Falling Cats, …

organized around 2 open questions

Q1. in subRiemannian geometry:
Is every geodesic smooth?

Q2. For the <u>planar</u> N-body prob:
Is every braid type realized by a collision-free zero angular momentum solution?

= Hot off the press =
posted yesterday!

An Answer to Q1.

<u>No</u>. $\exists$ a sR structure on $\mathbb{R}^3$ admitting a sR geodesic which is C^2 but not C^3. Explicitly

$$\gamma(s) = (s^{5/2}, s, 0) + o(s^{5/2})$$

with $s =$ arc length $0 \leq s \leq \varepsilon$.

Not all sub-Riemannian minimizing geodesics are smooth

Alessandro Socionovo, Yacine Chitour, Frédéric Jean, Roberto Monti, Ludovic Rifford, Ludovic Sacchelli, Mario Sigalotti

We consider the sub-Riemannian structure (Δ, g) in $\mathbb{R}^3$ with coordinates (x_1, x_2, x_3), generated by an orthonormal family of vector fields $\{X^1, X^2\}$ defined as

$$X^1 = \partial_1 \quad \text{and} \quad X^2 = \partial_2 + P(x)^2\partial_3,$$

where

$$P(x) = x_1^2 - x_2^m \qquad \forall x = (x_1, x_2, x_3) \in \mathbb{R}^3,$$

and m is an odd integer satisfying $m \geq 5$. Besides the motivations described above, the counterexample took this particular form after a study of several types of possible examples in [18], its structure (in particular with the square of P) being inspired by the Liu–Sussmann example [10].

this new news changes my talk & talk structure

what you see here is culled from the planned talk. The given talk deviated at various times.

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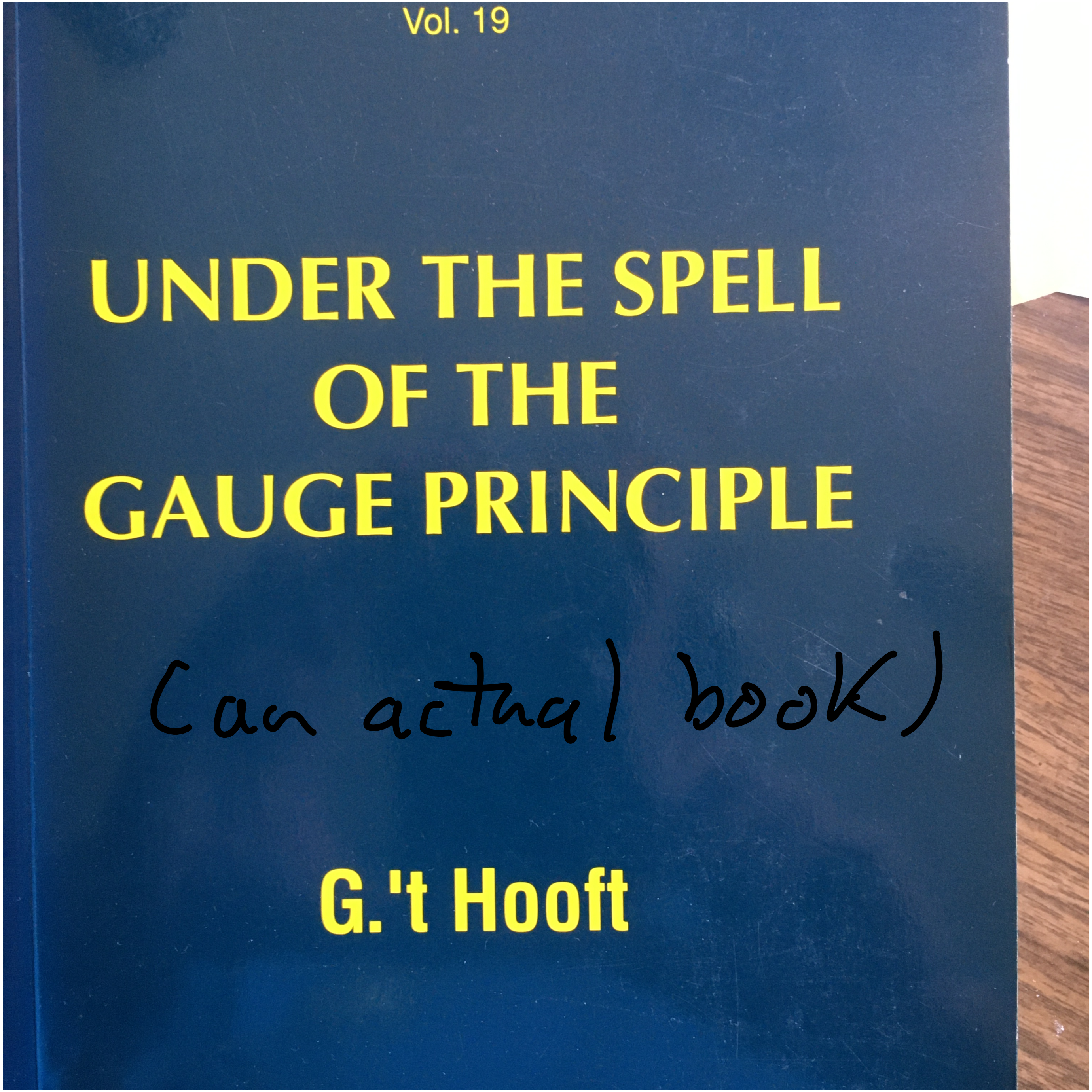
start w/ overview of my career:

UNDER THE SPELL
OF THE
PRINCIPLE OF LEAST ACTION

(pre-title)

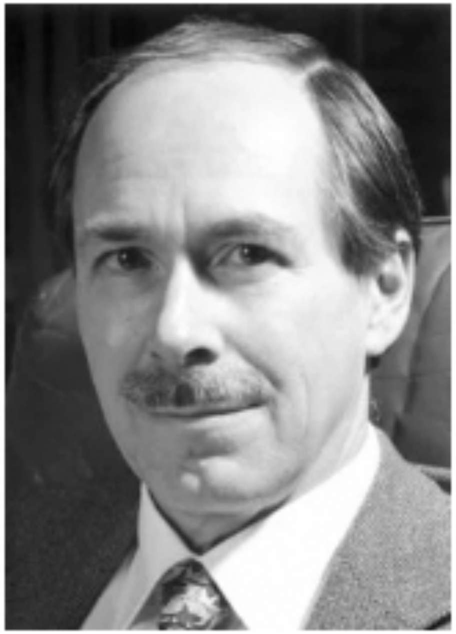
P. de Fermat, P.L. Maupertuis,
J.L. Lagrange, W.R. Hamilton


Vol. 19
UNDER THE SPELL
OF THE
GAUGE PRINCIPLE
(an actual book)
G.'t Hooft

subtitle

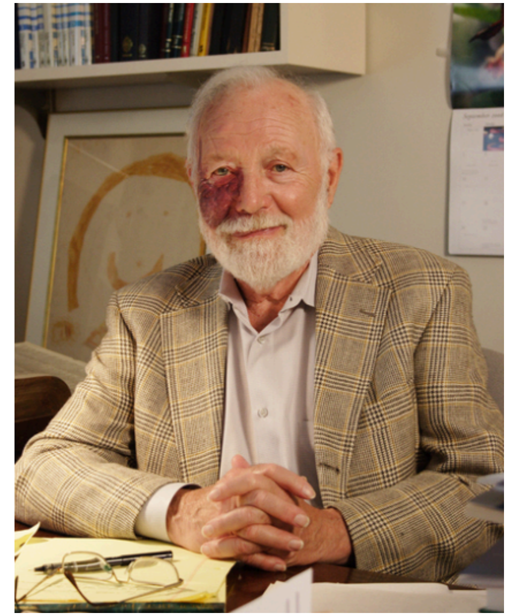
I witnessed a revolution. Gauge theory from physics invaded differential geometry and topology and remarkable advances were made.

I was a grad student in Berkeley from 1980 to 1985.

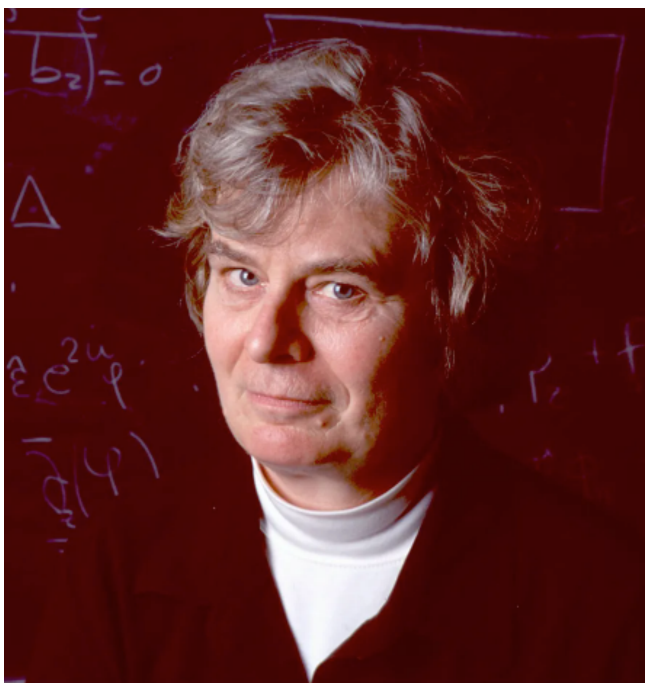


t'Hooft

Atiyah



Singer



Uhlenbek

Taubes

Donaldson

~~I witnessed a revolution. Gauge theory from physics invaded differential geome~~ ~~and topology and remarkable advances were made.~~
~~I was a grad student in Berkeley from 1980 to 1985.~~
Taubes was there. Singer was there. A PDE~~s~~ arising in gauge theory became employed to say remarkable things about 4 dimensional topology.

the anti-self-dual Yang-Mills eqn

Method: Given a compact 4-manifold, attach to it the moduli space of solutions to a certain non-linear PDE over M. One component of this moduli space was M itself! From properties of this moduli space one could conclude surprising theorems about M.

This PDE is the ``anti-self-dual Yang Mills equations''. G. t'Hooft gave the first example of solutions. The conformal group acts on the solutions, and as a result they concentrate. Taubes showed how to glue these concentrated 't'Hooft instantons onto any 4-manifold M, provided ``b_{2 +} = 0''

To define a gauge theory

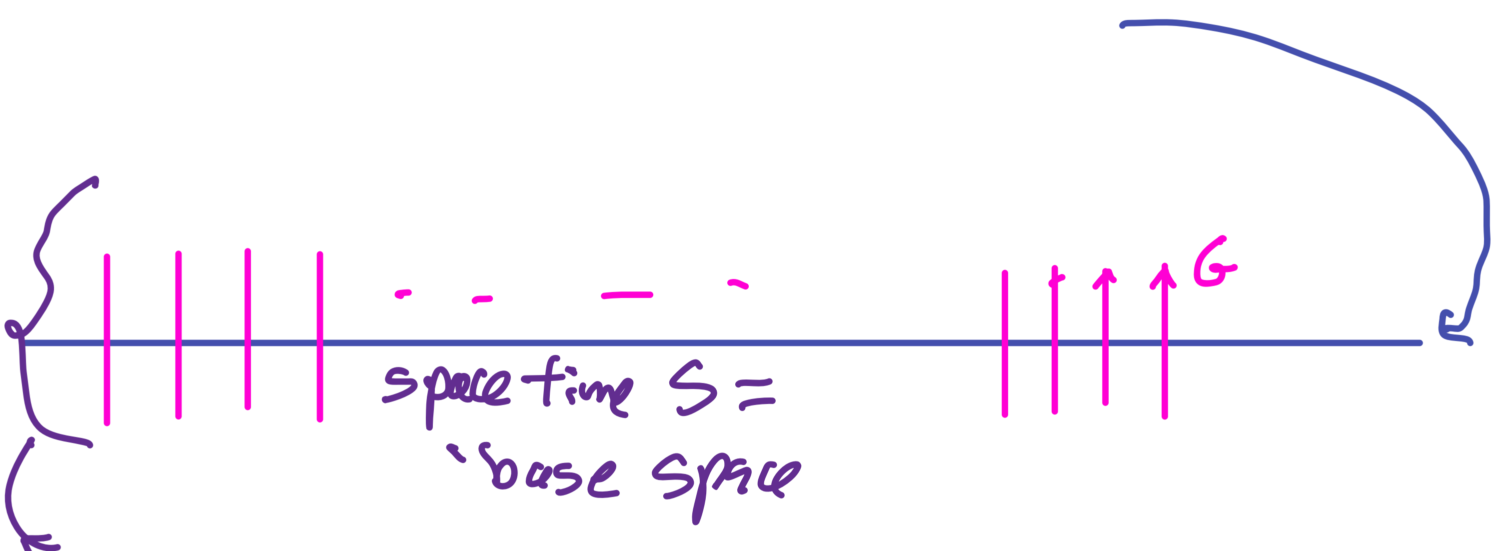
Choose a ~~compact~~ Lie Group G.

In high-energy physics ...

S^1 : electromagnetism } electroweak

$G = SU(2)$: weak }

$G = SU(3)$: strong

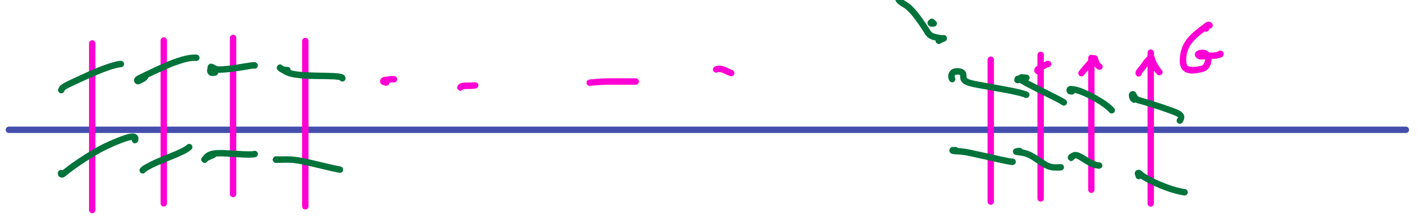


'total space' : locally $S \times G$

Connections on p. bundles

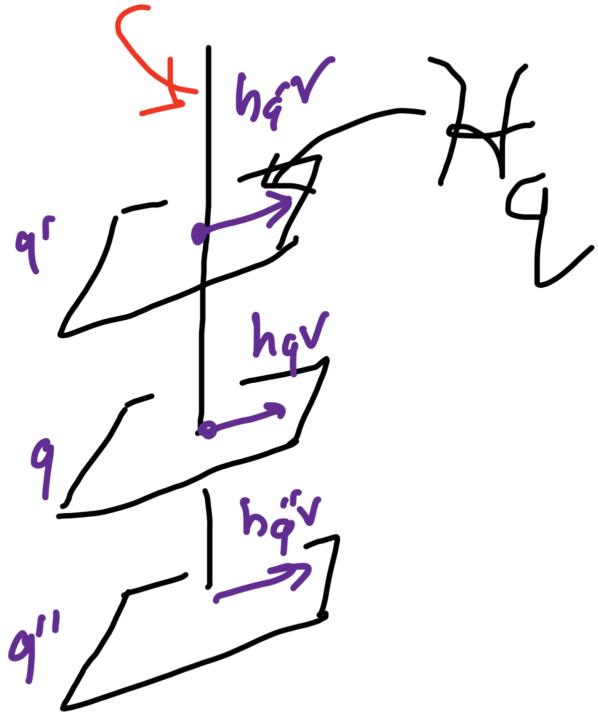
= gauge fields

= G-inv choice of horizontal



(1-d)

= way to lift horizontal paths from S to Q fiber $\pi^{-1}(m) = G\cdot q$



$$\begin{tikzcd} T_qQ \arrow[d, "d\pi_q"] \\ T_m \cancel{M} S \arrow[u, bend left, "h_q"] \end{tikzcd}$$

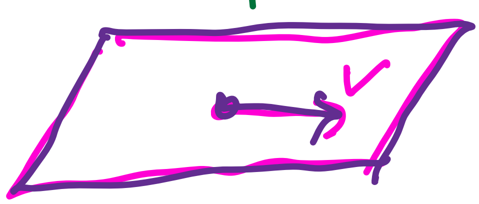
$TQ = V \oplus \mathcal{H}$ G-invariant splitting

defines 'connection'

horizontal space

vertical space
$\mathrm{Ker}(d\pi)$;

$$\mathcal{H}_q = \mathrm{im}\, h_q = \mathrm{Ker}\, A_q$$



Generalities: A loop in base, when lifted to total space, need not close up!

Parameterize loop $c(t)$ in S w/ $c(0) = c(1)$.

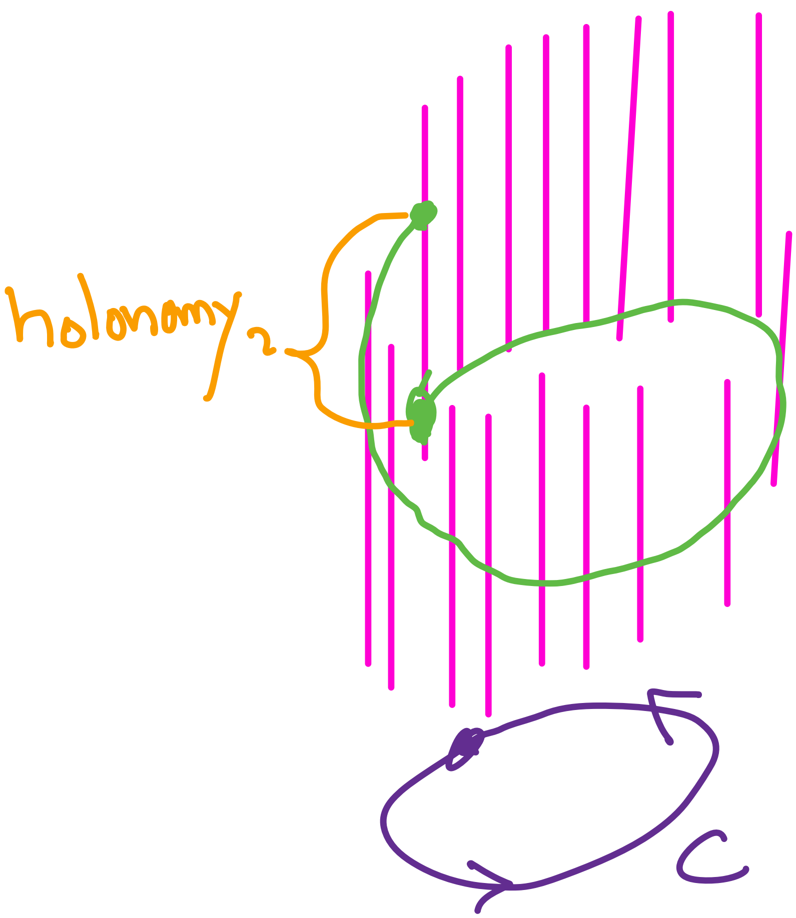
Then lift $q(t)$ satisfies

- $\pi(q(1)) = \pi(q(0))$

so $q(1) = g \cdot q(0)$ some $g \in G$.

This g is called the holonomy of the loop c (as based at $q(0)$)

Picture



An example we can draw:

$$Q = \mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$$
$$= S \times G$$

so $G = \mathbb{R}$ additive group

Connection: planes

$$D_{(x,y,z)} = \mathrm{Ker}\{dz - a(x,y)\,dy\}$$
$$= \mathrm{span}\{\partial_1,\ \partial_2 + a(x,y)\,\partial_3\}.$$

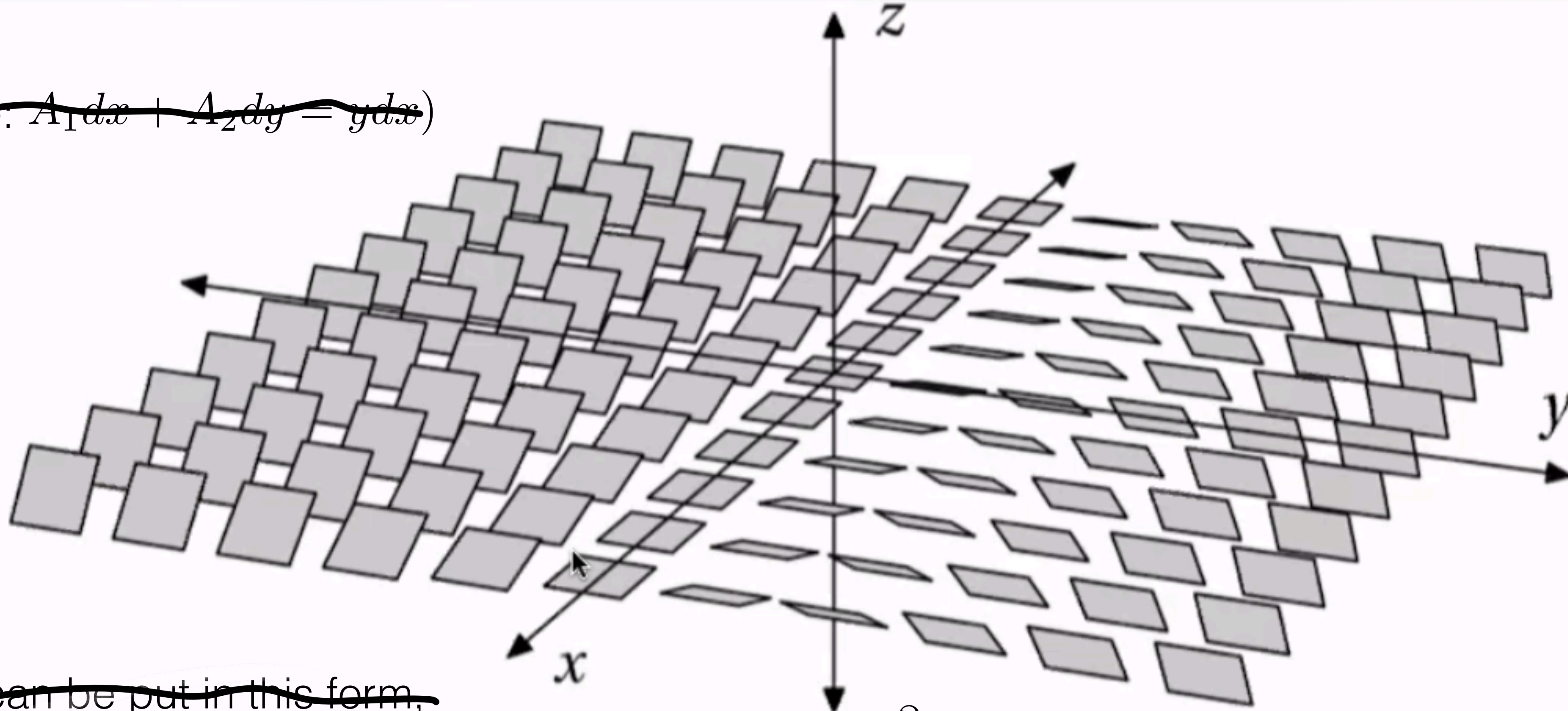
$$\left(\partial_1 = \frac{\partial}{\partial x} = (1,0,0)\right.$$
$$\left.\text{etc} \ldots\right)$$

Two-plane fields in 3-space: $\{\underline{dz - A_1(x,y)dx - A_2(x,y)dy} = 0\}$

`distribution', D

one-form, θ

~~(here: $A_1 dx + A_2 dy = y dx$)~~



~~D can be put in this form,~~

~~provided: the two-planes don't go vertical:~~ $\cancel{\frac{\partial}{\partial z} \notin D(x,y,z)}$

~~and they are invariant under z-translations~~

Here: $G = \mathbb{R}$; $S = \mathbb{R}^2$, $Q = \mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$.

For connection: field of two planes spanned by

$$\begin{cases} X = \partial_1 = (1, 0, 0) \\ Y = \partial_2 + a(x,y)\partial_3 = (0, 1, a_3) \end{cases}$$

<u>or</u> annihilated by $dz - a(x,y)dy$

Horizontal curve:

= curve tangent to

$$\mathcal{H} = \mathrm{Span}(X, Y)$$

so if $\gamma(t) = (x(t), y(t), z(t))$ is horizontal then:

$$\dot{\gamma} = u_1 X + u_2 Y$$

u_1, u_2 fns of t

or:

$$\dot{x} = u_1$$
$$\dot{y} = u_2$$
$$\dot{z} = a(x,y)\,u_2$$

Equivalently: draw any C^1 curve in $\mathbb{R}^2$, say $c = (x(t), y(t))$.

define $\gamma = (x(t), y(t), z(t))$

by $z(t) = z(0) + \int_0^t a(x(t), y(t))\,\dot{y}(t)\,dt$

or $dz = a\,dy$

if c is a loop in $\mathbb{R}^2$: say, $c(0) = c(1)$

then

$\Delta z = z(1) - z(0)$

$$\cdots = \oint a\, dy \qquad \text{Stoke's Thm}$$

$$= \iint_{\text{disc}} d(a\, dy)$$

$$= \iint B\, dx \wedge dy$$

where $B = \dfrac{\partial a}{\partial x}$

N.B.

$$[X, Y] = [\partial_1, \partial_2 + a\partial_3]$$

$$= (\partial_1 a)\partial_3$$

$$= B(x,y)\partial_3$$

<u>So</u>

<u>Holonomy</u> about C

$= \int$ curvature

$$\text{curvature} = B\,dx\,dy = d(a\,dy)$$

here.

$- C =$

J. Fluid Mech. (1989), *vol.* 198, *pp.* 557–585
Printed in Great Britain

Richard

Geometry of self-propulsion at low Reynolds number

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(Received 15 April 1987 and in revised form 12 July 1988)

The problem of swimming at low Reynolds number is formulated in terms of a gauge field on the space of shapes. Effective methods for computing this field, by solving a linear boundary-value problem, are described. We employ conformal-mapping techniques to calculate swimming motions for cylinders with a variety of cross-

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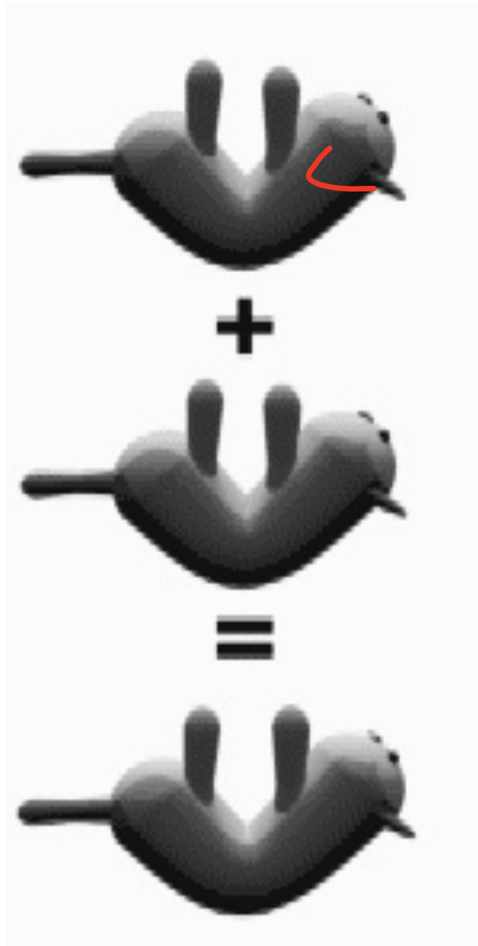
Jair
Koiller

Vol. 5

GEOMETRIC PHASES IN PHYSICS

Alfred Shapere
Frank Wilczek



Falling cat wiki

idea

heavy

$$M(t, x, \bar{\lambda}) = \inf_{u \in U} H(t, x, u, \bar{\lambda}). \tag{5.1.4}$$

The necessary condition for Lagrange problems now takes the form below, which we shall deduce explicitly from (4.2.i) in Section 5.2. A direct proof is indicated in Section 7.3I.

5.1.i (Theorem). *Under the hypotheses listed above, let* $x(t) = (x^1, \ldots, x^n)$, $u(t) = (u^1, \ldots, u^m)$, $t_1 \le t \le t_2$, *be an optimal pair, that is, an admissible pair* x, u *such that* $I[x, u] \le I[\tilde{x}, \tilde{u}]$ *for all pairs* $\tilde{x}, \tilde{u}$ *of the class* Ω *of all admissible pairs. Then the optimal pair* x, u *has the following properties:*

(P1′) *There is an absolutely continuous vector function* $\bar{\lambda}(t) = (\lambda_0, \lambda_1, \ldots, \lambda_n)$, $t_1 \le t \le t_2$ *(multipliers), which is never zero in* $[t_1, t_2]$, *with* λ_0 *a constant in* $[t_1, t_2]$, $\lambda_0 \ge 0$, *such that*

$$d\lambda_i/dt = -H_{x^i}(t, x(t), u(t), \bar{\lambda}(t)), \qquad i = 1, \ldots, n, \quad t \in [t_1, t_2] \text{ (a.e.)}.$$

(P2′) *For every fixed* t *in* $[t_1, t_2]$ (a.e.), *the Hamiltonian* $H(t, x(t), u, \bar{\lambda}(t))$ *as a function of* u *only (with* u *in* U*) takes its minimum value in* U *at* $u = u(t)$:

$$M(t, x(t), \lambda(t)) = H(t, x(t), u(t), \lambda(t)), \qquad t \in [t_1, t_2] \text{ (a.e.)}.$$

(P3′) *The function* $M(t) = M(t, x(t), \bar{\lambda}(t))$ *is absolutely continuous in* $[t_1, t_2]$ *(more specifically,* $M(t)$ *coincides a.e. in* $[t_1, t_2]$ *with an AC function), and*

$$\begin{aligned} dM/dt &= (d/dt)M(t, x(t), \lambda(t), u(t)) \\ &= H_t(t, x(t), u(t), \bar{\lambda}(t)), \qquad t \in [t_1, t_2] \text{ (a.e.)}. \end{aligned}$$

(P4′) *Transversality relation*:

$$\lambda_0\, dg - M(t_1)\, dt_1 + \sum_{j=1}^{n} \lambda_j(t_1)\, dx_1^j + M(t_2)\, dt_2 - \sum_{j=1}^{n} \lambda_j(t_2)\, dx_2^j = 0$$

for every $(2n+2)$*-vector* $h = (dt_1, dx_1, dt_2, dx_2) \in B'$, *that is,*

$$\lambda_0\, dg + \left[M(t)\, dt - \sum_{j=1}^{n} \lambda_j(t)\, dx^j \right]_1^2 = 0. \tag{5.1.5}$$

The transversality relation is identically satisfied if t_1, x_1, t_2, x_2 are fixed, that is, for the boundary conditions which correspond to the case that both end points and times are fixed ($dt_1 = dx_1^i = dt_2 = dx_2^i = 0$, $i = 1, \ldots, n$). For Lagrange problems of course $g = 0$, $dg = 0$.

Here x, u is an admissible pair itself, so that the differential equations

$$dx^i/dt = f_i(t, x(t), u(t)), \qquad i = 1, \ldots, n, \tag{5.1.6}$$

1987-9

Steve Zelditch

(R.I.P.)

"talk to him'.

Alex Pines : NMR, quantum chemist.

Jerry & Alan 'talk to him'

Shankar Sastry. EECS: control +

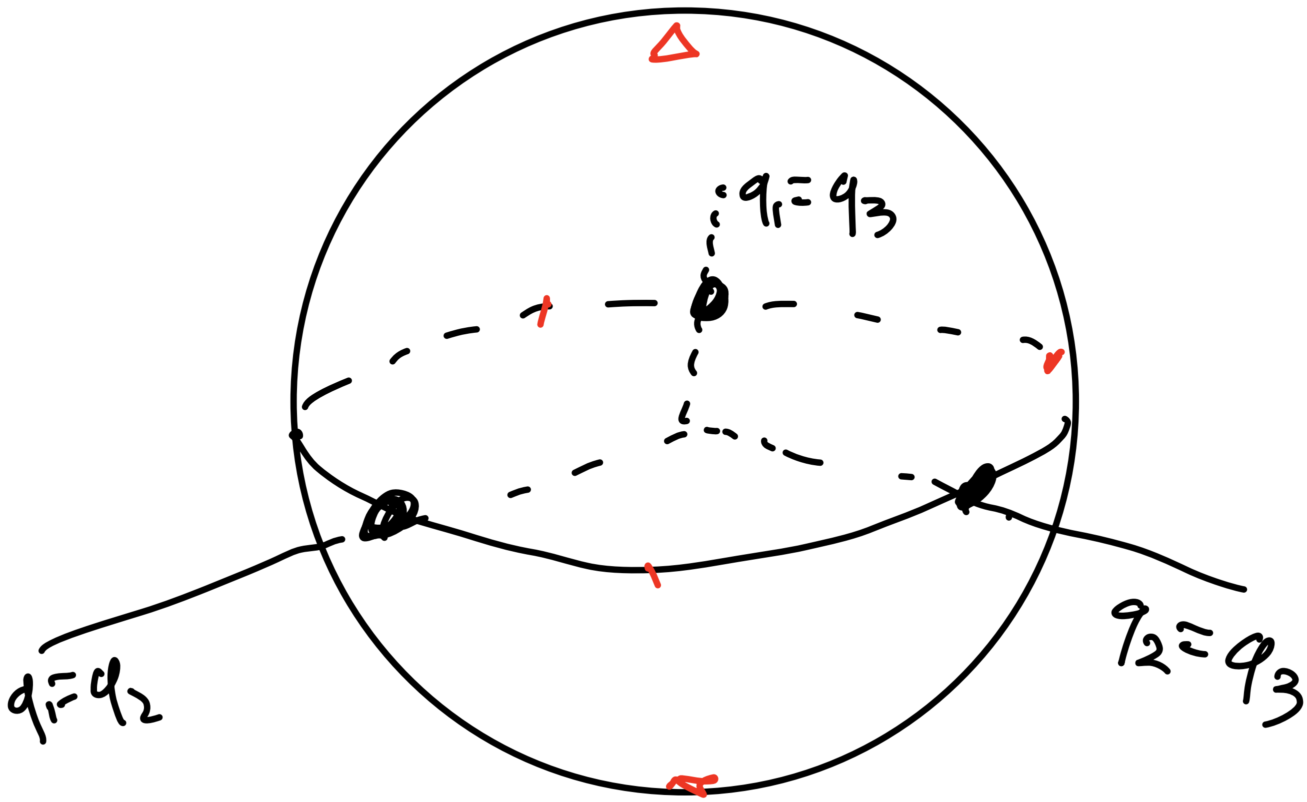
Wu-yi Hsiang

Chris Golé

⋮

Shape space;

cat consisting of 3 mass points.



shape sphere
braids, etc