2}(F(x))^2$$

Thm B [A-M, M-P 2002, ctd] $(x(s), \dot{x}(s))$ solves Hamilton's eqns

for this H:

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial p} = p \\ \dot{p} &= -\frac{\partial H}{\partial x} = -F(x)F'(x) \end{aligned} \tag{1}$$

and obeys the energy constraint:

$$H(x(s), \dot{x}(s)) = \frac{1}{2}$$

We call such a curve x(s) an ``F- curve'' .

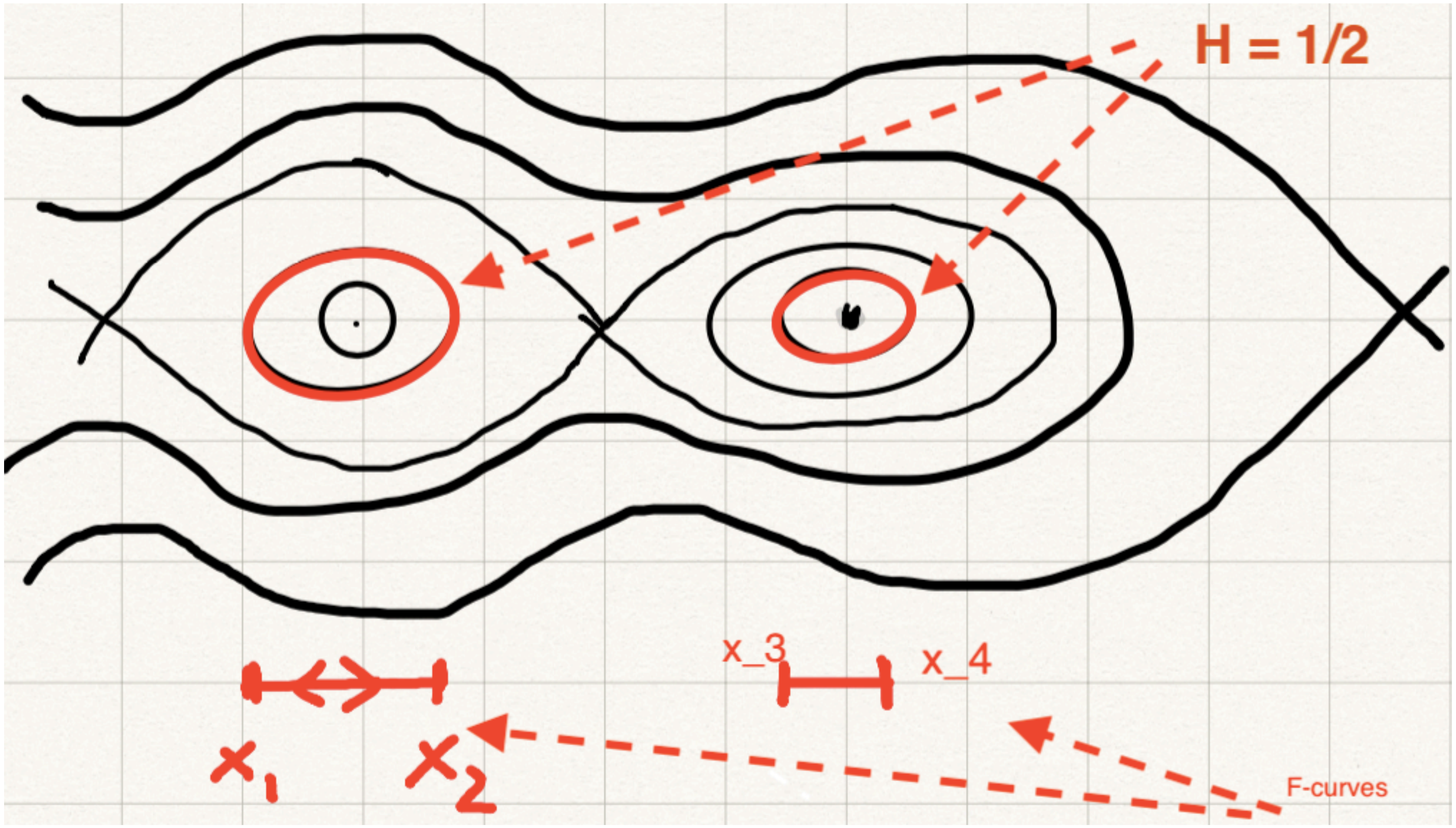
The geodesic flow is completely determined by the F-curves.

The F-curve associated to ***any*** deg. k poly F(x) arise as the x-projection of some geodesic on the jet space

Will give a `magnetic field' proof of these theorems below

Level curves of $$H(x,p) = \frac{1}{2}p^2 + \frac{1}{2}(F(x))^2$$

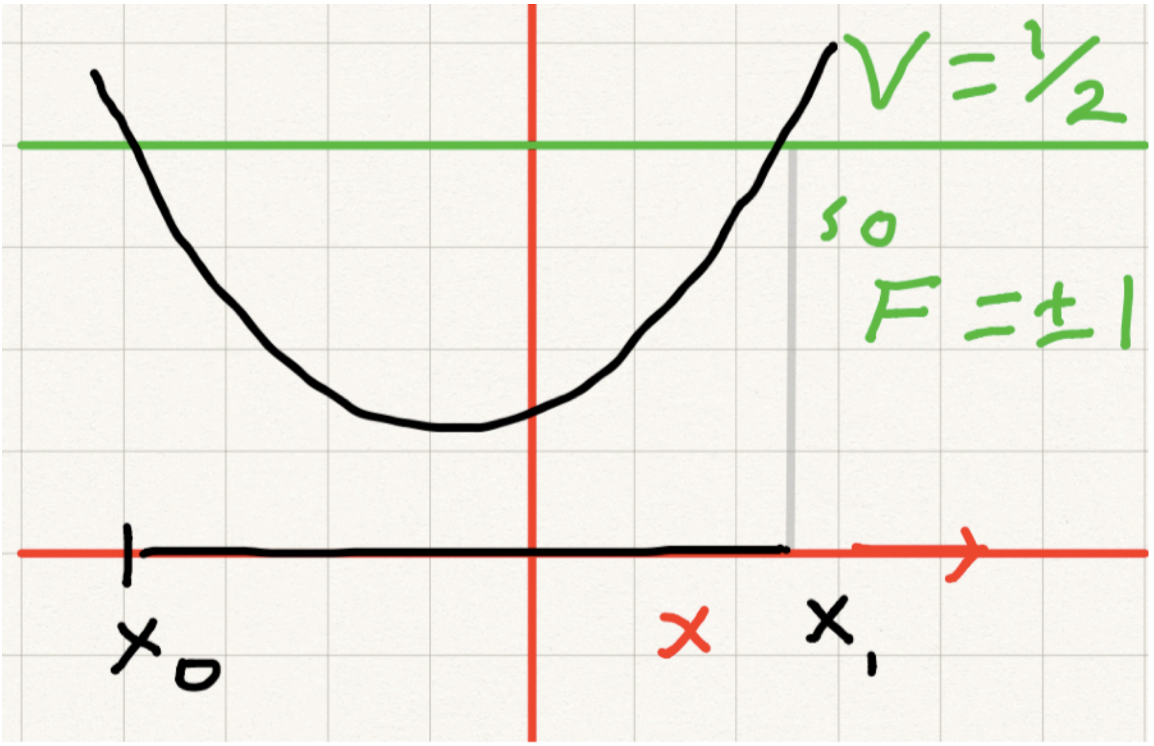
Level set
H = 1/2



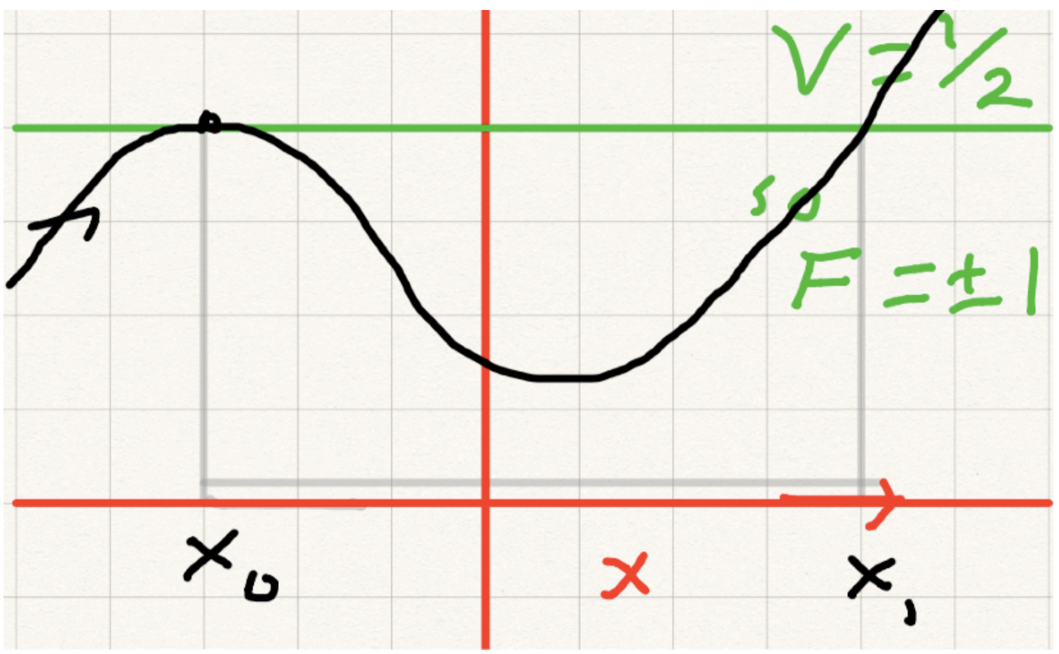
projects onto the ***Hill region:*** $\{x : \frac{1}{2}F(x)^2 \leq \frac{1}{2}\}$ or:

$$\{|F(x)| \leq 1\} = [x_1, x_2] \cup [x_3, x_4] \cup \ldots \cup [x_{2i-1}, x_{2i}], i \leq k$$

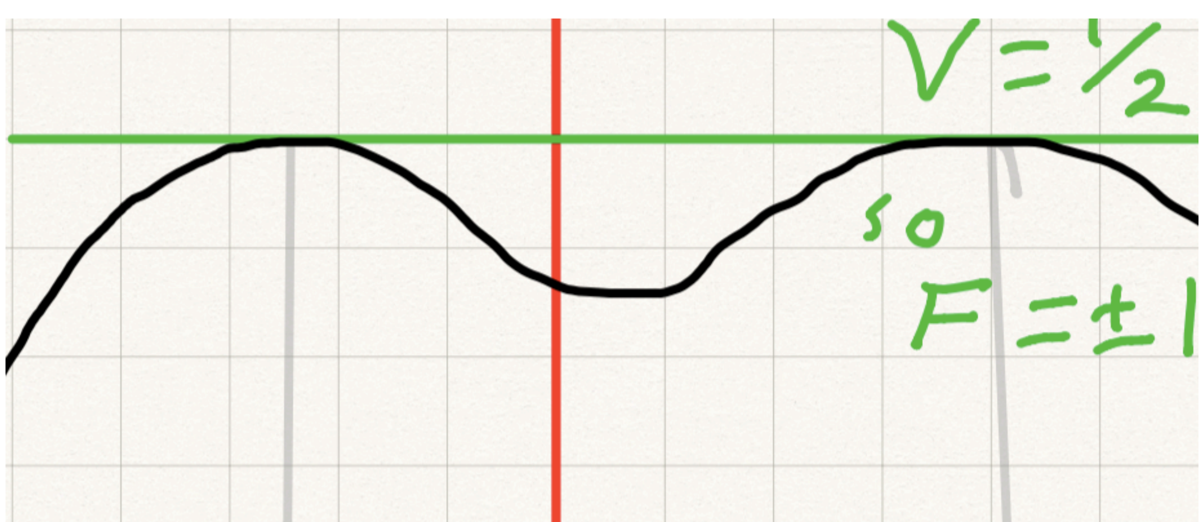
and is the union of at most k closed bounded intervals
whose endpoints x_i satisfy $F(x_i) = \pm 1$
Each interval is swept out by an F-curve.



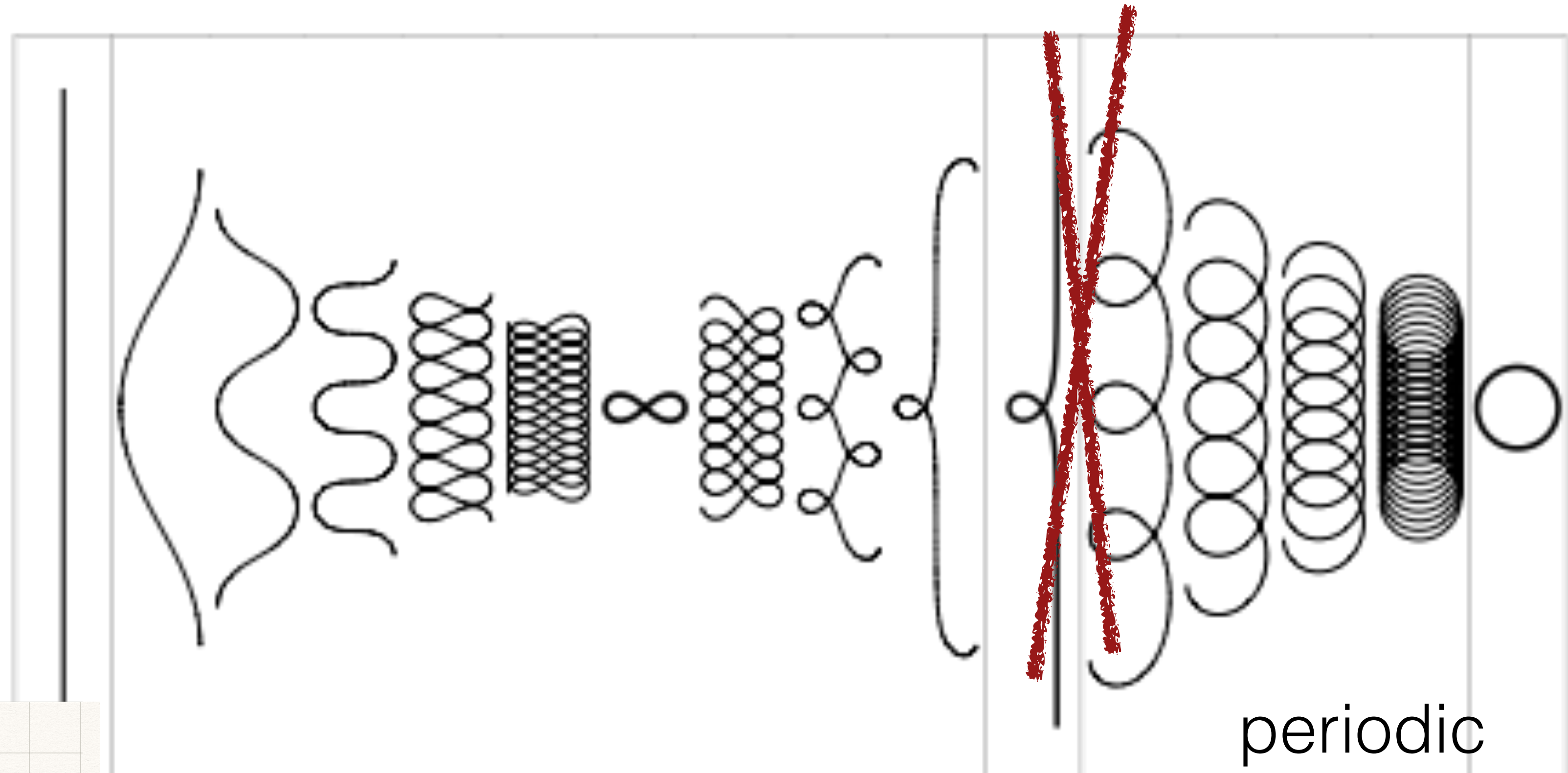
x(s) periodic.
Traverses interval once in time L/2, there and back in period L

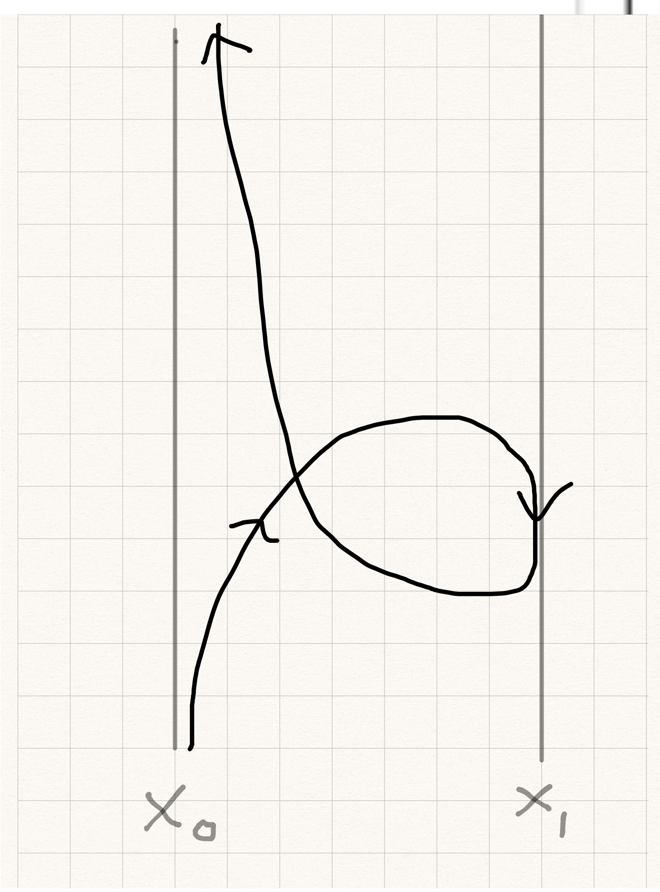


x(s) critical .
Of homoclinic type.
Takes infinite time to reach x_0, bouncing off x_1

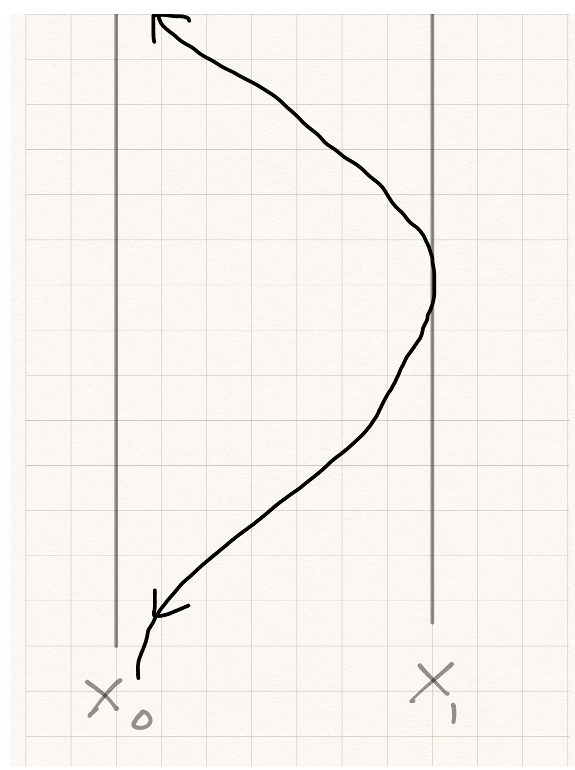


x(s) critical .
Heteroclinic from x_0 to x_1.
Takes infinite time to traverse interval once

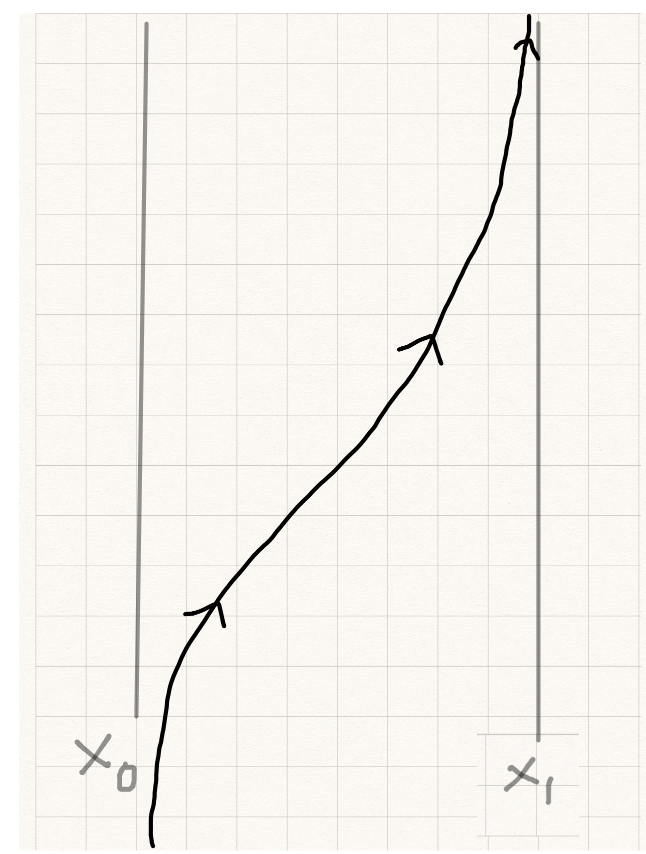




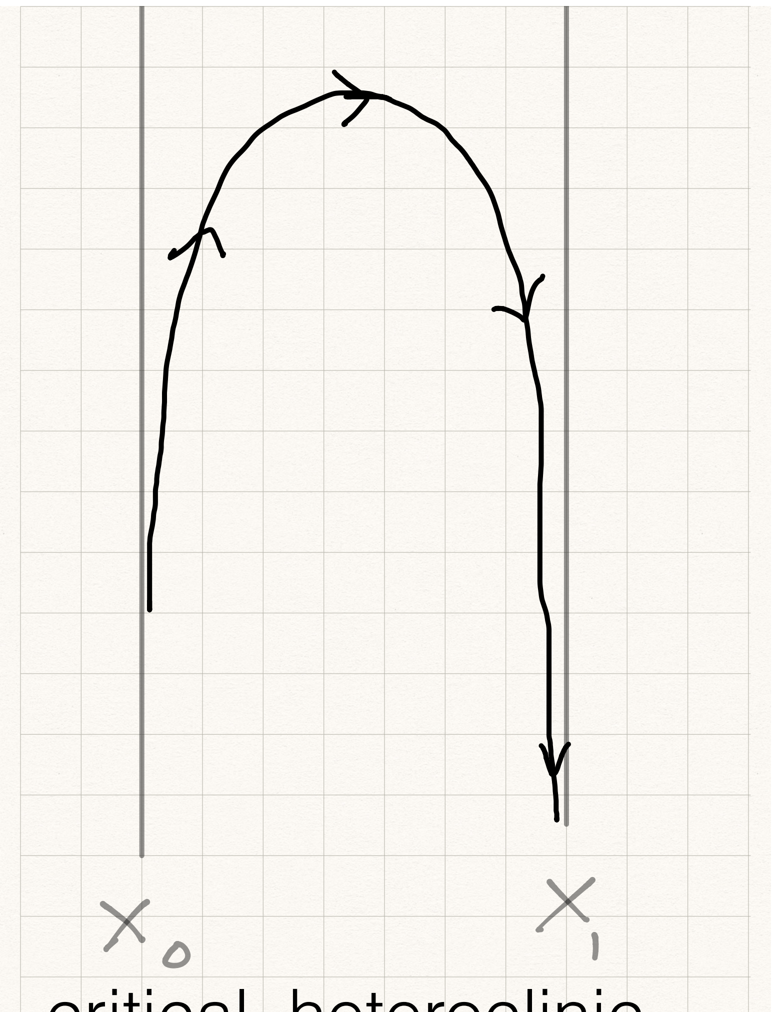
critical, homoclinic



critical, homoclinic



critical, heteroclinic



critical, heteroclinic,
of `turn-around type':
$F(x_0) = -F(x_1)$

Remark : ¿ Why is the vertical (u_k) direction special ?

Answer: The singular curves in J^k are precisely the u_k lines.

MAIN RESULTS

Conjecture [Doddoli, M-] Geodesics whose F-curves are critical and not of `turn-around type' are global minimizers.

For k =2 this is the **theorem of Ardentov-Sachkov** above.
Up to scale there is just one such curve, the one for $F(x) = x^2 - 1$

Theorem. [Doddoli, M-]
(i) For k >2, the geodesic whose x coordinate x(s) is an F-curve for

$F(x) = x^k - 1$ is a global minimizer.

(ii) Geodesics whose F-curves are periodic are **not** global mins.
Indeed, they **fail to minimize** past one period L.
specifically: if x(0) = x(L) is an endpoint of for the F-curve's interval then
s = L is conjugate to s = 0 along the corresponding geodesic in J^k

Plan of attack for an *eventual* proof for the full conjecture:

Two ideas :

1) Build an *intermediate sR manifold* $\mathbb{R}^3_F$ *depending on F*

$$J^k \xrightarrow{\pi_F} \mathbb{R}^3_F \xrightarrow{pr_F} \mathbb{R}^2$$

$$pr_F \circ \pi_F = \pi : J^k \to \mathbb{R}^2$$

All projections are sR submersions between sR manifolds

The intermediate space will be of ``magnetic type':

Characterize its geodesics and global minimizers

Use: the horiz lifts of global mins are global mins.

2): Use the method of *Hamilton-Jacobi* to find *calibrations* S on the intermediate space, thus generating (quasi-) global mins

The solution S will be associated to a + b F -curves

Step 1. i) A polynomial change of coordinates :

$$(x, u_0 = y, u_1, \ldots, u_k) \mapsto (x, \theta_0, \theta_1, \ldots, \theta_k)$$

$$\theta_0 = u_k \ , \qquad \theta_1 = -u_{k-1} + xu_k \quad , \qquad \ldots$$

yields an alternate expression for our frame:

$$X_1 = \frac{\partial}{\partial x}$$

$$X_2 = \frac{\partial}{\partial \theta_0} + \Sigma_{j=1}^{k} \frac{x^j}{j!} \frac{\partial}{\partial \theta_j}$$

ii) Suppose given

$$F = a_0 + a_1 x + a_2 \frac{x^2}{2!} + \ldots + a_k \frac{x^k}{k!}$$

Define the projection π_F to 3-space, with coords X, Y, Z by

X = x

$$Y = \theta_0 := u_k$$

$$Z = a_0\theta_0 + a_1\theta_1 + \ldots + a_k\theta_k$$

The projection π_F is linear in these coordinates so its differential easy to compute. Get:

$$\pi_{F*}X_1 = \frac{\partial}{\partial X} = \frac{\partial}{\partial x} \qquad = E_1$$

$$\pi_{F*}X_2 = \frac{\partial}{\partial Y} + F(x)\frac{\partial}{\partial Z} \quad = E_2$$

which is an o.n. frame for the sR structure of magnetic type:

Distribution: $dZ - F(x)dY = 0$

metric: $dX^2 + dY^2$ restricted to distribution.

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magnetic analogy: vector potential: $F(X)dY$

magnetic field: $F'(X)dX \wedge dY = K(X)dX \wedge dY$

Example: Martinet case. F(x) = x^2

General F, ct'd. The geodesics for $\mathbb{R}^3_F$ are generated by:

$$H = \frac{1}{2}p_X^2 + \frac{1}{2}(p_Y + F(X)p_Z)^2$$

Since:

$$\dot{p}_Y = \dot{p}_Z = 0$$

we can view $p_Y = a, p_Z = b$ as constant parameters.

Then

$$H = \frac{1}{2}p_x^2 + \frac{1}{2}(a + bF(x))^2$$

(X = x)

generates a family of F-curves x(s), not just for F, but for the ***pencil*** of polynomials a + b F(x).

Step 2. Hamilton-Jacobi method

solve $H(q, dS(q)) = \frac{1}{2}$ for S

eqn is equiv to $\|\nabla_{hor} S(q)\| = 1$

Integral curves q(t) for $\dot{q} = \nabla_{hor} S(q)$

are minimizing geodesics.

Our case of $\mathbb{R}^3_F$

$$\nabla_{hor} S = \frac{\partial S}{\partial x} E_1 + (\frac{\partial S}{\partial Y} + \frac{\partial S}{\partial Z} F(x)) E_2 \qquad \mathbb{R}^3_F$$

Hamilton-Jacobi eqn:

$$(\frac{\partial S}{\partial x})^2 + ((\frac{\partial S}{\partial Y} + \frac{\partial S}{\partial Z} F(x))^2 = 1$$

Ansatz: S(x,Y, Z) = b Z + a Y + f(x)

yields

$$f'(x)^2 + (a + bF(x))^2 = 1$$

Compare with the energy H=1/2 eq implied by the geodesic equations:

$$(\dot{x})^2 + (a + bF(x))^2 = 1$$

$$\dot{x} = p_x, a = p_Y, b = p_Z$$ & $$\dot{Y} := \dot{u}_k = (p_Y + P_Z F(x))$$

Suggests:

$$\dot{x} = f'(x), \dot{u}_k = a + bF(x)$$

which are the first two components of the ODE:

$$\dot{q} = \nabla_{hor} S(q)$$

The last (Z) component arises by horiz lift: $\dot{Z} = F(x)\dot{u}_k = F(x)(a + bF(x))$

Solve the boxed eq by taking a square root and integrating:

$$f(x) = \pm \int_{x(0)}^{x} \sqrt{1-(a+bF(\xi))^2}d\xi$$

Must have $1-(a+bF(\xi))^2 \geq 0$

which means that x & x(0) must lie within a single interval $[x_i, x_{i+1}]$

of the Hill region (*) associated to the 1-deg of freedom Hamiltonian

$$H = \frac{1}{2}p_x^2 + \frac{1}{2}(a+bF(x))^2$$

at energy level 1/2.

The criticality or regularity of a + b F(x)
at the endpoints of the interval of def, $[x_i, x_{i+1}]$

govern whether or not the horizontal gradient flow of S is complete or not on the **slab**

$$\{x, Y, Z) : x \in [x_i, x_{i+1}]\} \subset \mathbb{R}^3_F$$

$$\dot{x} = \pm\sqrt{1 - (a + bF(x))}^2$$

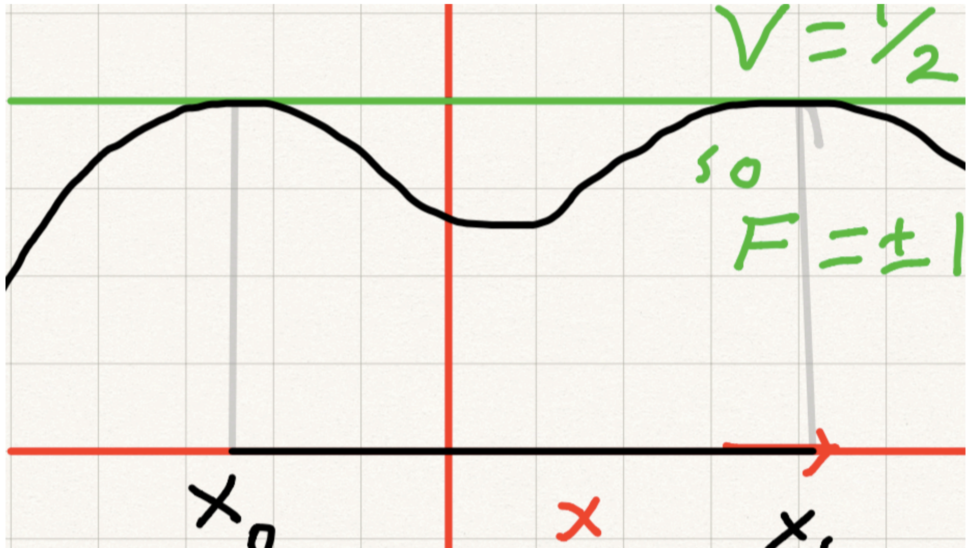
$$\dot{Y} = a + bF(x)$$

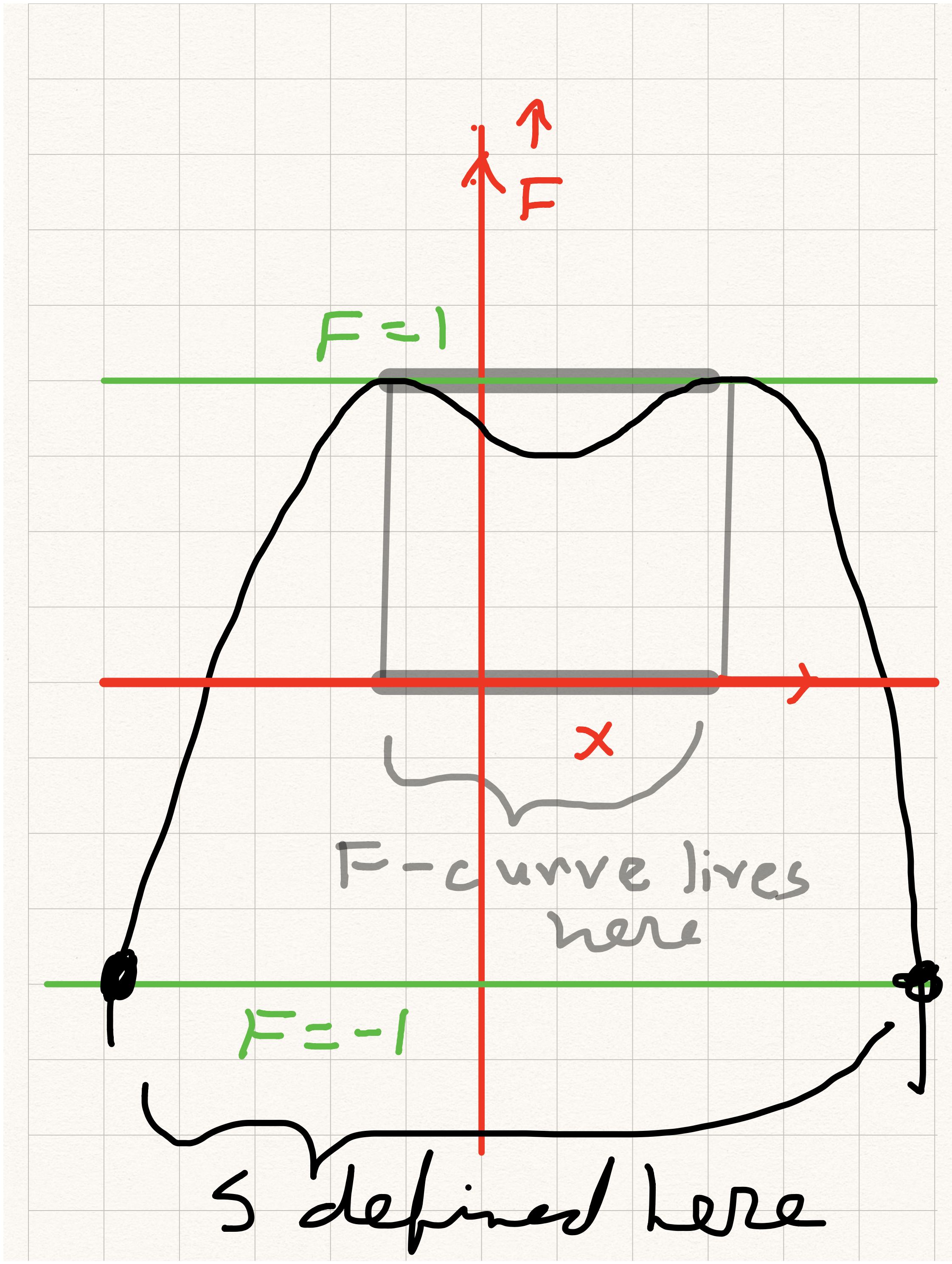
$\dot{Z}$ by horizontally lifting the x, Y curve…

EXAMPLE:

Heteroclinic case, no `turn-back':

a + b F(x) =1 at endpoints x_i, x_{i+1}





Result: the geodesic corresponding to this heteroclinic F-curve is a **global minimum** within the larger slab within which S is defined.

END

perhaps… but if …

…time permitting - a bit of blather on

-Buseman,
-being bi-asymptotic to two singular lines,
-dim count ons space of pairs of singular lines…

END