How does a falling cat,
dropped from upside down with no spin,
right itself?

What does this problem have to do with gauge theory ?

With the three-body problem ?

Unifying Mathematical theme: geometry of a principal G-bundle

$$G \to Q \to S$$

group

a space on which G acts

the quotient space Q/G

	group G	total space P	base space S
High energy physics	U(1) or U(2) or SU(3) electromagnetism or electro-weak or quarks (QCD)	X x G =	space-time
Differential Geometry	SO(n)	Frame bundle of the n-manifold	an n-manifold
Mechanics and Control Theory (Cats, N-	rigid motions, so SO(n) or E(n)	configuration space of the mechanical or	shape space! =Q/G

Groups ?

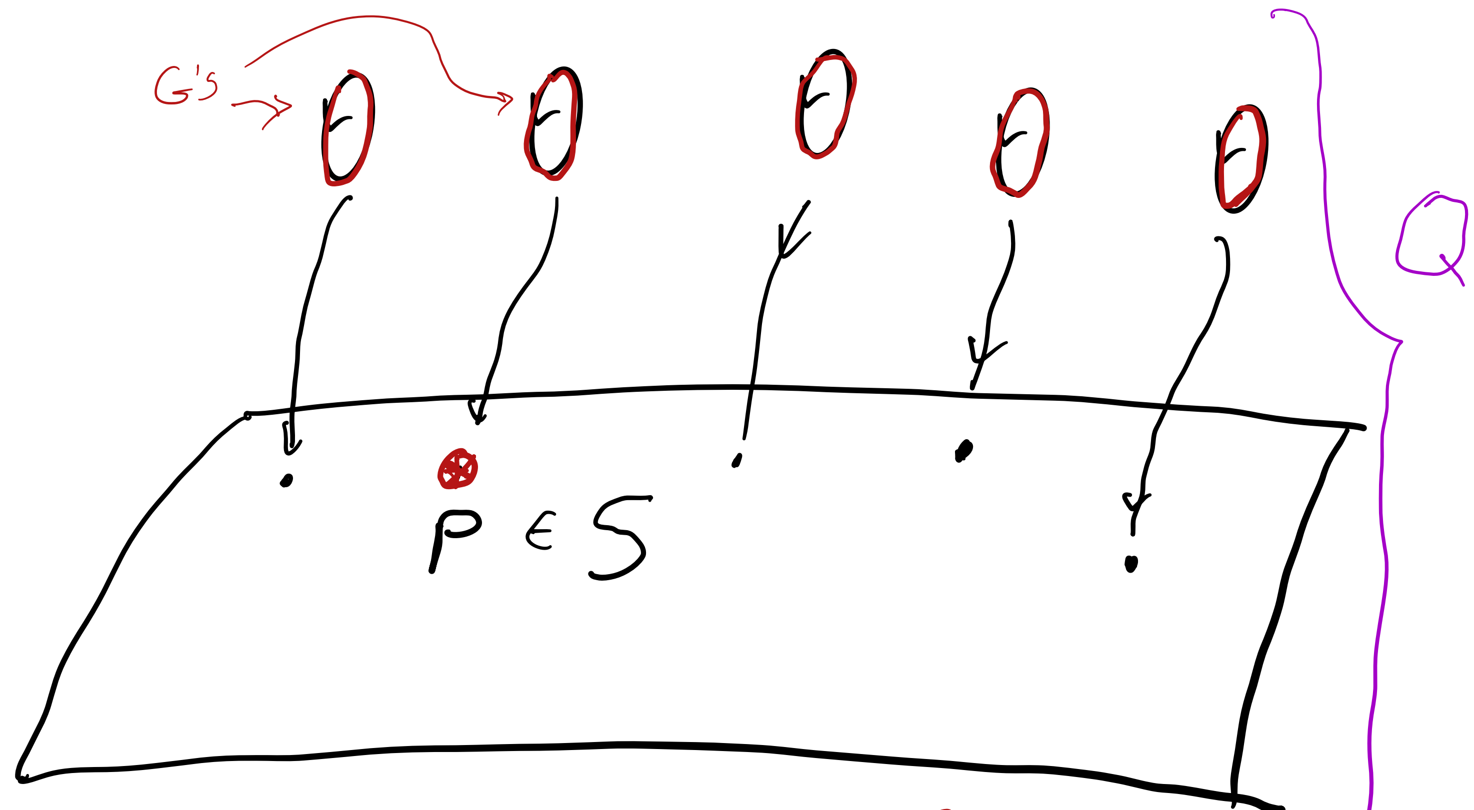
Stand up!
move…

For the falling cat
and for the 3-body problem

G = group of rigid motions
= rotations and translations

Principal G-bundles:

At every point p of a space S . . .



. . . attach a group G

(Here G is circle S^1)

Q = configuration space
= space of `located cats' in space

or , for the 3-body problem:

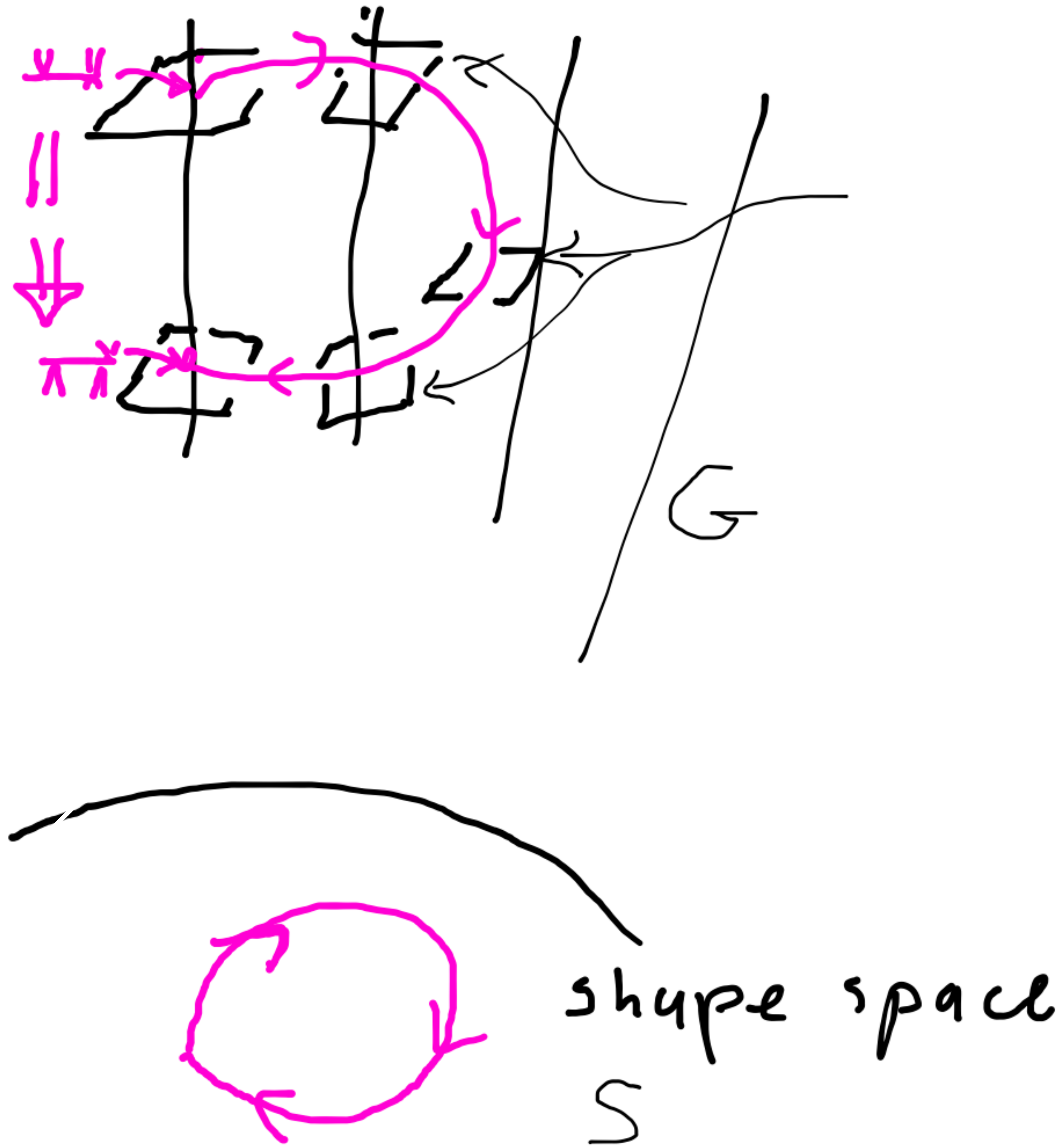
Q = configuration space of the 3- body problem:
= triples of points in the plane (planar 3-body)

G acts on Q by rotating and translation
the `frozen cat'
or, in the 3-body problem, by rotating and translating
the triangle formed by the 3 bodies

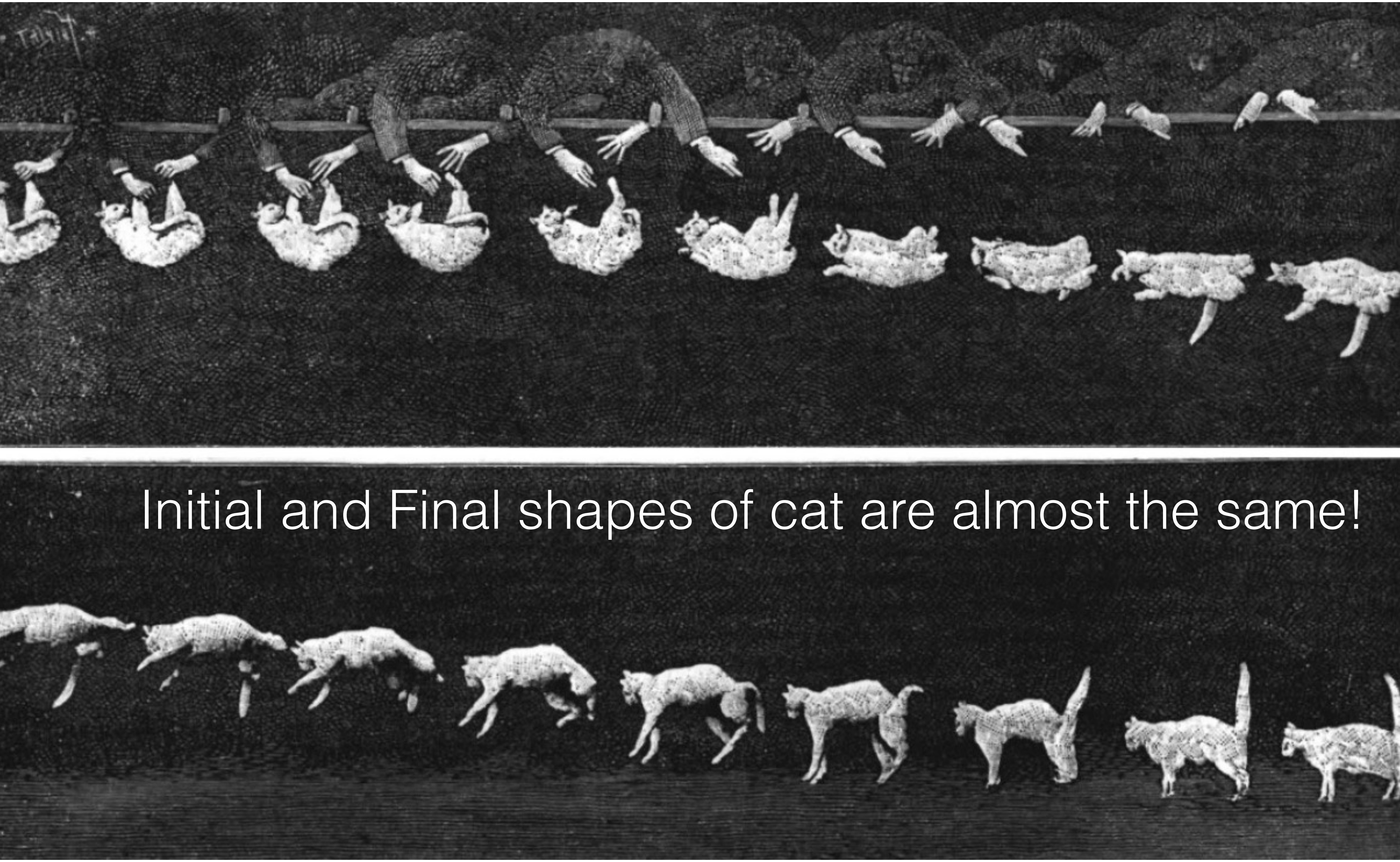
Utility of principal bundle picture for understanding the strategies of the falling cat for righting herself

a reorientation strategy
IS a loop in shape space

holonomy



loop in shape space



(from `Falling Cat ‘ wiki page; a
copy of a photo taken in 1894)

What is a shape?

.. more kinesthetics..
hands.. (*)

A `shape' is a G-orbit!

i.e

an equivalence class of configurations
under the action of G

A shape is a point in `shape space': S:=Q/G.

Quotient space S = Q/G

= shape space;

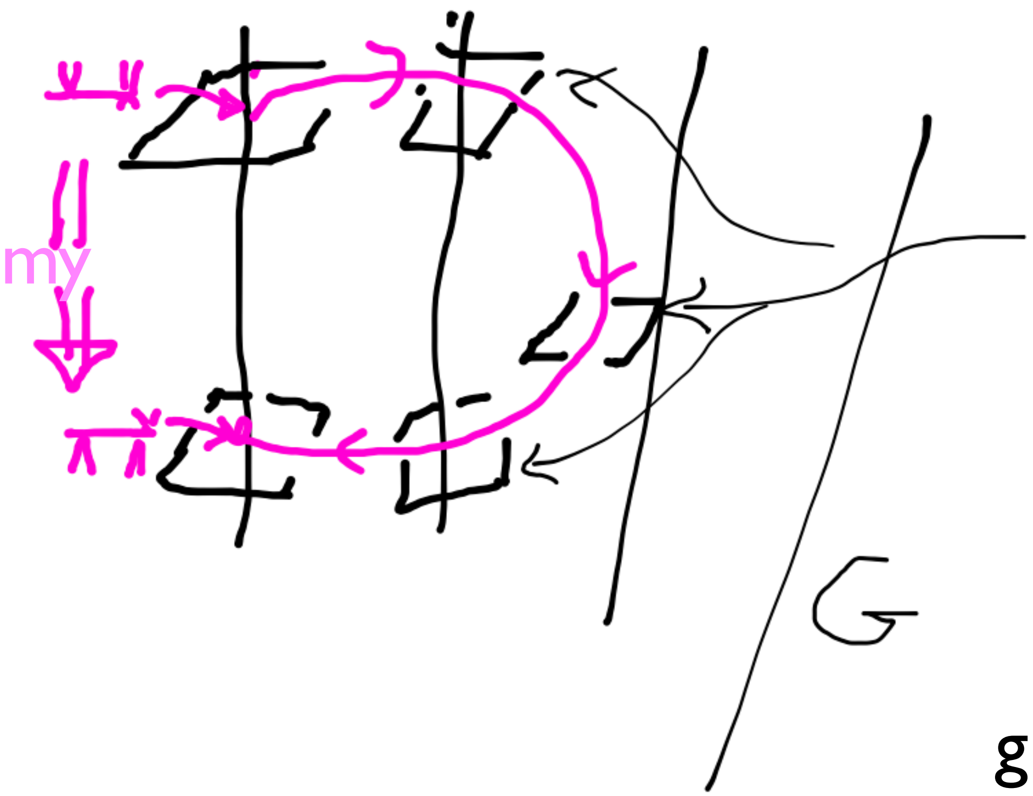
so space of shapes of cats,

or *shapes of triangles* for the three-body problem.

Planar 3-body shape space: $\mathbb{R}^3$

Bundle w connection. Total space = Config. space

= space of located shapes

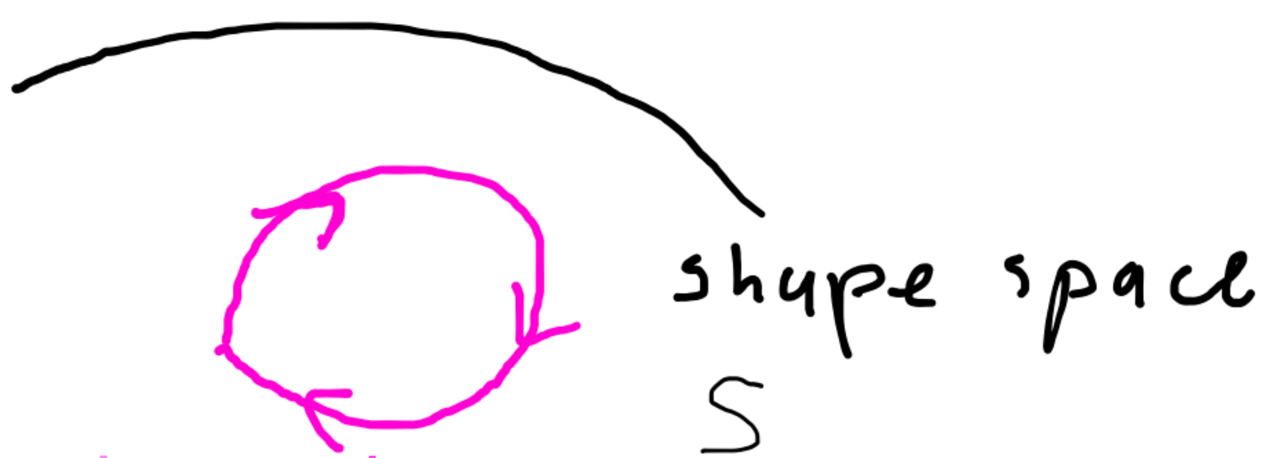


holonomy

connection `A', given by physics

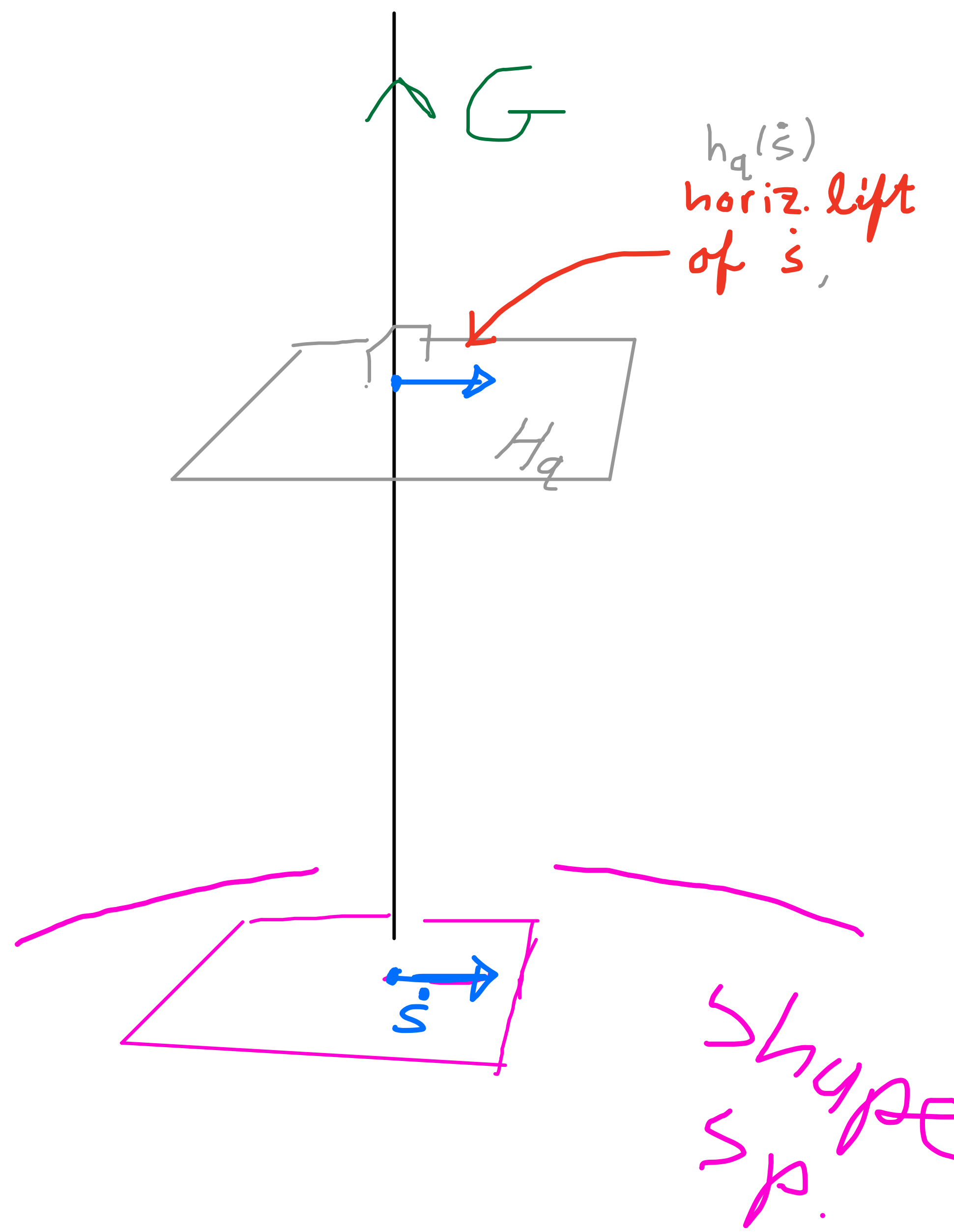
structure group =

group of rigid motions



loop in shape space =

reorientation strategy or gait



connection:
encodes horizontal
space's H

Horizontal motions =
paths perpindicular to the group orbits,
or perpindicular to the `vertical spaces''

where a `vertical vector ' is a vector tangent to a G-orbit

To show:

Horizontal motions = motions with
total angular momentum zero

Perpindicular relative to what metric on Q?

Model Q, the configuration space for
the `located cat' as a collection of point masses.
So:

$$q = (q_1, q_2, \ldots, , q_N), q_a \in \mathbb{R}^3$$

represents a point of Q. Think of
the q_a's as `marker points. (`Foot', `head', ..,)
They have masses m_a. Define an inner product
on Q for which the squared length
of velocities v_a is twice their kinetic energy
K:

$$K(\dot{q}) = \frac{1}{2}\Sigma m_a |v_a|^2, v_a = \dot{q}_a \quad = \frac{1}{2}\langle v, v\rangle$$

so that :

$$< q, q' >= \Sigma m_a q_a \cdot q'_a$$

Mass metric

To work out

$$H_q = (V_q)^{\perp}$$

we need V_q, the tangent space to the group orbit through q:

$$G(q) = \{Rq := (Rq_1, \ldots, Rq_N) : R \in SO(3) \text{ a rotation}\}$$

Infinitesimal rotations are given by cross products:

$$\frac{d}{d\epsilon} R(\epsilon) q_a = \omega \times q_a, \qquad \omega, q_a \in \mathbb{R}^3$$

So:

$$V_q = \{\text{``}\omega \times q\text{''} : \omega \in \mathbb{R}^3\}$$

where

$$\text{``}\omega \times q\text{''} = (\omega \times q_1, \omega \times q_2, \ldots, \omega \times q_N)$$

Suppose that v in Q is mass-metric perpindicular to all these vertical vectors:

$$0 = < v, \text{``}\omega \times q'' >$$
$$= \Sigma m_a v_a \cdot (\omega \times q_a)$$
$$= \Sigma m_a \omega \cdot (q_a \times v_a)$$
$$= \omega \cdot (\Sigma m_a q_a \times v_a)$$
$$= \omega \cdot (\Sigma q_a \times m_a v_a)$$

for all $\omega \in \mathbb{R}^3$

This is true iff $\Sigma q_a \times m_a v_a = 0$

But this says the total angular momentum is zero!

$$\Sigma q_a \times m_a v_a = \text{the angular momentum associated to } q, v$$

!!

Prop. A deformation, or `motion’ q(t) of a located shape q(0) is mass-metric perpindicular to the group orbits if and only if its total angular momentum is zero.

This fact connects the geometry to the physics !

Same principle in Riemannian geometric terms

If I have a 'Riemannian submersion' $\pi : Q \to S$
then any geodesic which is perpindicular to a fiber
at one point is perpindicular to the fibers at all points

J. Fluid Mech. (1989), *vol.* 198, *pp.* 557–585
Printed in Great Britain

Geometry of self-propulsion at low Reynolds number

By ALFRED SHAPERE† AND FRANK WILCZEK‡

† Institute for Advanced Study, Princeton, NJ 08540, USA

‡ Institute for Theoretical Physics, University of California, Santa Barbara, CA 93106, USA

(Received 15 April 1987 and in revised form 12 July 1988)

The problem of swimming at low Reynolds number is formulated in terms of a gauge field on the space of shapes. Effective methods for computing this field, by solving

The three-body problem

Galileo

1632; `Dialogo ..” :

The laws of physics are
invariant under my group.

My group contains your group G.

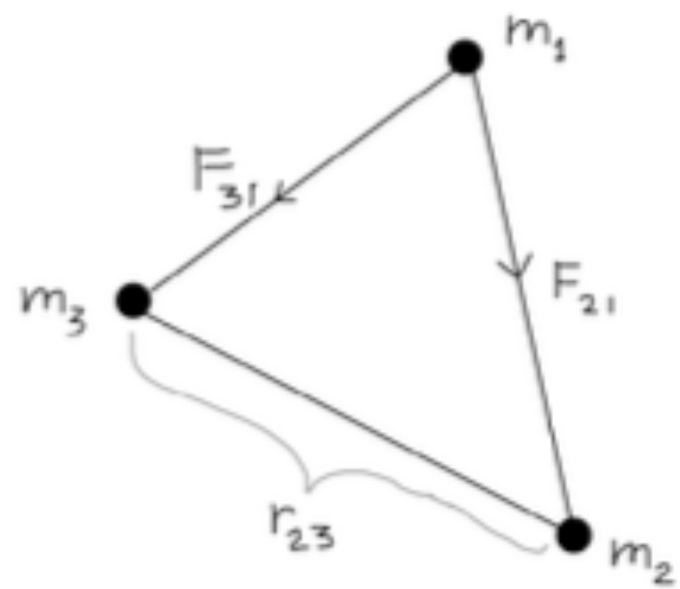
G= group of rigid motions of space
=translations, rotations, ~~reflections~~

Newton, Principia, 1687:

The laws of physics can be written as differential equations which are invariant under Galileo's group.

For three bodies moving in the plane or space under the influence of their mutual gravitational attraction these differential equations are

$$m_1\ddot{q}_1 = \quad F_{21} + F_{31},$$
$$m_2\ddot{q}_2 = \quad F_{12} + F_{32},$$
$$m_3\ddot{q}_3 = \quad F_{23} + F_{13},$$



$$F_{ba} = -\frac{Gm_a m_b}{r_{ab}^2}\hat{q}_{ab} \quad \text{with} \quad \hat{q}_{ab} = \frac{q_a - q_b}{r_{ab}},$$

where

$$r_{ab} = |q_a - q_b|$$

Planar 3-body problem:

$$Q = \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \qquad = \mathbb{C}^3$$

Newton's 3 body ODEs descend to shape space:

$$S = Q/G$$

which equals…

$$\mathbb{R}^3$$

-the space of oriented congruence classes of planar triangles

!!

Some Details..

mod out by translations $\mathbb{C}^3/\text{translations} = \mathbb{C}^2$

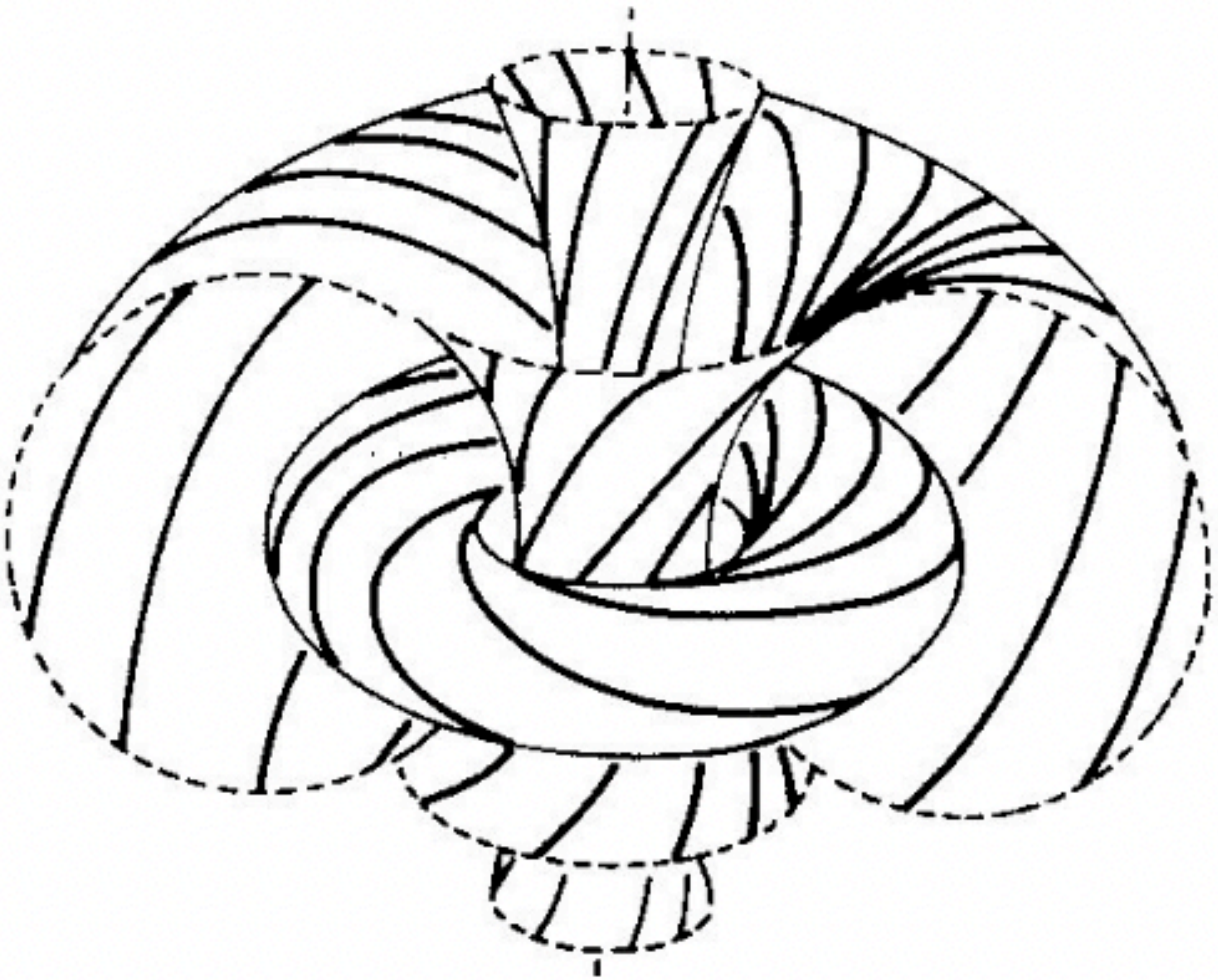
then rotations $\mathbb{C}^2/\mathbb{S}^1 = \mathbb{R}^3$

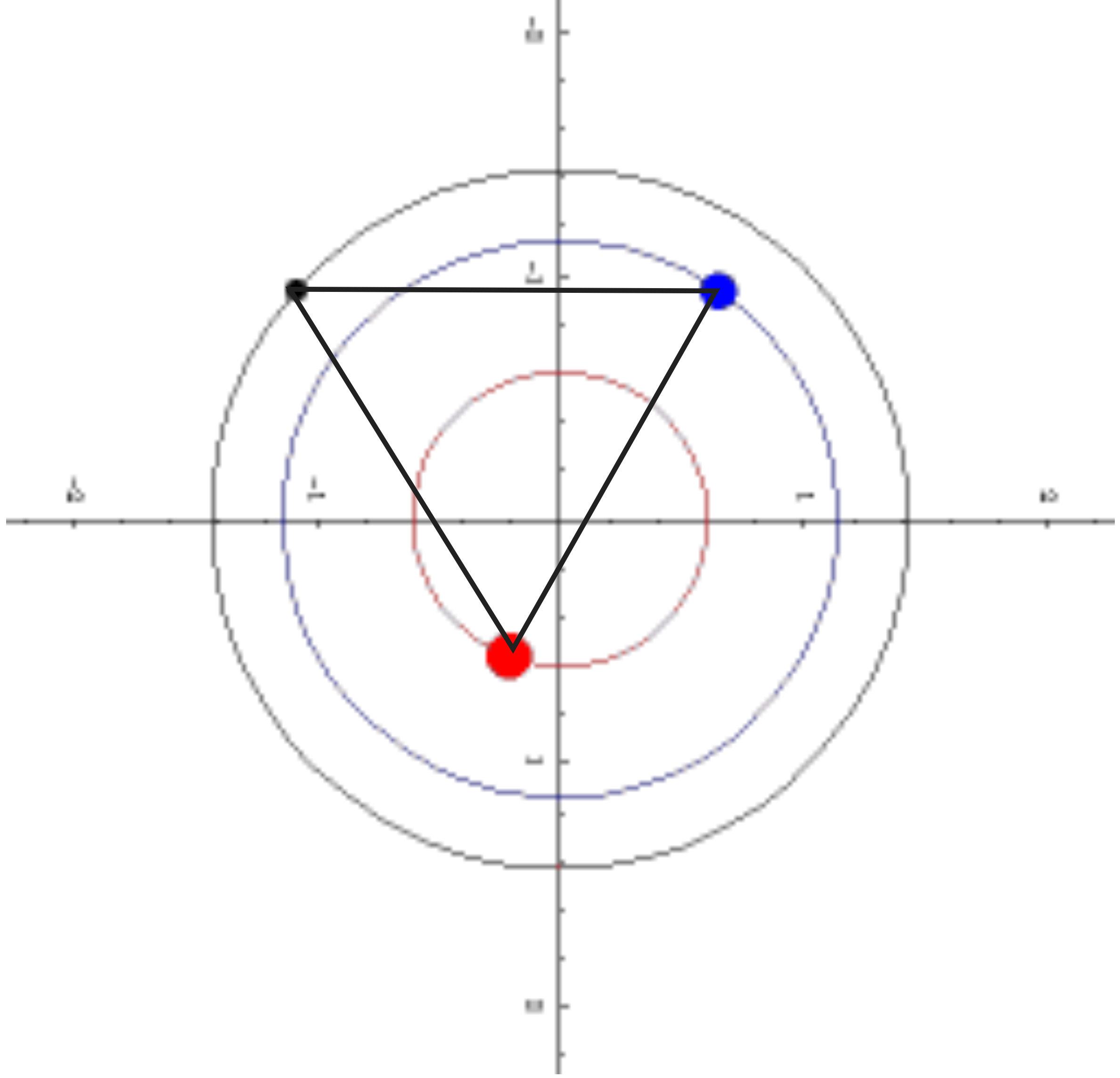
metrically: $\mathbb{R}^3 = Cone(\mathbb{S}^2(1/2))$

$\mathbb{S}^3 \subset \mathbb{C}^2$

Hopf!

$$\begin{array}{c}\mathbb{S}^3\\ \downarrow\\ \mathbb{S}^2\end{array}$$

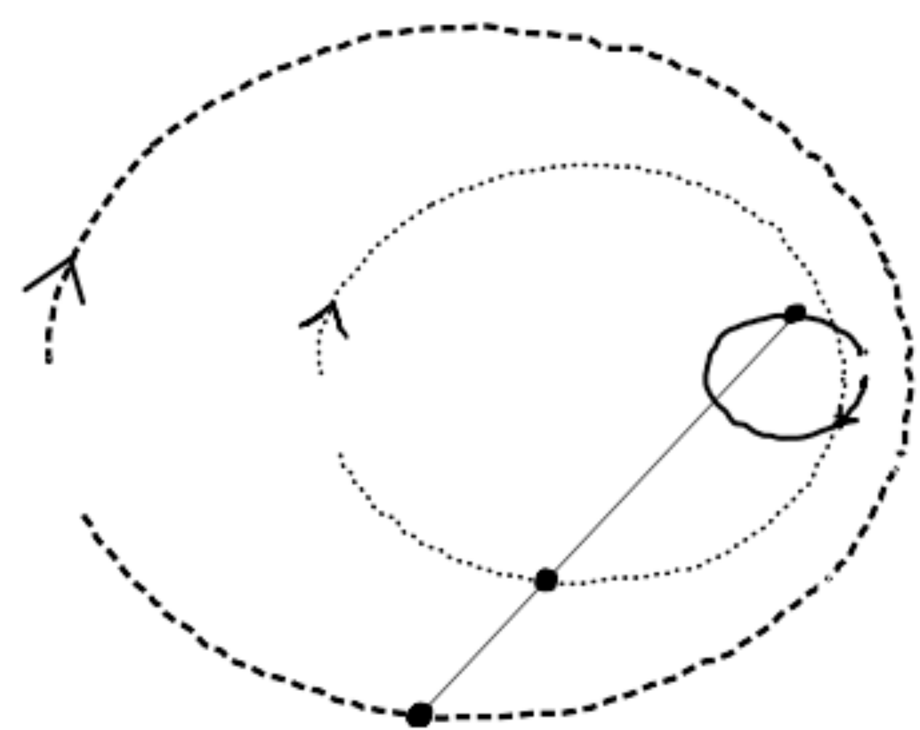




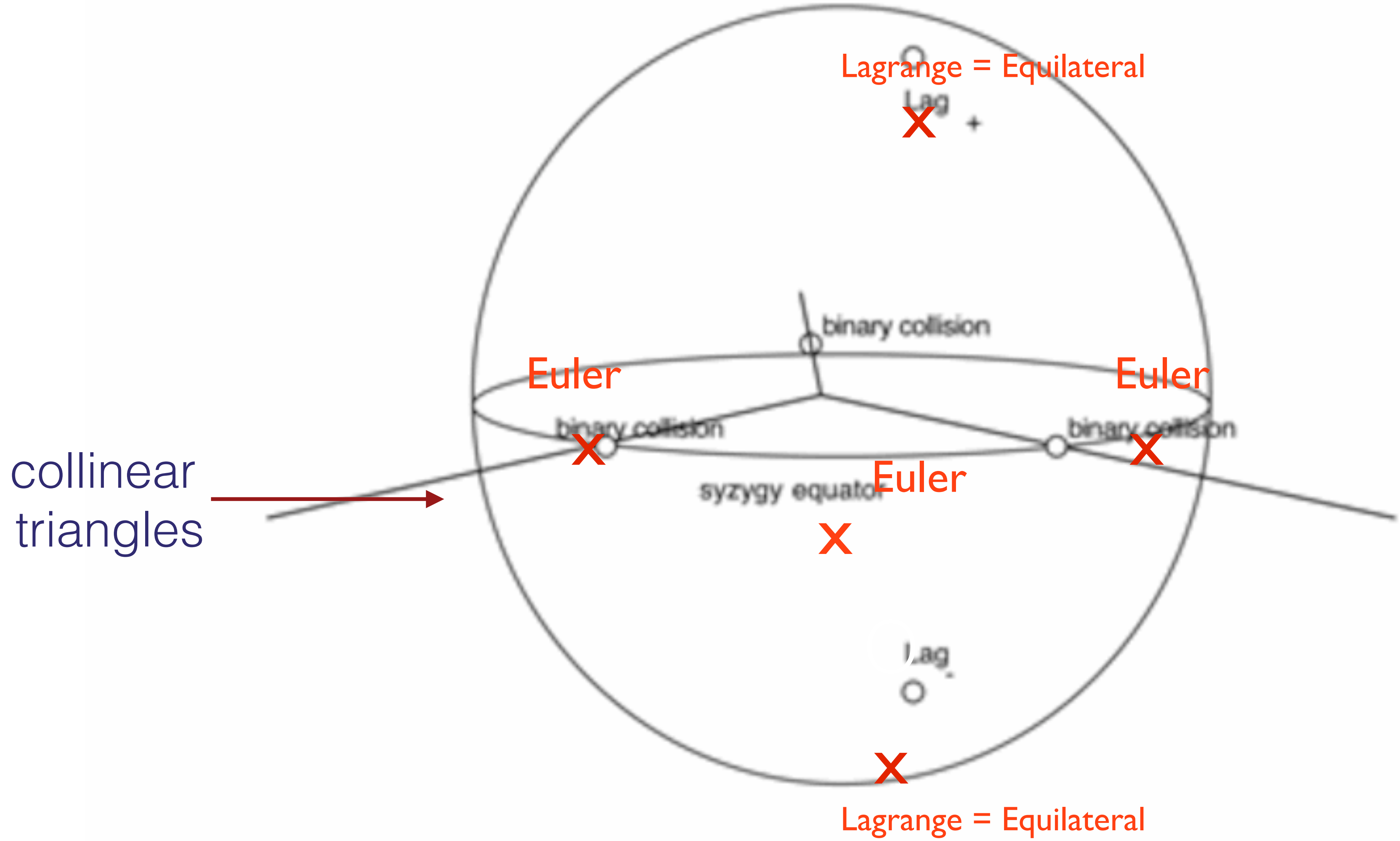
Lagrange's solutions.
Equilateral triangles.

1772

Euler's collinear solutions.   1767



Projected to the shape sphere, the corresponding curves do not move: they are points!



SHAPE SPHERE $\subset$ SHAPE SPACE

Oriented similarity classes of triangles

Oriented congruence classes of triangles

$$G \to Q \to S$$

group

config. space on which G acts

quotient space by G

How we (re)discovered the figure eight

We used the variational principle:

The extremals of a certain function A (= `action') on the PATH SPACE of Q solve Newton's equations.

$$A(q(\cdot)) = \int_0^T L(q(t), \dot{q}(t))dt$$

$$\mathrm{L(q,\ v) = K(v) + U(q)}$$

(Lagrangian)

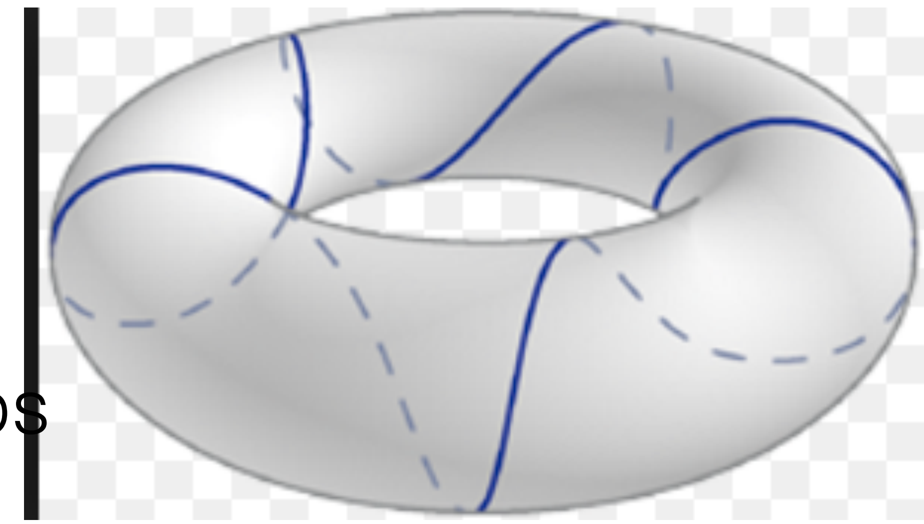
$$U = G(\frac{m_1 m_2}{r_{12}} + \frac{m_2 m_3}{r_{23}} + \frac{m_1 m_3}{r_{13}})$$

(neg. of potential energy)

Domain of A: paths q:[0,T] to Q with q(0) =q_0 fixed and q(T) = q_1 fixed.

We took inspiration from Riemannian geometry and topology:

Thm.
On a *compact* Riemannian manifold every ***free homotopy class (*)*** of loops is realized by a periodic *geodesic*.



Pf. Direct method of the calculus of variations: (1) fix such a class. (2) minimize the lengths of loops over all loops representing this class

length: $= \int_0^T L(q(t), \dot{q}(t))dt; \qquad L(q,v) = \sqrt{<v, v>_q}$

We applied this topological `direct method' idea to Newton's three-body eqns

a. Replace `length' by `action'

b. REALIZE THAT: A free homotopy class of loops in the ***collision-free*** planar 3-body configuration space

=

A conjugacy class in the pure braid group on 3 strands

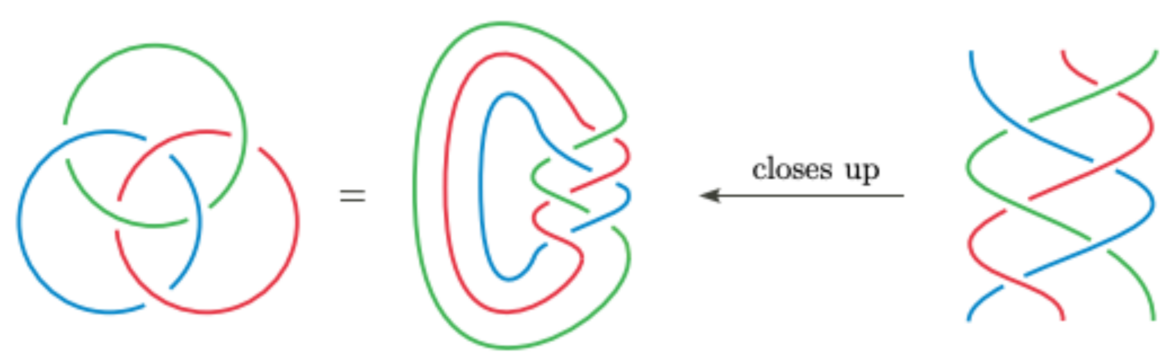
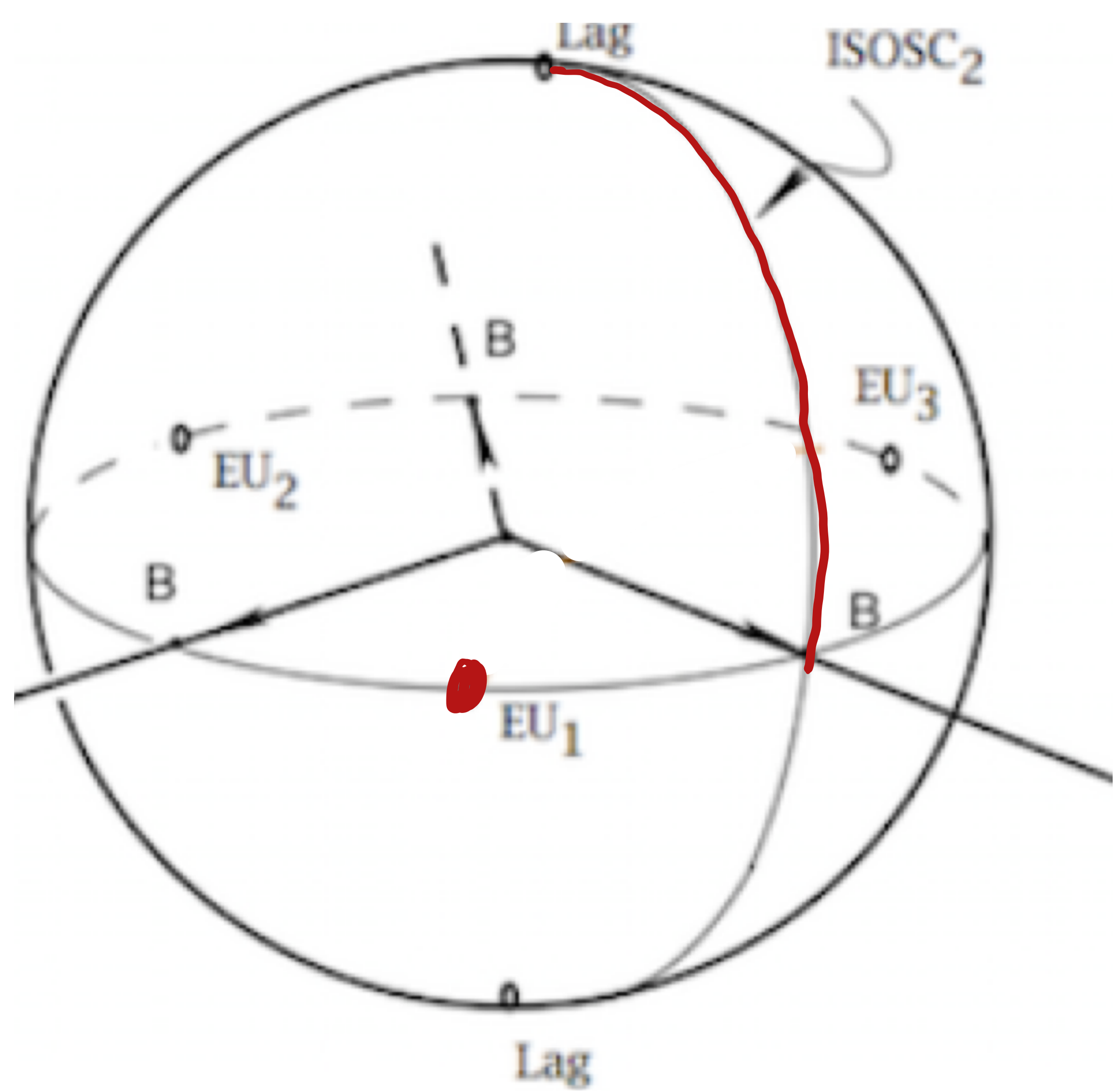


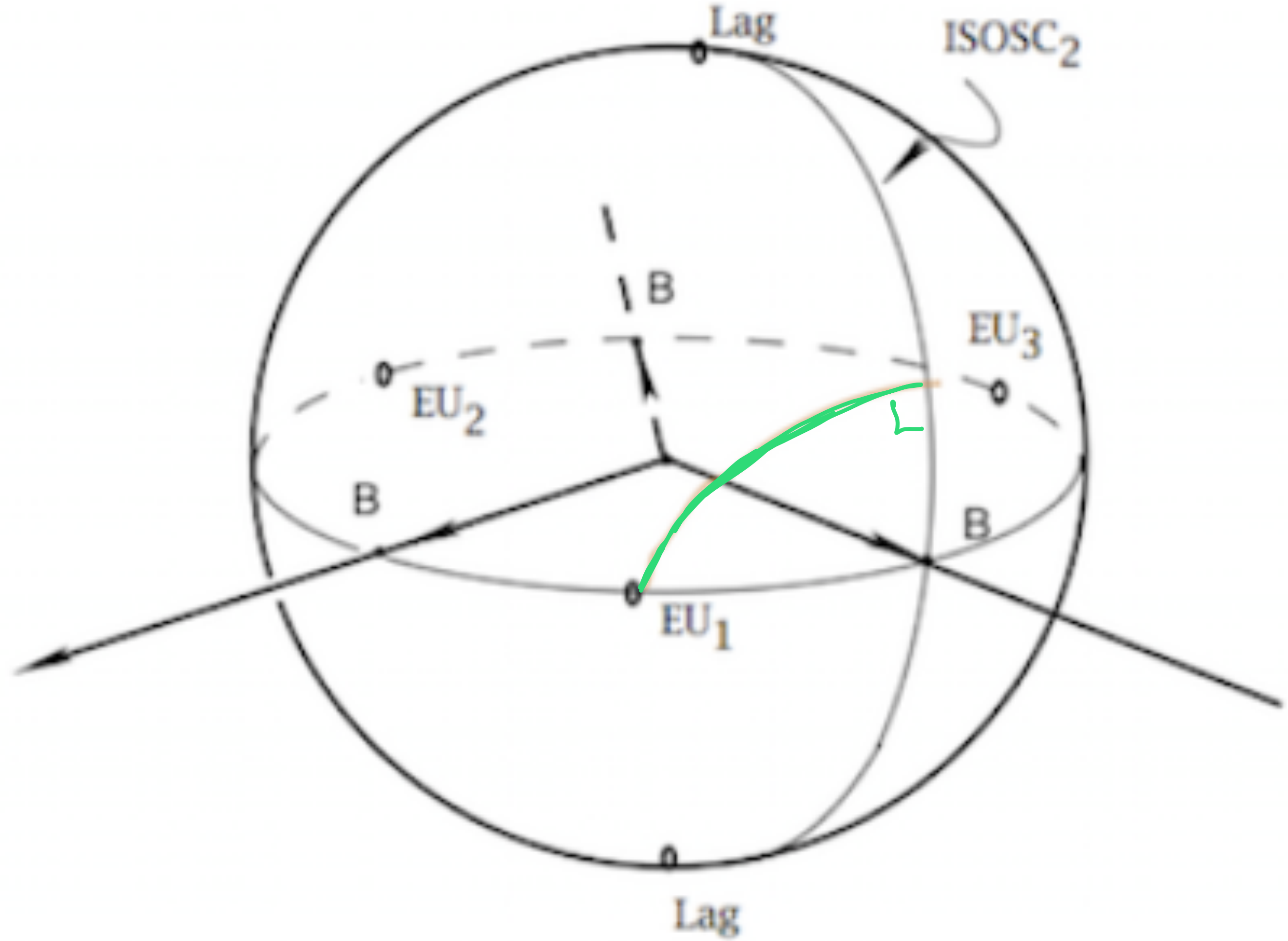
FIGURE 30. Borromean rings as the closure of a string link

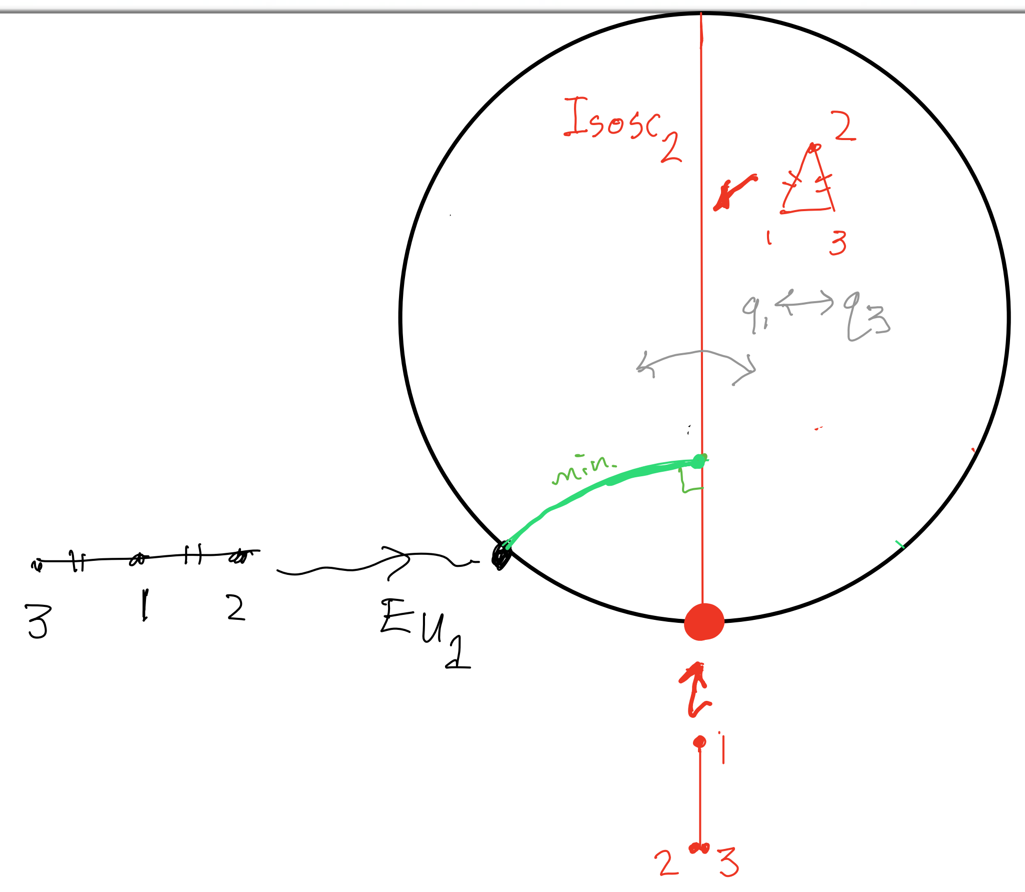
c. Push the variational principal down to shape space (don't insist loop closes up; rather only closes up modulo rotations) so as to minimize over loops realizing a given `projective pure braid'

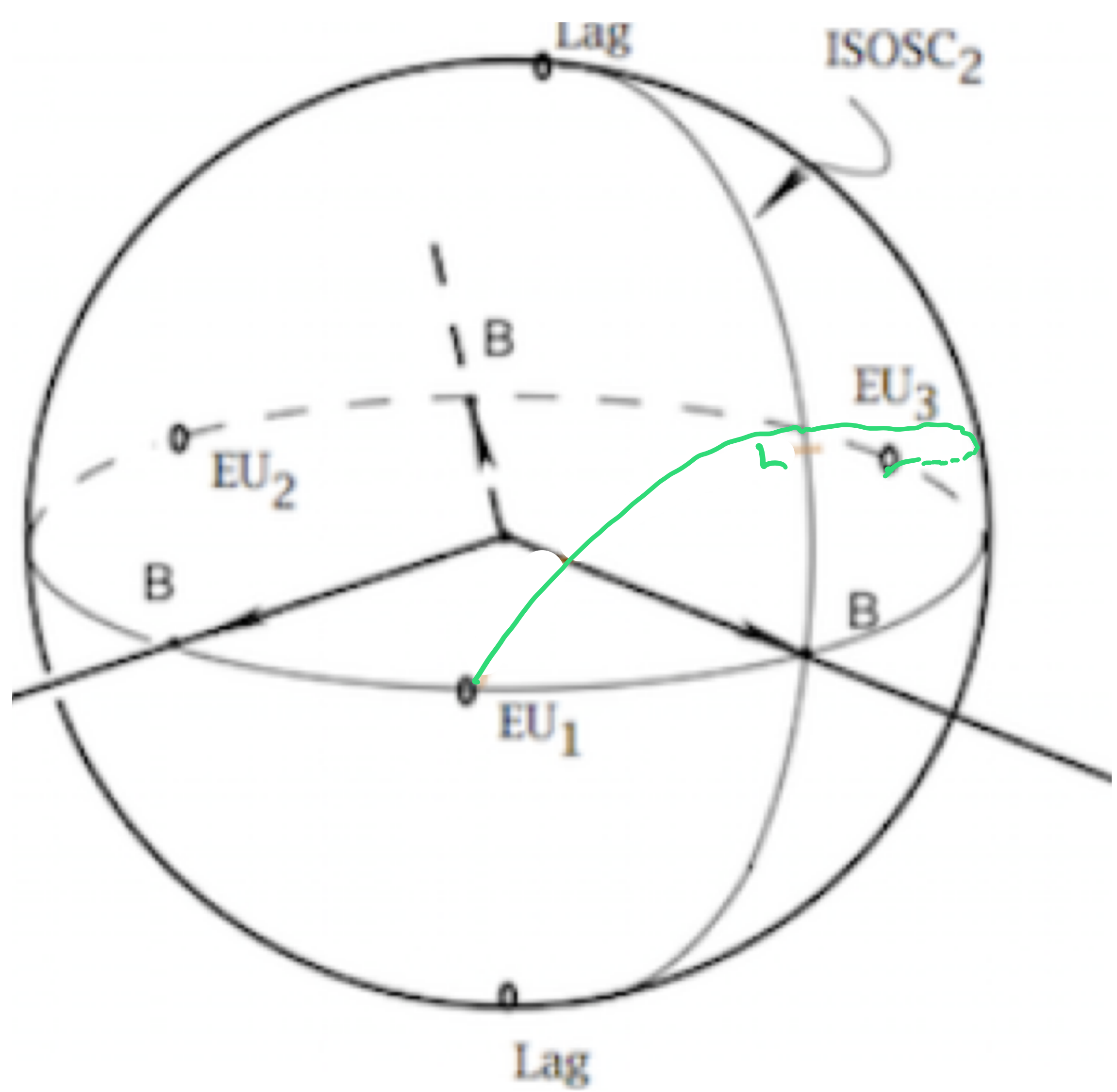
This strategy fails….
due to `tight binary loops'
converging to collision and
destroying topological constraint (*)

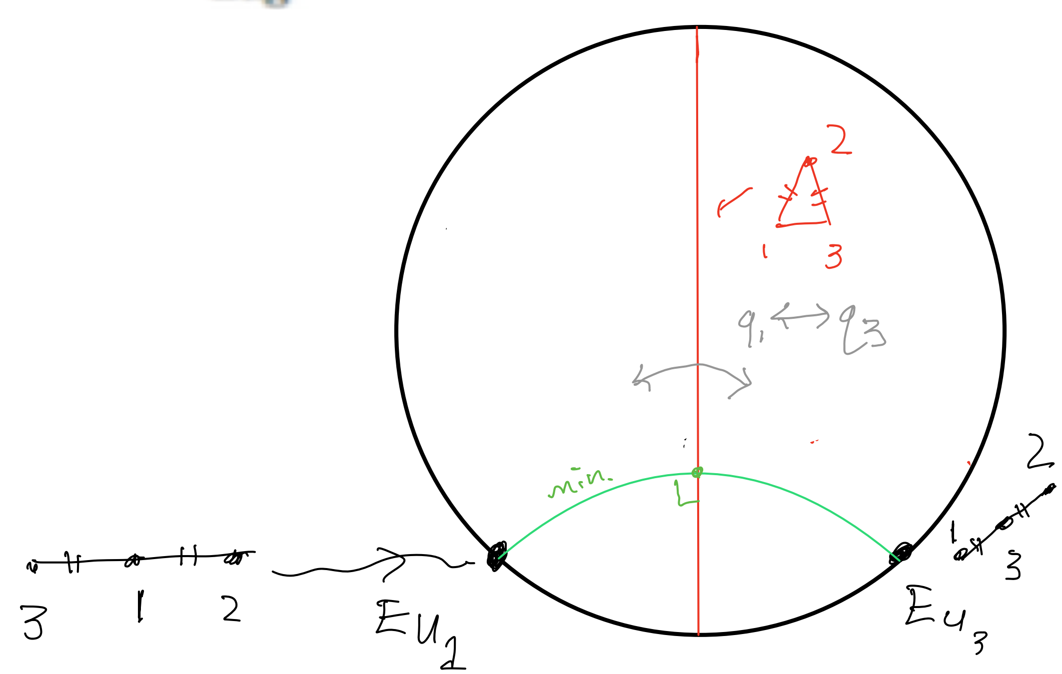
The strategy can be saved - made to work - if we take all three masses to be equal and impose additional discrete symmetries on the competing paths, symmetries arising from mass interchange.

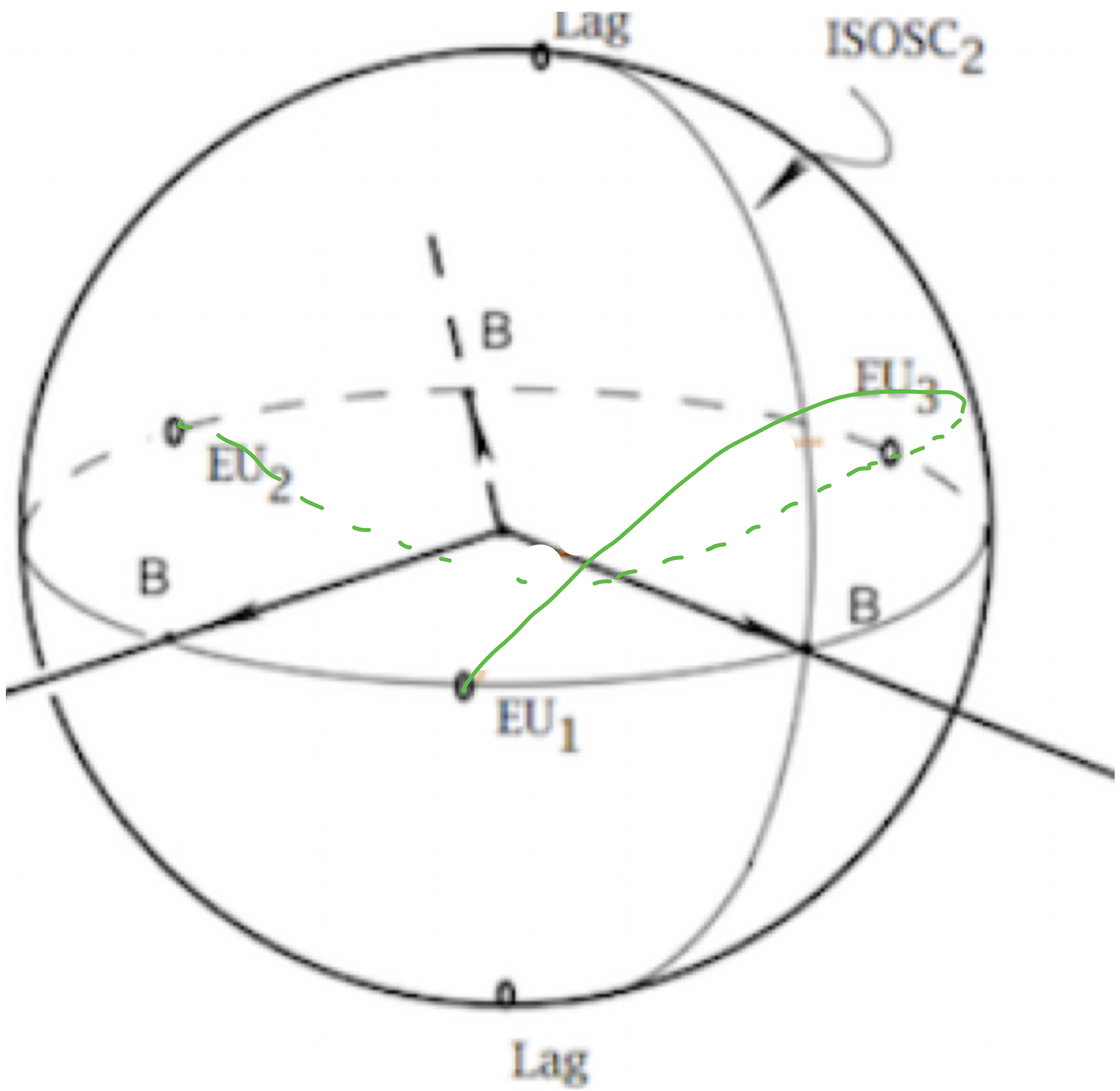

Lag
ISOSC2
B
EU3
EU2
B
B
EU1
Lag


Lag
ISOSC2
B
EU2
EU3
B
B
EU1
Lag

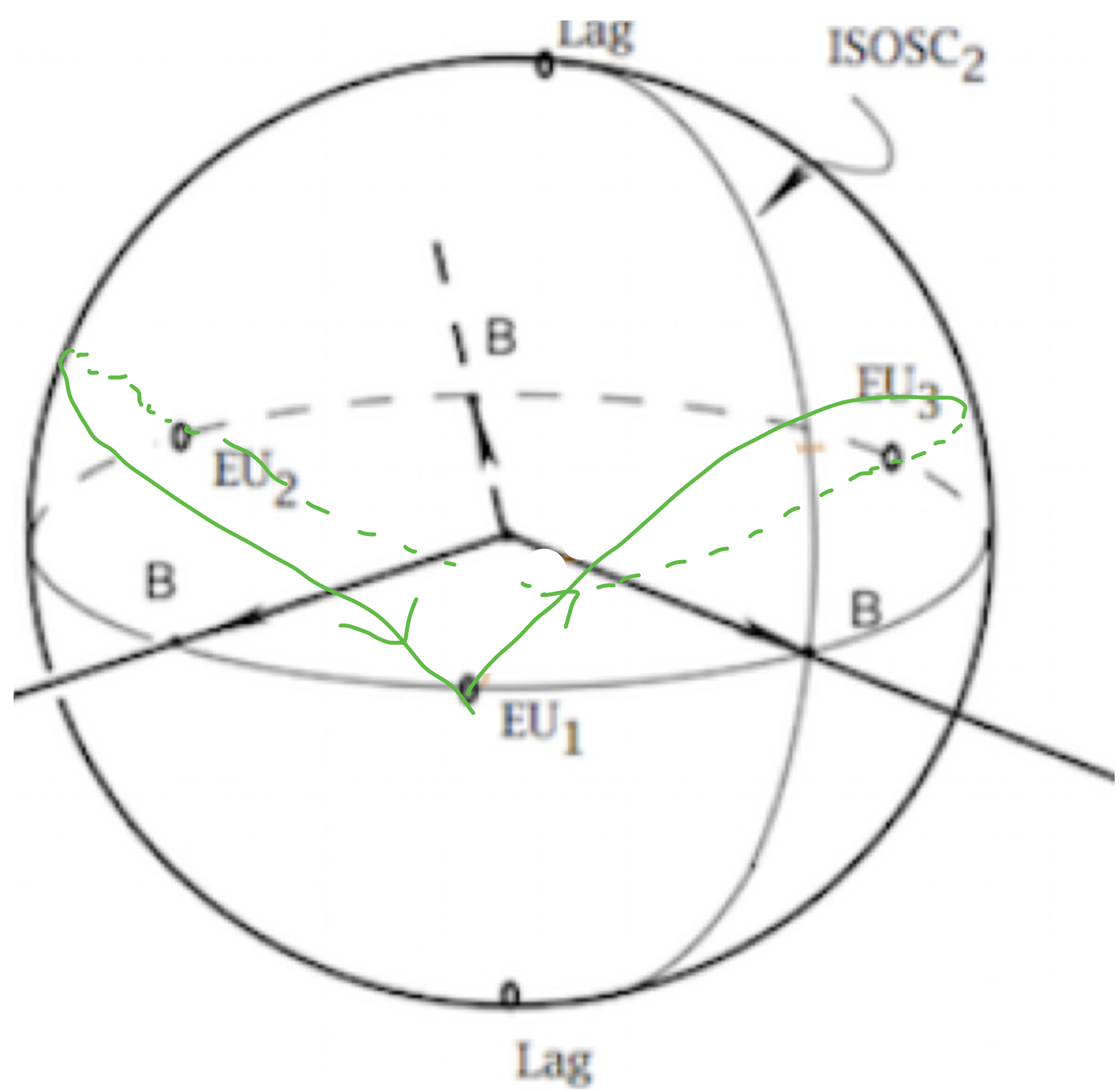

Isosc2
2
1 3
q1 ↔ q3
min.
3 1 2
EU1
1
2 3

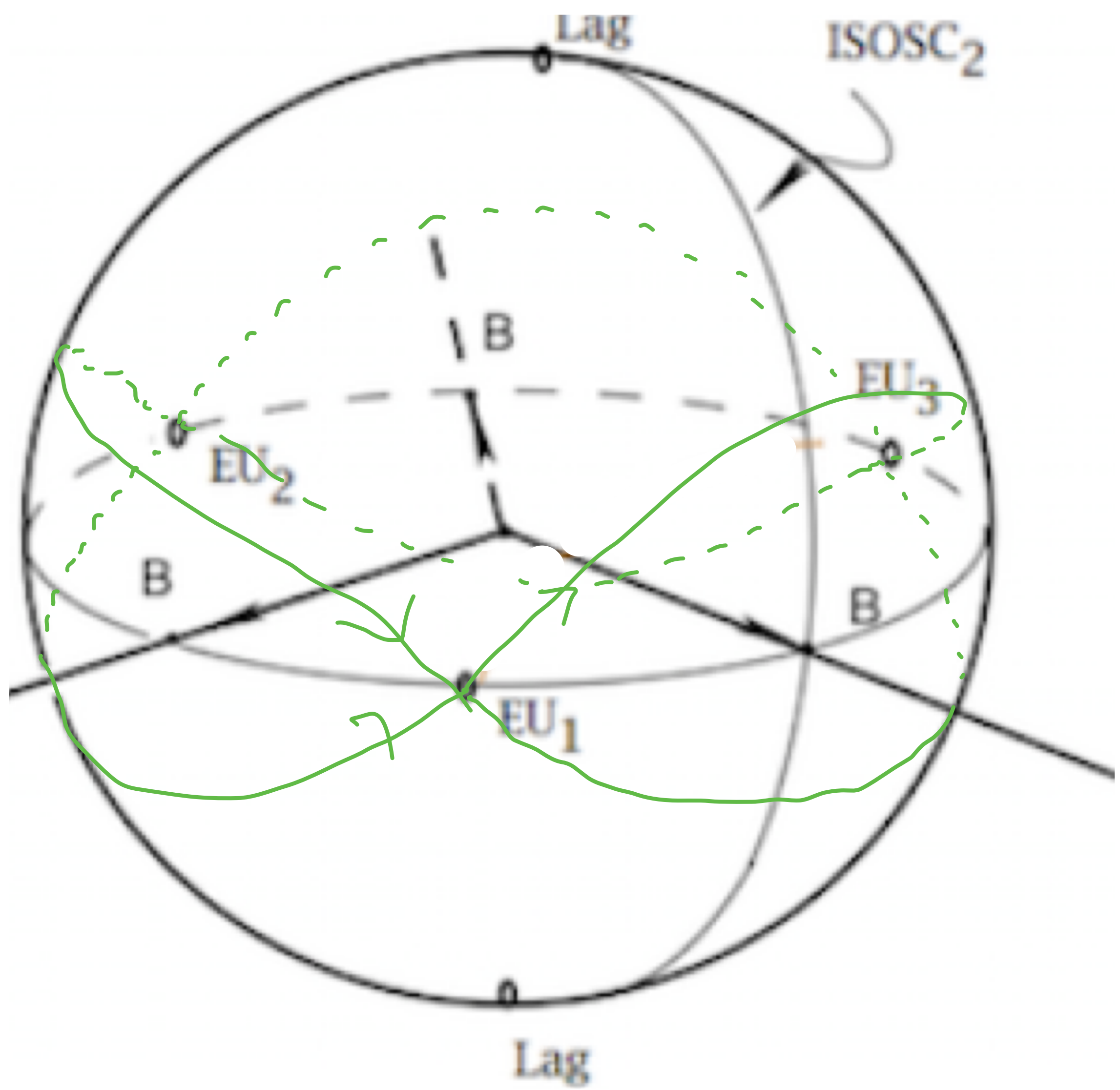

Lag
ISOSC2
B
EU3
EU2
B
B
EU1
Lag


2
1
3
q1 ↔ q3
min.
2
1
3
3
1
2
Eu1
Eu3

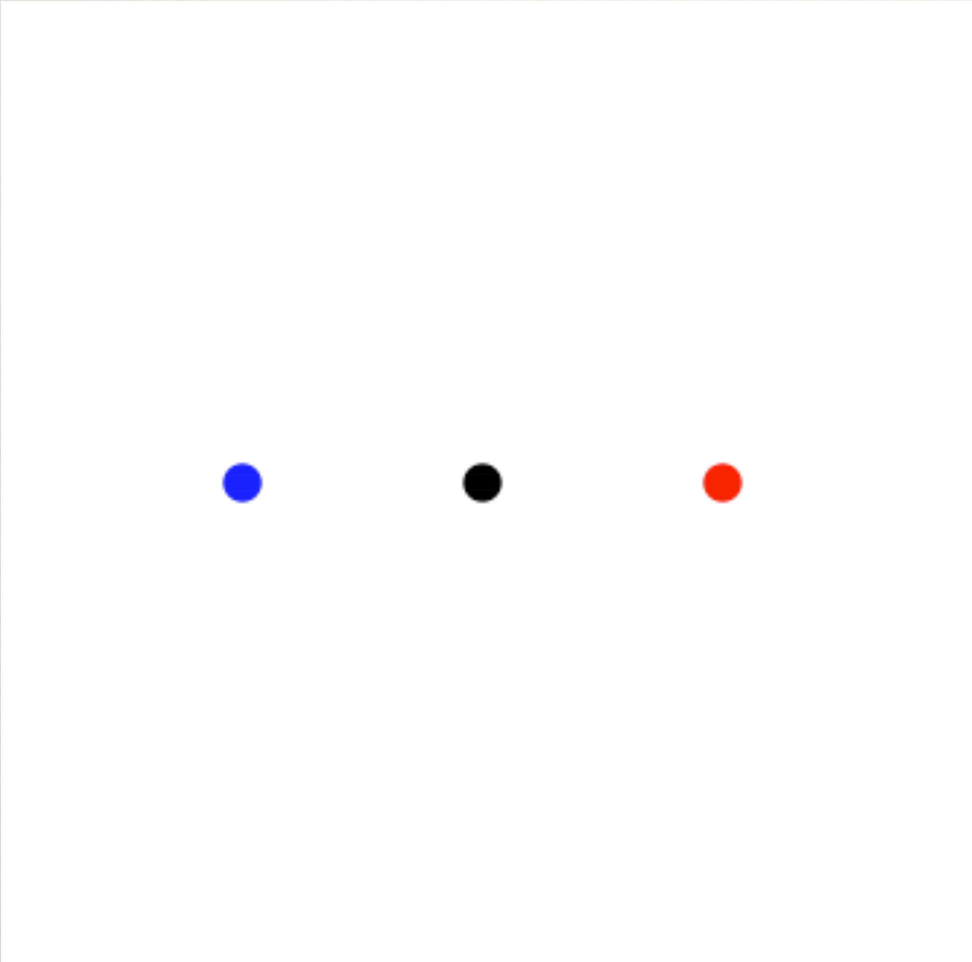
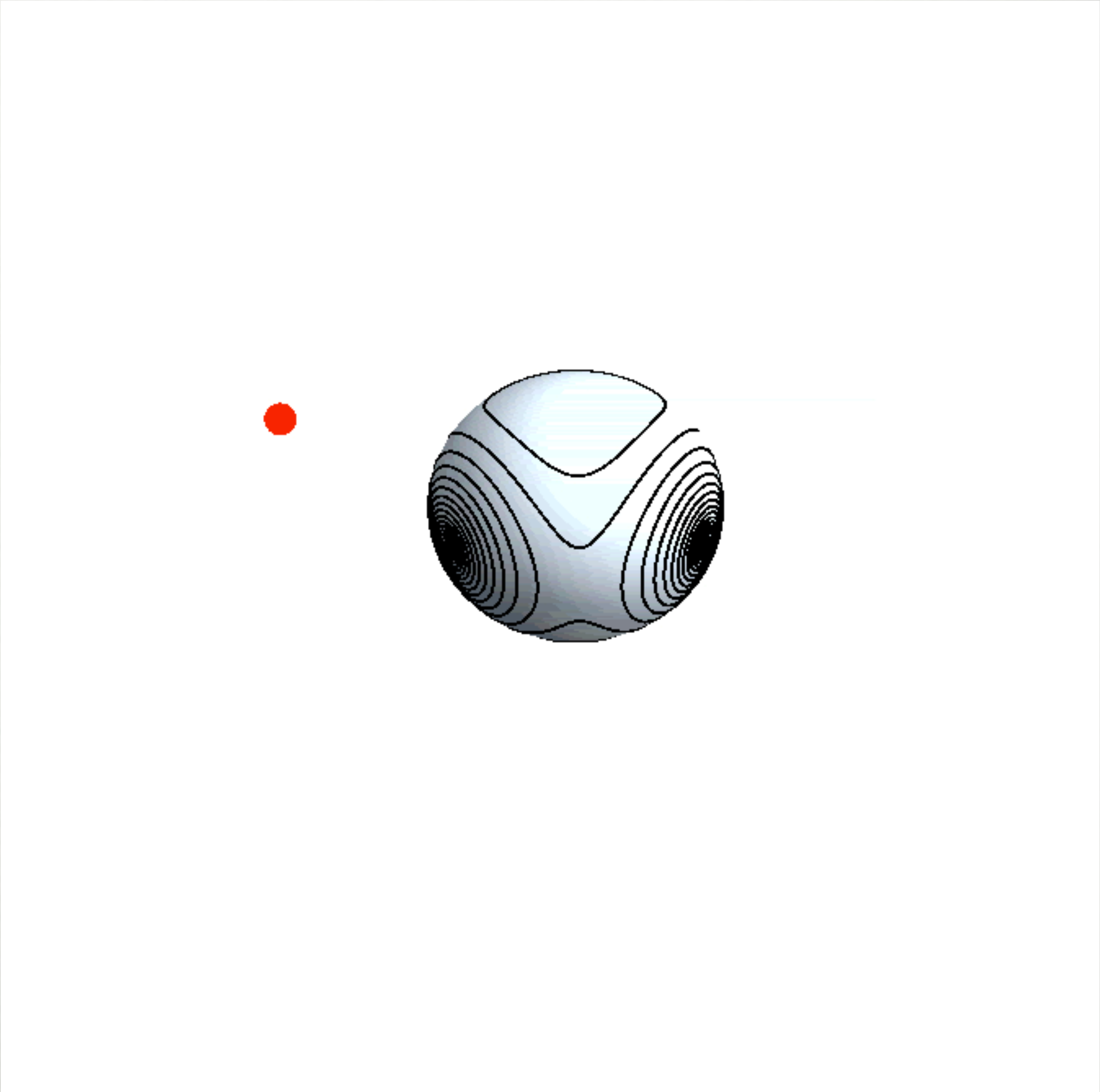


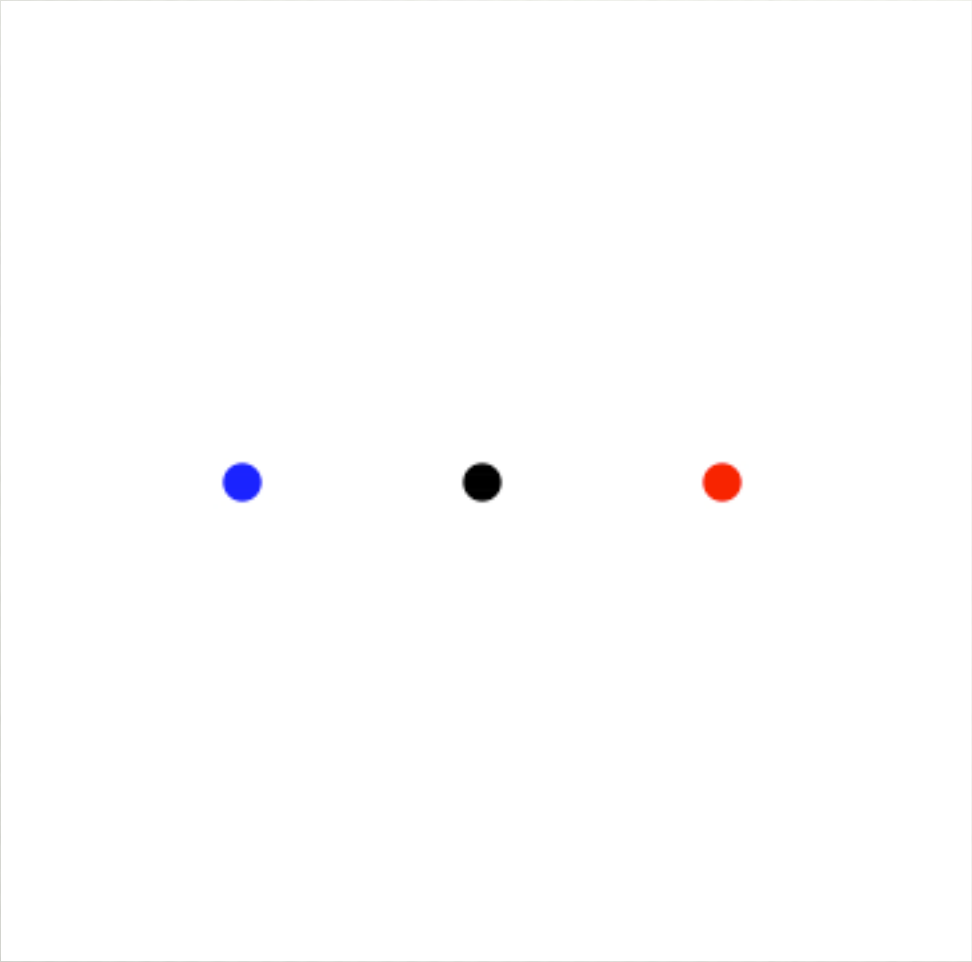
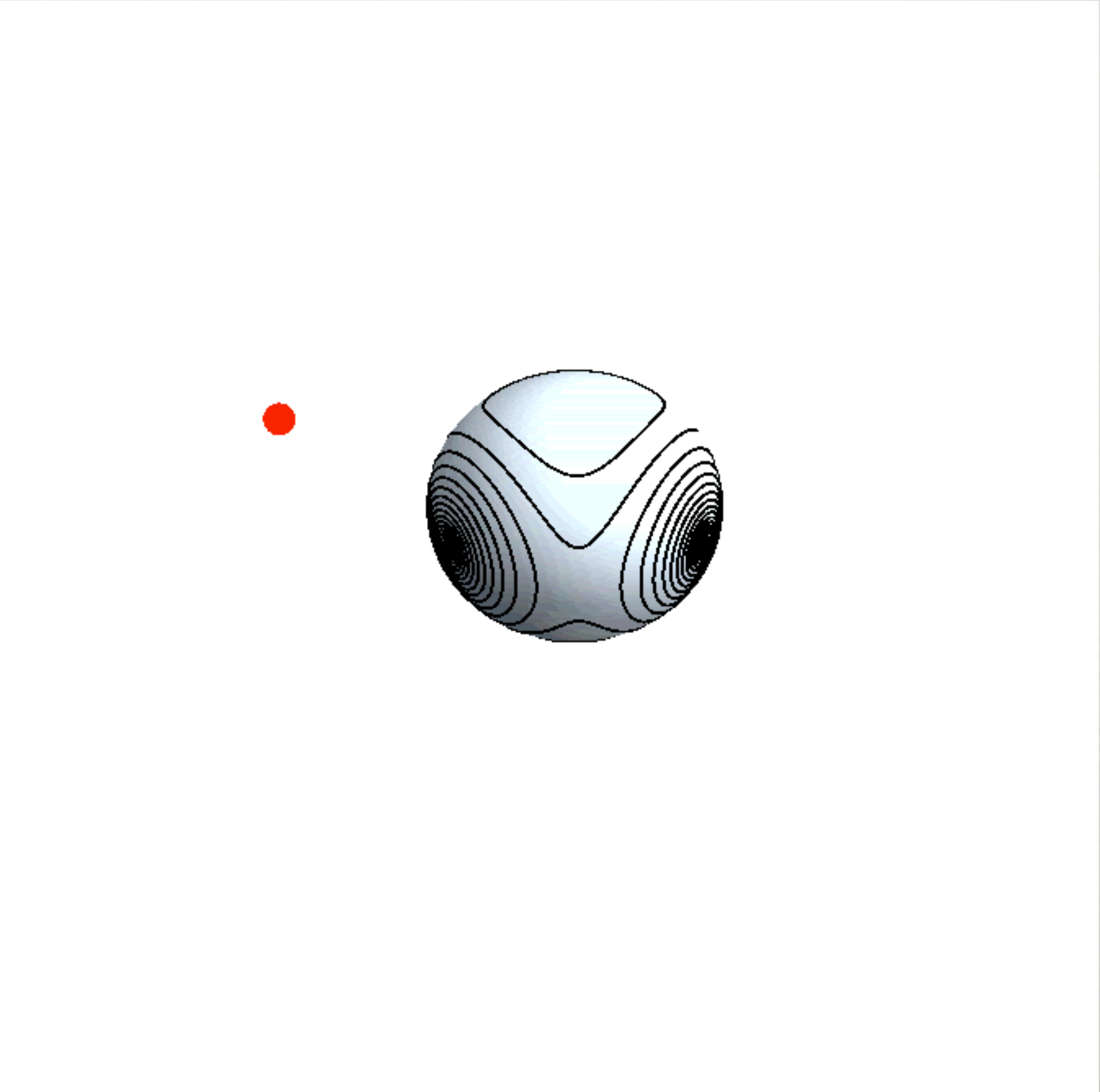
Lag
ISOSC2
B
EU3
EU2
B
B
EU1
Lag


Lag
ISOSC2
B
EU3
EU2
B
B
EU1
Lag



Lag
ISOSC2
B
EU3
EU2
B
B
EU1
Lag

Li-Liao :

movies.htmumericaltank.sjtu.edu.cn

Movies of the Collisionless Periodic Orbits in the Free-fall Three-body Pr on Shape Sphere

Xiaoming LI and Shijun LIAO

Shanghai Jiaotong University, China

Parameters:

Body mass: m_1, m_2, m_3

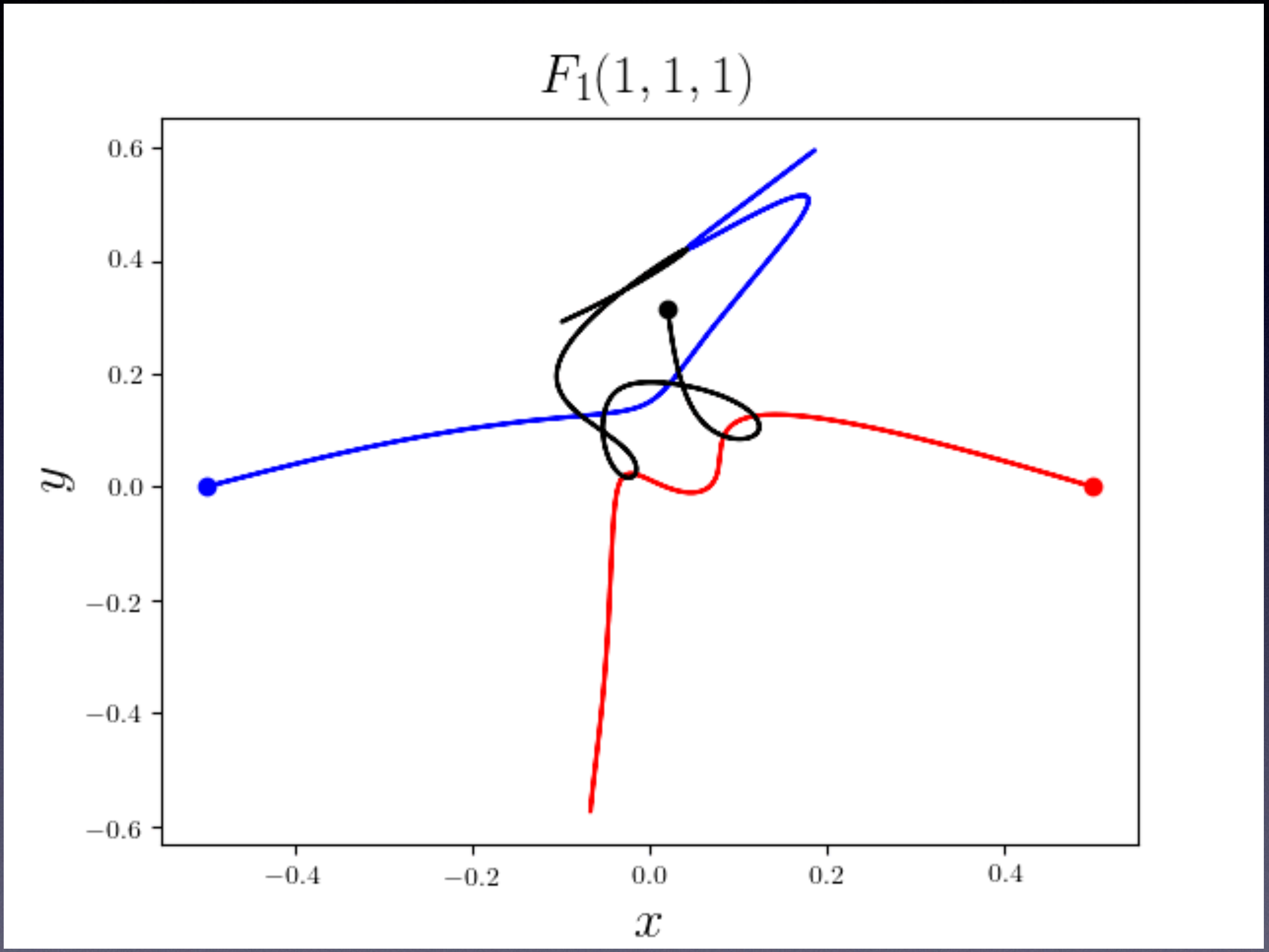
Newton's gravitational constant: G = 1

Initial positions: (-0.5, 0), (0.5, 0), (x, y)

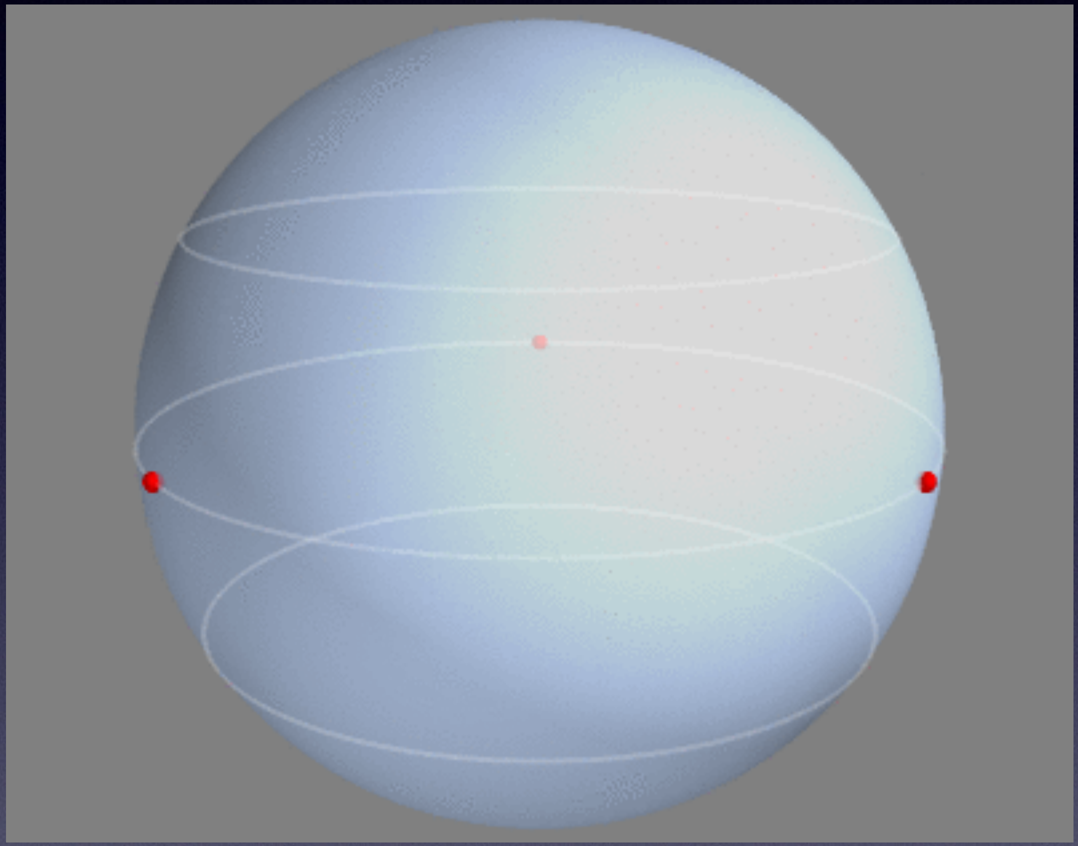
Initial velocities: (0,0), (0,0), (0,0)

T is the period

$F_i(m_1,m_2,m_3)$	x	y	T	Real Space	Shape Sphere
$F_1(1,1,1)$	0.0207067154	0.3133550361	2.1740969264	movie	movie
$F_2(1,1,1)$	0.2053886532	0.1952668419	1.6896364928	movie	movie
$F_3(1,1,1)$	0.0562664280	0.4691503375	4.5419125588	movie	movie
$F_4(1,1,1)$	0.1846729355	0.5753740774	5.1586391029	movie	movie
$F_5(1,1,1)$	0.0880412663	0.5488924176	4.9647695145	movie	movie
$F_6(1,1,1)$	0.3142334050	0.5384825297	4.8672002993	movie	movie
$F_7(1,1,1)$	0.0741834378	0.5324424488	5.4455591108	movie	movie
$F_8(1,1,1)$	0.2871136862	0.5252008584	5.1183291764	movie	movie



$F_1(1,1,1)$
0.6
0.4
0.2
0.0
−0.2
−0.4
−0.6
y
−0.4
−0.2
0.0
0.2
0.4
x



Quotient map: $\mathbb{C}^3 \to \mathbb{R}^3$ from Configurations to shapes

$\mathbb{C}^3 \to^{modtranslations} \mathbb{C}^2 \to^{modrotations} \mathbb{R}^3$ is:

$\mathbb{C}^3 \to^{Jacobi} \mathbb{C}^2 \to^{Normalization} \mathbb{C}^2 \to^{`Hopf'} \mathbb{R}^3$

Jacobi:

$(q_1, q_2, q_3) \mapsto (q_2 - q_1, q_3 - (\frac{m_1}{m_1+m_2} q_1 + \frac{m_2}{m_1+m_2} q_2)) = (Y_0, Y_1)$

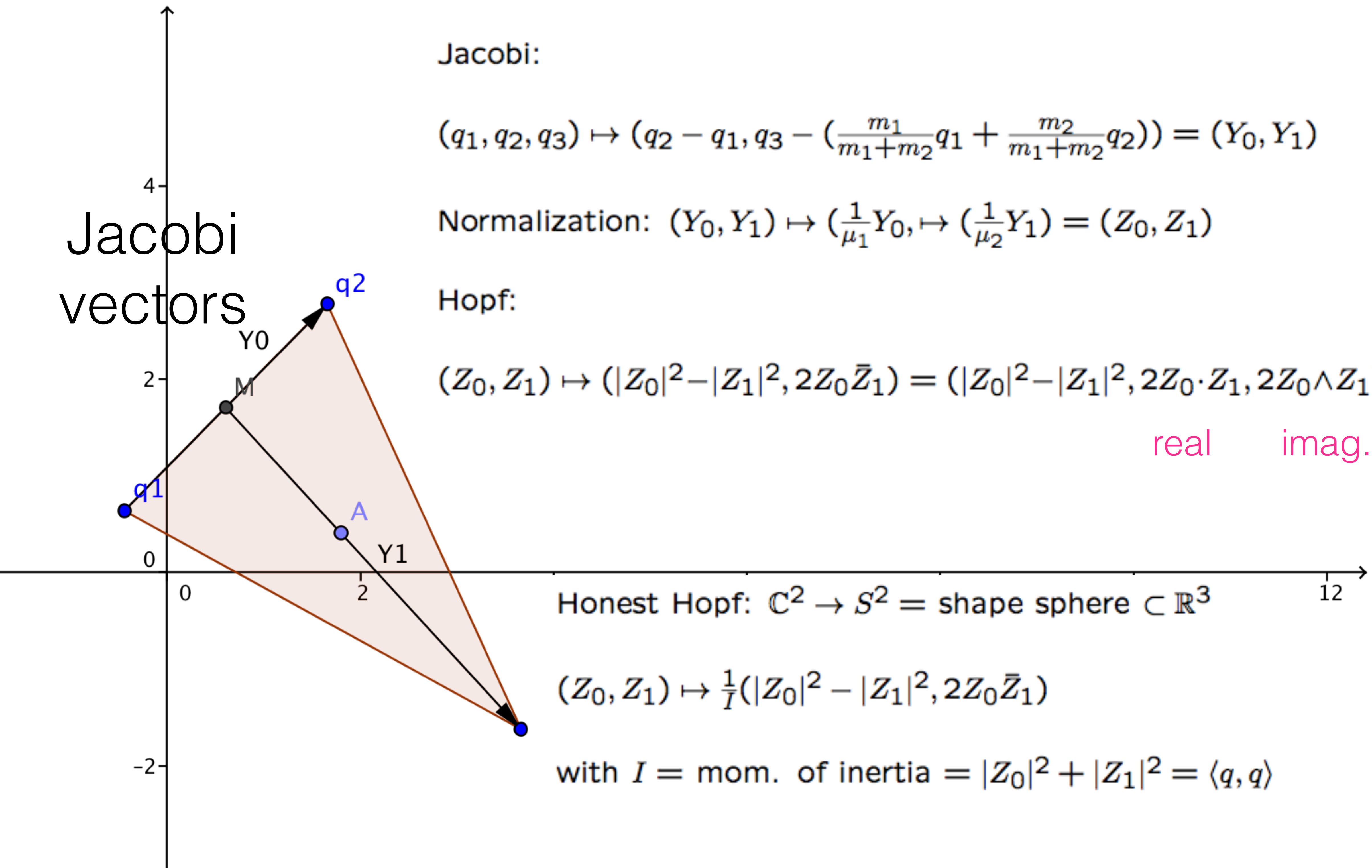
Normalization: $(Y_0, Y_1) \mapsto (\frac{1}{\mu_1} Y_0, \mapsto (\frac{1}{\mu_2} Y_1) = (Z_0, Z_1)$

Hopf:

$(Z_0, Z_1) \mapsto (|Z_0|^2 - |Z_1|^2, 2 Z_0 \bar{Z}_1) = (|Z_0|^2 - |Z_1|^2, 2 Z_0 \cdot Z_1, 2 Z_0 \wedge Z_1)$

real imag.

Jacobi vectors



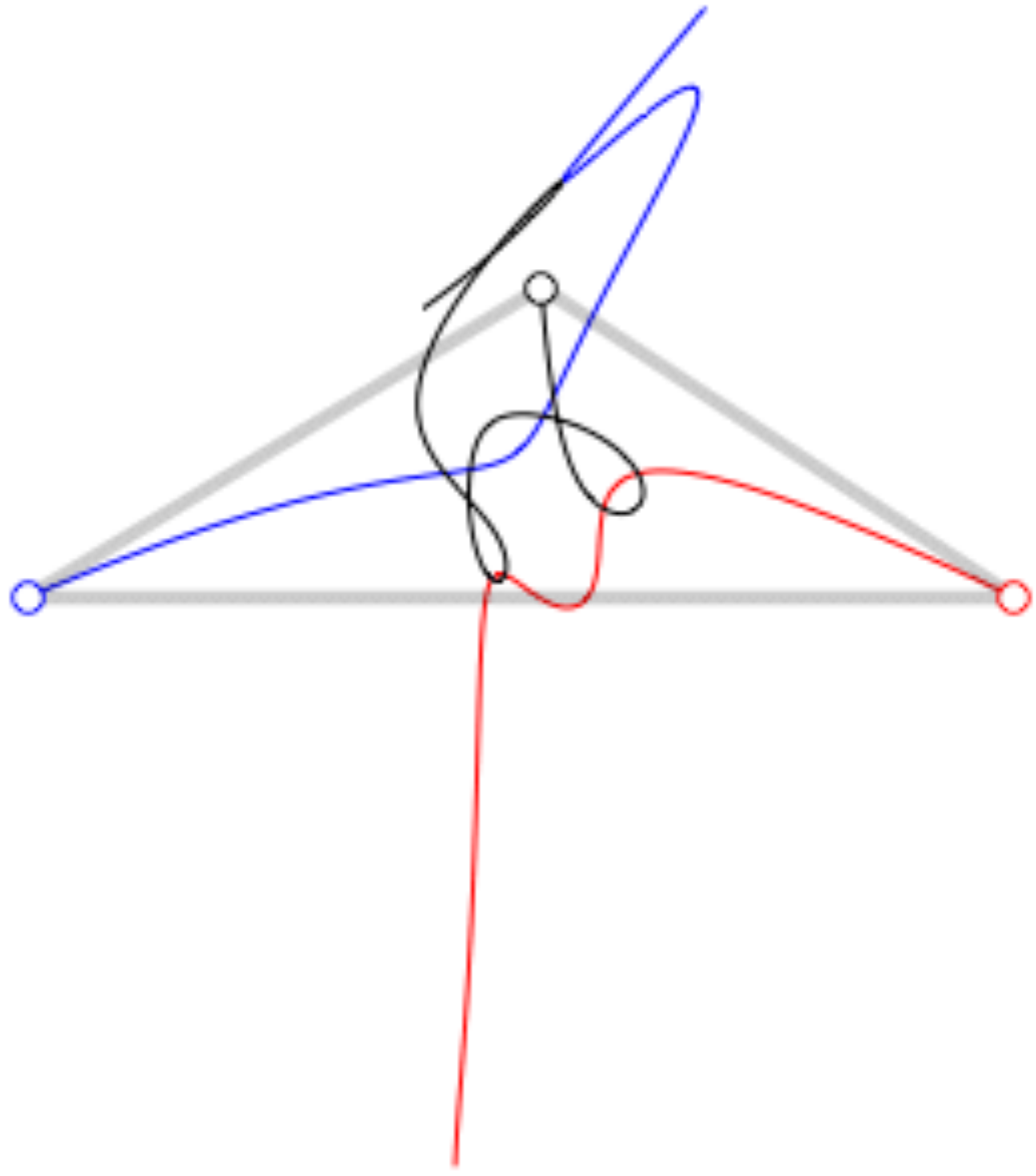
Honest Hopf: $\mathbb{C}^2 \to S^2 =$ shape sphere $\subset \mathbb{R}^3$

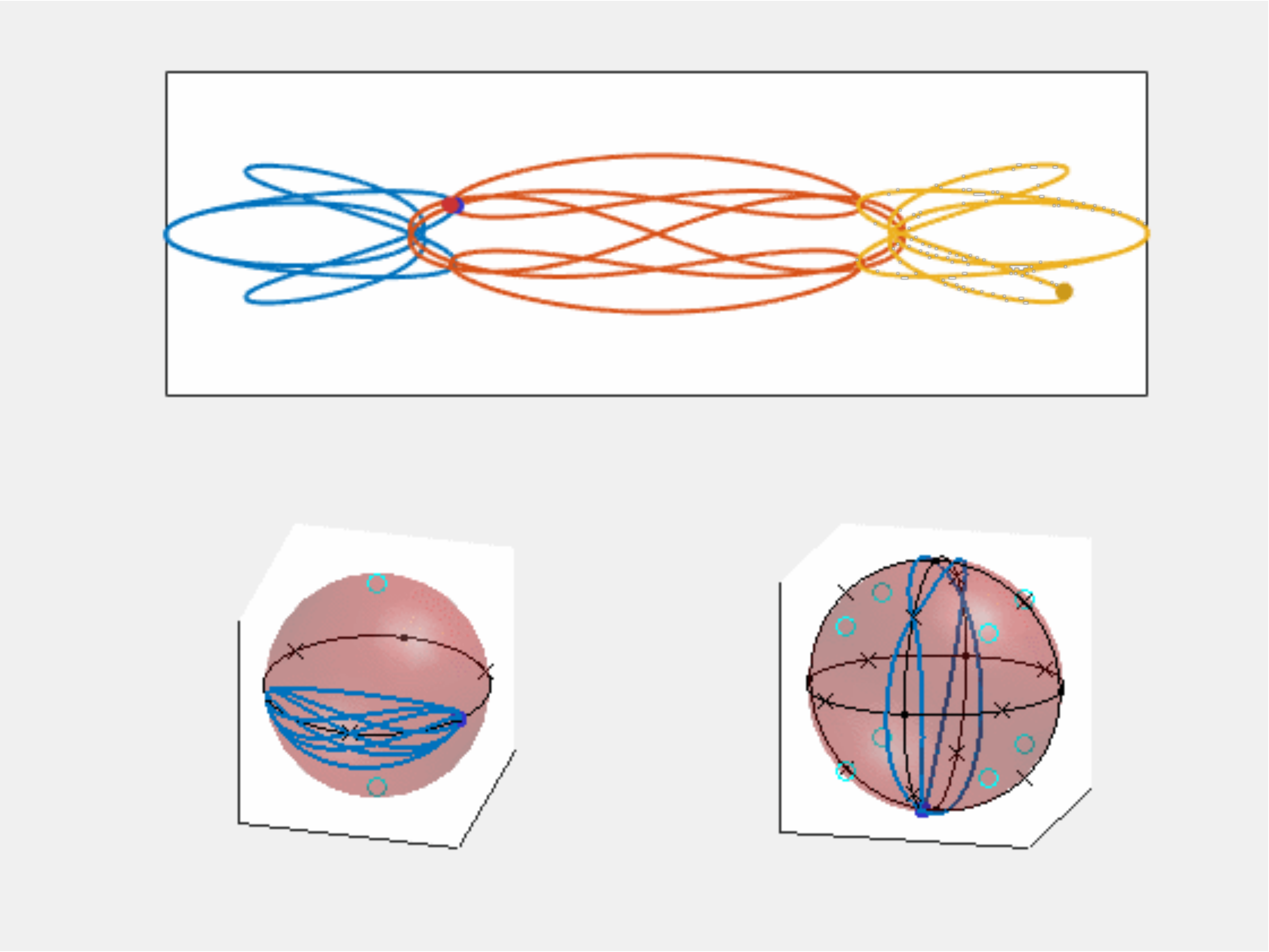
$(Z_0, Z_1) \mapsto \frac{1}{I}(|Z_0|^2 - |Z_1|^2, 2 Z_0 \bar{Z}_1)$

with $I =$ mom. of inertia $= |Z_0|^2 + |Z_1|^2 = \langle q, q \rangle$

Fini

Overflow:

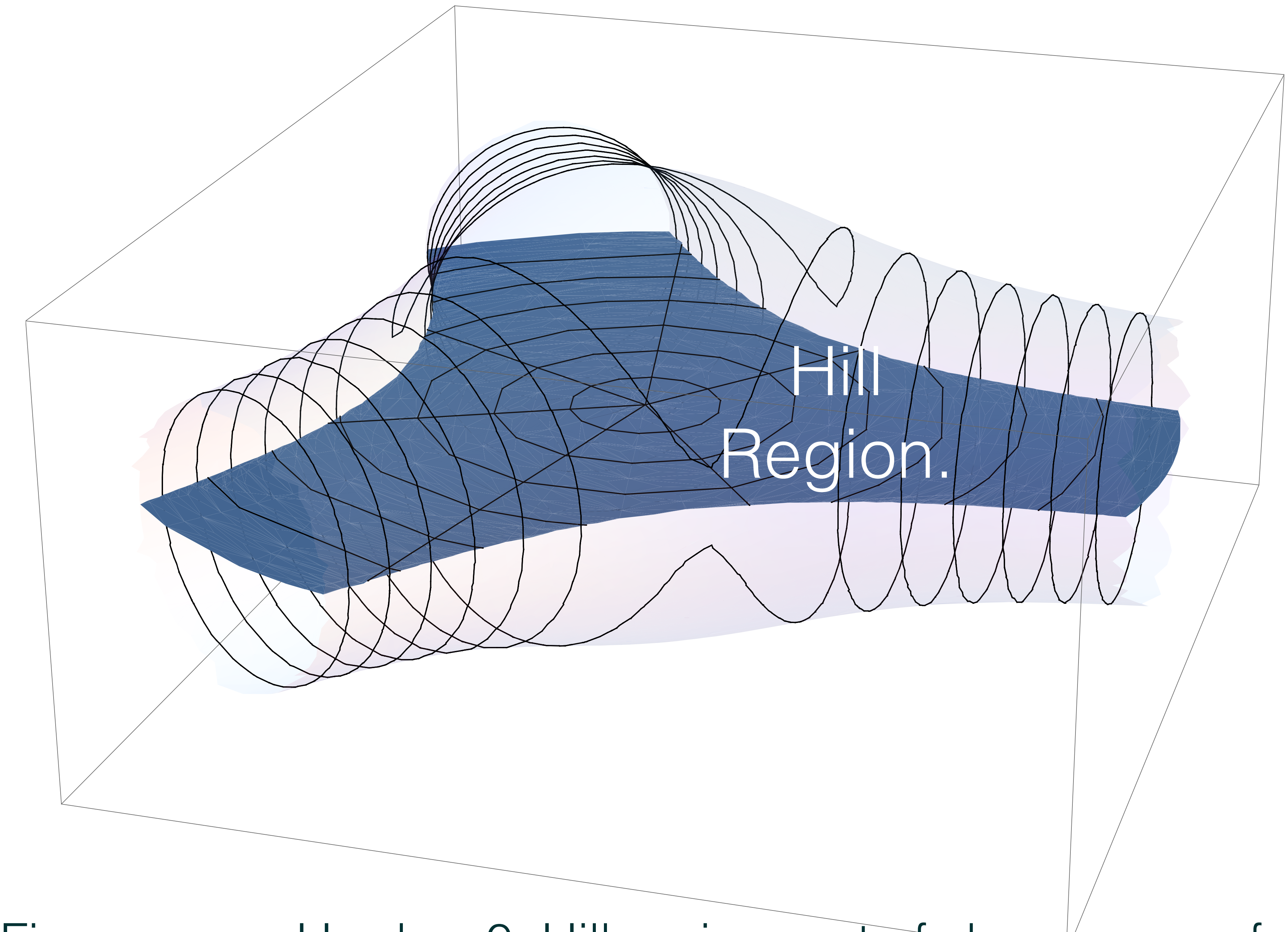




`Burrau' or
Pythagorean 3-4-5
three body problem (*)

Overflow!

(*): Greg Laughlin, UCSC made film w
Burlisch-Stoer integrator



Fix energy = H =-h < 0. Hill region:part of shape space for which there is a v and H(q,v) = -h. Domain where motion occurs. Identical to region with U(q) > +h

Under the spell of the gauge principle' -t'Hooft

Under the spell of the variational principle

-Maupertuis, Hamilton, Lagrange, Feynman…

(me)

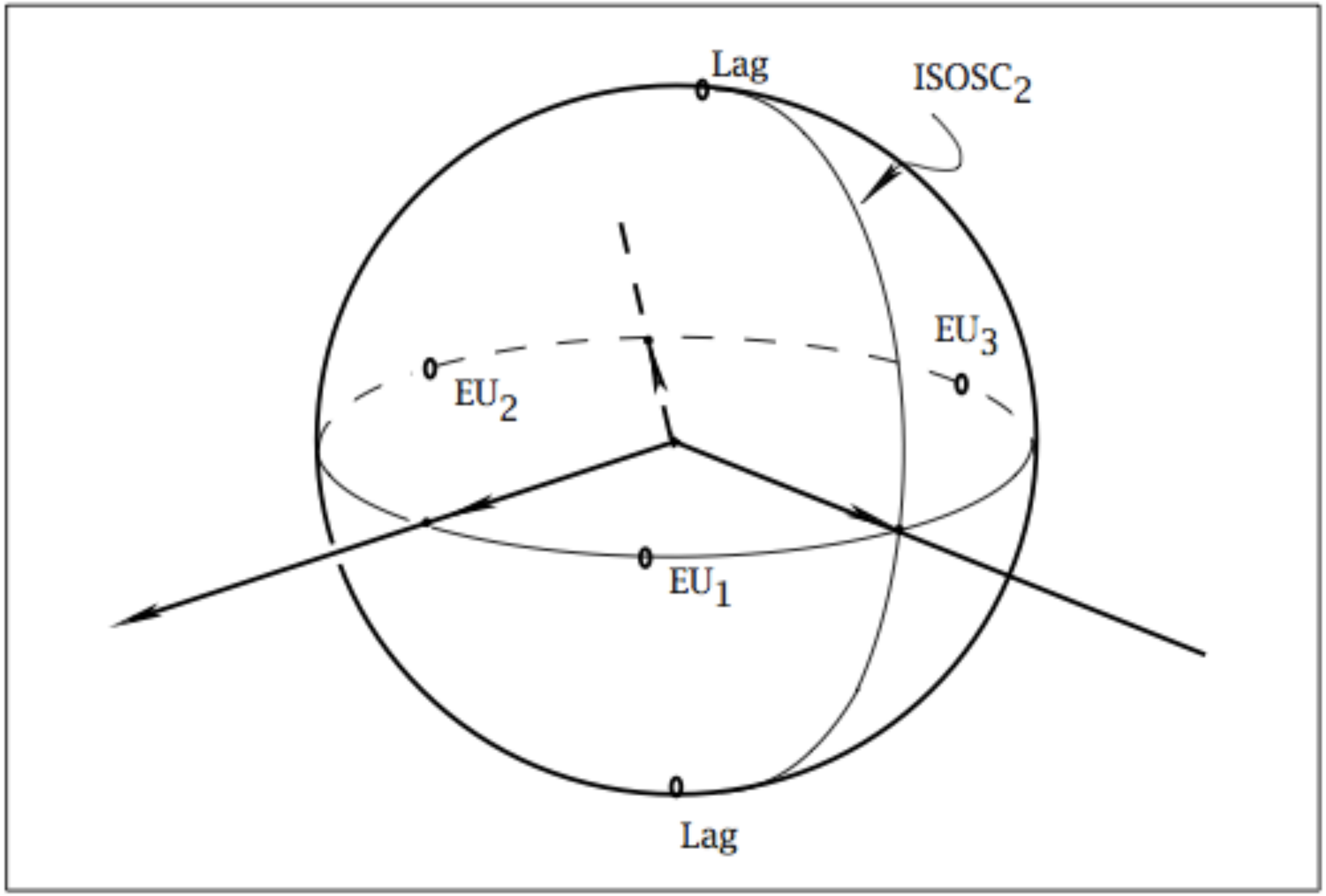


Figure 4. The shape sphere.

Figure 4. The shape sphere.