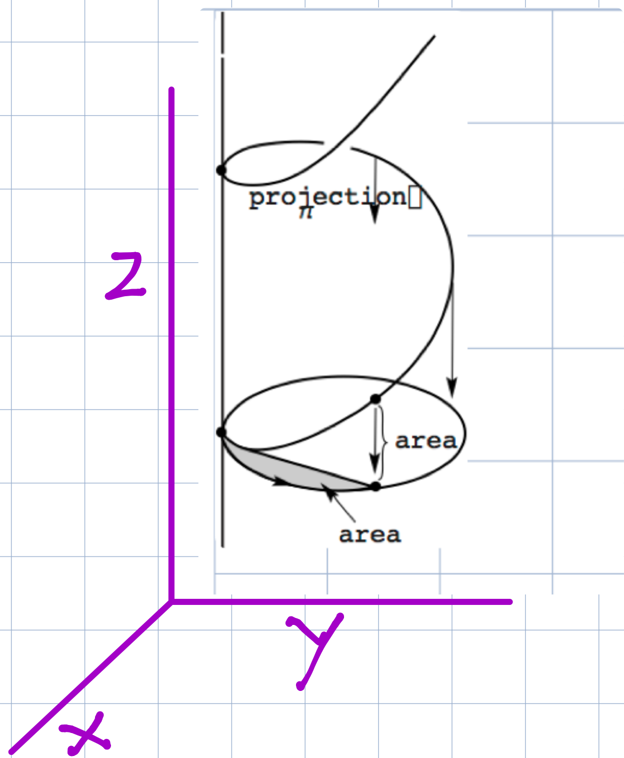


Theorem. On the Heisenberg group

1. Geodesics
are horizontal lifts of
Euclidean lines and circles in
the plane.

2. Metric lines
are horizontal lifts
of Euclidean lines in the plane.

as a manifold: $\mathbb{R}^3$

dy dx

as a metric space:

Call a smooth path c(t) = (x(t), y(t), z(t)) ``horizontal'' if :

$$\frac{dz}{dt} = \frac{1}{2}\left(x(t)\frac{dx}{dt} - y(t)\frac{dy}{dt}\right)$$

and call the length of such a path :

$$\ell(c) := \int_c \sqrt{dx^2 + dy^2} := \int_a^b \sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2}\, dt$$

Then d(A, B) = inf { length of c: c horizontal, c joins A to B }

as a Lie group: (x,y,z)(x', y', z') = (x + x', y+ y', z+ z')
+ (0,0, (1/2)(xy' - yx'))

d is a left-invariant metric on the Heisenberg group:
d(gA, gB) = d(A, B)

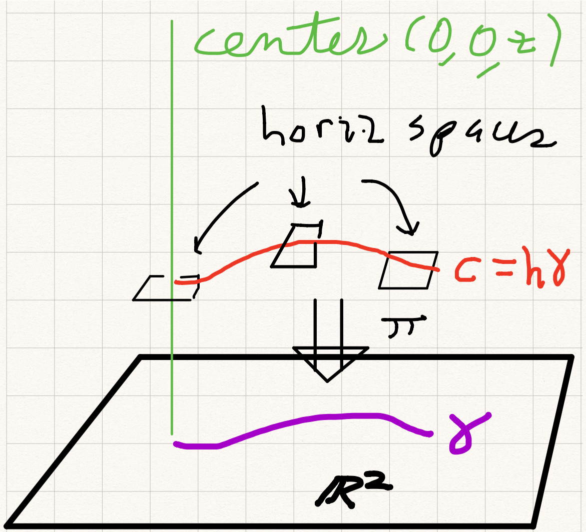
The projection

$$\pi : \mathbb{R}^3 \to \mathbb{R}^2; \pi(x, y, z) = (x, y)$$

is a **submetry:**

DEF: a **submetry** is an onto map
F between metric spaces such that F(B(p, r)) = B(F (p), r)

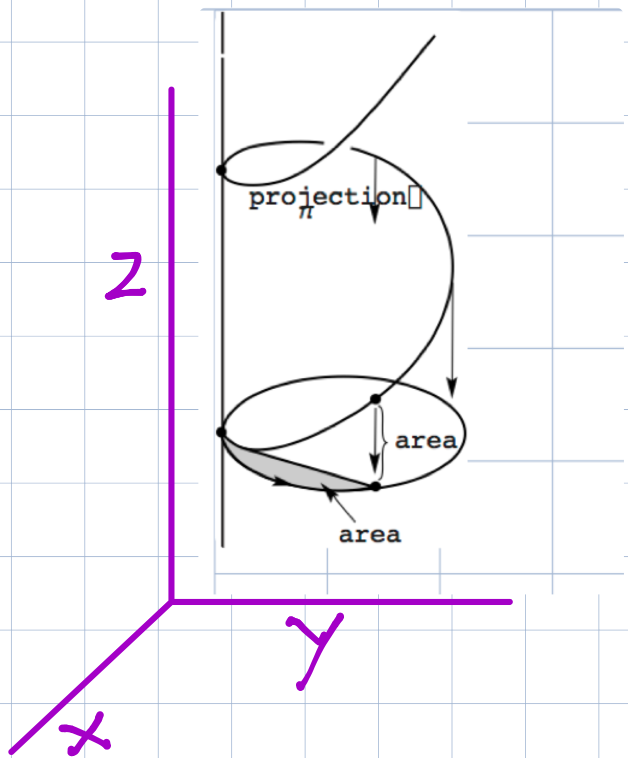
$$\ell(c) = \ell_{\mathbb{R}^2}(\pi(c)) \implies \pi \text{ is a submetry}$$



c is the horizontal lift of γ

$\gamma(t) = (x(t), y(t)) \implies c(t) = (x(t), y(t), z(t))$ where

$$\frac{dz}{dt} = \frac{1}{2}(x(t)\frac{dy}{dt} - y(t)\frac{dx}{dt})$$



Theorem. On the Heisenberg group

1. Geodesics
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Other sR geometries admitting submetries onto the Euc plane:

1 Those arising out of planar magnetic fields

2 Carnot groups.

3 Other Lie groups with homomorphisms onto the plane

In all cases there is a horizontal lift operation
The horizontal lifts of Euclidean lines are always metric lines.

Are there any metric lines besides lifts of Euclidean lines?

Generalities.

sR structure on a smooth manifold Q consists of

D , a linear sub-bundle of TQ, called a \`distribution'
and < , > a fiber-linear inner product on D

Def: a \`\`*horizontal*'' path is an (abs cts) path in Q (a.e.) tangent to D

the *length* of such a path c is the integral of the length
of its derivative v = dc/dt, this instantaneous norm computed using
the fiber inner product

d(A,B) as before
= inf { length of c: c horizontal, c joins A to B }

Theorem. [Chow-Rashevskii] If D bracket-generates TQ
and if Q is connected then
d defines an honest metric on Q and the induced metric
topology on Q agrees with its manifold topology

Exer. A min. geodesic from A to B is a horizontal curve connecting A and B whose length realized the distance d(A, B)

Geodesics fall into two categories: **normal** and **abnormal.**

Take the case rank(D)= 2, for simplicity

so D is spanned (locally) by two smooth orthonormal vector fields

$$X = \sum X^\mu(q)\frac{\partial}{\partial q^\mu} \qquad\qquad Y = \sum Y^\mu(q)\frac{\partial}{\partial q^\mu}$$

and a path q(t) in Q is *horizontal* if $\dot{q} = u_1(t)X(q(t)) + u_2(t)Y(q(t))$

in which case its length is : $\ell(q(\cdot)) = \int \sqrt{u_1(t)^2 + u_2(t)^2}dt$

Form the fiber-linear functions $P_1, P_2 : T^*Q \to \mathbb{R}$

$$P_1 = P_X = \sum p_\mu X^\mu(q) \qquad P_2 = P_Y = \sum p_\mu Y^\mu(q)$$

and the fiber-quadratic function $H = \frac{1}{2}(P_1^2 + P_2^2) : T^*Q \to \mathbb{R}$

Def. A normal sR extremal is the projection to Q of a nonzero characteristic for the hypersurface { H = 1/2}

Def. An abnormal sR EXTREMAL is the projection to Q of a nonzero characteristic of the codim n-2 dim submanifold { H = 0}

WHERE **Def. a characteristic** for a submanifold $\Sigma \subset T^*Q$

is an abs cts curve z(t) in Σ whose derivative dz/dt lies

the kernel of ω_Σ the restriction to Σ of the canonical symplectic form

equivalently dz/dt $\in T_{z(t)}\Sigma \cap (T_{z(t)}\Sigma)^{\omega\perp}$

Equivalently: a **normal** extremal is the projection to Q
of a solution to Hamilton's equations for H which has energy H = 1/2

this equivalency holds because 1/2 (or any positive number)
is a **regular value** for H

$\ker \omega_\Sigma = \mathbb{R} X_H$

these Hamilton's eons are smooth ODEs on T*Q so normal geodesics are smooth

while : an **abnormal** extremal is the projection to Q
of an abs. continuous curve z(t) = (q(t), p(t)) lying in

$$\Sigma = D^{\perp} := \{(q,p) : p(D_q) = 0\}$$

with p(t) never zero,
and with z(t) a characteristic in the previous sense for this Σ

these eqns are a mix of algebraic and differential eqns and
their solutions - the abnormal extremals - need not be smooth

Every minimizing geodesic is an arc of either a normal or abnormal extremal

It could be both (with two different cotangent lifts then)

Sufficiently short arcs of
the normal extremals are always minimizing geodesics,

Very short arcs of abnormal extremals might **not** be minimizing geodesics

[M-] It can happen that a minimizing geodesic is **really** abnormal:
it is an abnormal extremal, and it is **not** a normal extremal:
it does not ``satisfy the geodesic equations''

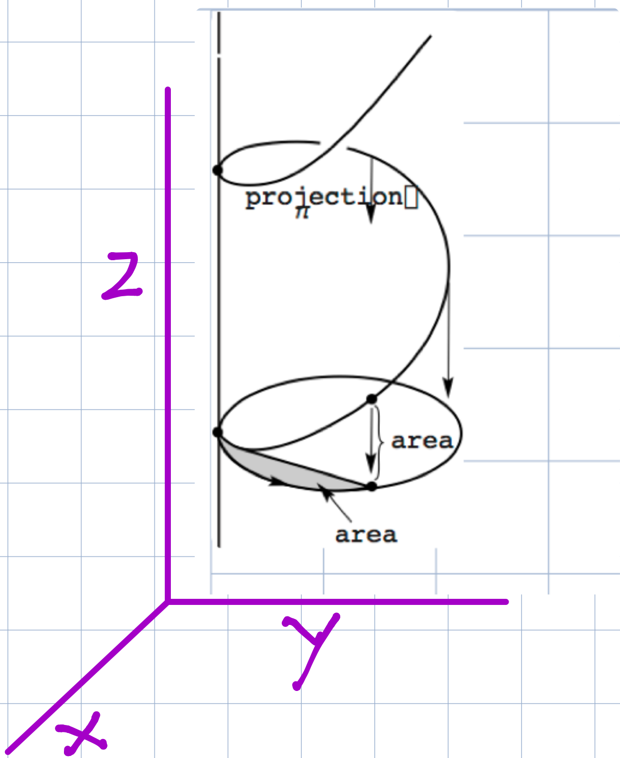
For rank 2 distributions, this phenomenon of ``strictly abnormal minimizers'' just described is *topologically stable* and *generic*

Generalizations

1. Carnot groups
2. Planar magnetic fields yielding sR submetry $\mathbb{R}^3 \to \mathbb{R}^2$
3. Lie groups endowed with onto homomorphism to Euc plane.

in these three cases there is a horizontal lift operation
and the horizontal lift of a Euclidean line is a metric lines.

In these examples are there any metric lines besides these these horiz. lifts of Euclidean lines?



recall Heisenberg

....

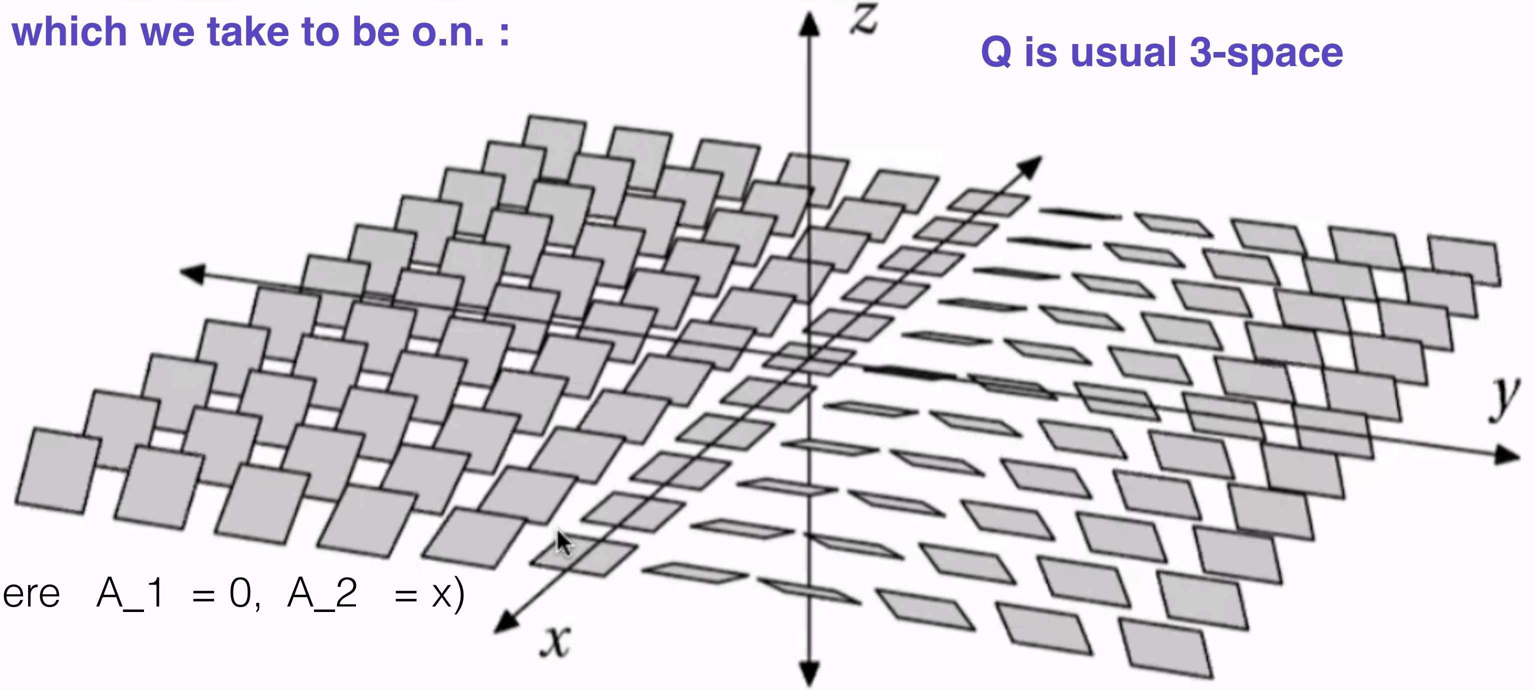
special case of:

D is framed by a pair of vector fields:

$$X = \frac{\partial}{\partial x} + A_1(x,y)\frac{\partial}{\partial z} \qquad\qquad Y = \frac{\partial}{\partial y} + A_2(x,y)\frac{\partial}{\partial z}$$

which we take to be o.n. :

Q is usual 3-space



(here A_1 = 0, A_2 = x)

$$B(x,y) = \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}$$ **= planar magnetic field**

Original Heisenberg example: A = (1/2) (xdy - y dx)

ie

A_1 = -(1/2) y

A_2 = + (1/2) x

so dA = dx ^ dy

, B = 1

Horizontal lift **:**

horizontality condition: dz - A_1 dx - A_2 dy = 0

If c is a plane curve, c(t)= (x(t), y(t)) ,

then integrate the horizontality eqn :

$$z(s) = z(0) + \int_{c([0,s])} A_1(x,y)dx + A_2(x,y)dy$$

to obtain (x(t), y(t), z(t)) a horizontal curve projecting onto c(t).

Call this `hc'' , the horizontal lift of c.
= unique **up to a z-translation**
(choice of z(0))

$$\gamma = hc$$

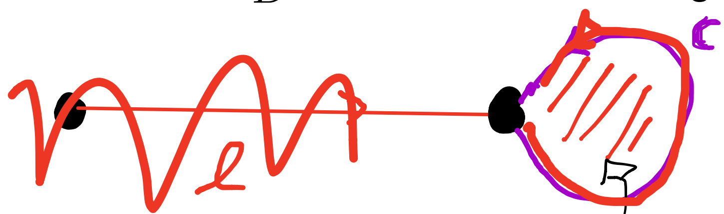
$$\pi(\gamma) = c$$

$$\pi : \mathbb{R}^3 \to \mathbb{R}^2$$

$$\ell(\gamma) = \ell_{\mathbb{R}^2}(c)$$

If c is closed, then the difference between its final and initial height is

$$\Delta z = \int_c A_1 dx + A_2 dy = \int \int_D B(x, y) dx dy$$ = Flux of Magnetic Field



encloses Flux

$\phi = \Delta z$

$$B(x, y) = \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}$$

= Magnetic Field

(via Lie brackets: [X, Y] = B(x,y) Z; $Z = \frac{\partial}{\partial z}$

Heisenberg case: flux = area,

so sR geodesic problem is equivalent to the isoperimetric problem

whose solutions are lifts of lines and circles,

Defining eqn for lines and circles, a la elem. differential geometry:

$$\kappa = \text{const.}$$

where $\kappa = \text{curvature of plane curve c.}$

or

$$\kappa(s) = \lambda B(x(s), y(s))$$

$$B \equiv 1$$

Geodesic equations, general magnetic case:

$$\lambda_0 \kappa(s) = \lambda B(c(s))$$

$$\lambda_0, \lambda \text{ constants with } \lambda_0 \lambda \neq 0$$

$$\kappa(s) = \text{ curvature of plane curve } c(s) = (x(s), y(s))$$

s = arc length parameterizeration of c

B(x,y) = planar magnetic field `pointing out of board''

Normal geodesics: $\lambda_0 \neq 0$ $\quad \frac{\lambda}{\lambda_0} =$ mass *charge/ (speed)^2

Abnormal extremals : $\lambda_0 = 0$ curve: zero locus of magnetic field.

Straight lines : $\lambda = 0$

These good geodesic eqns can be derived from the earlier H :

$$H = \frac{1}{2}\{(p_x + A_1(x,y)p_z)^2 + (p_x + A_2(x,y)p_z)^2\}$$

no z's

so $\dot{p}_z = 0$ **follows from Hamilton's eqns**

View the const. parameter p_z as electric **charge** $p_z = \lambda$

Then H is the Hamiltonian of a particle of mass 1 and this charge moving in the xy plane under the influence of the magnetic field B(x,y) where

$$B(x,y) = \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}$$

$$\{P_1, P_2\} = -B(x,y)p_z$$

In Heisenberg: B(x, y) = 1, identically

Observe:

Straight lines in the plane are solutions (with charge 0), for **any** B(x,y)

Their horizontal lifts are always metric lines

since $$\pi : Q = \mathbb{R}^3 \to \mathbb{R}^2$$

satisfies $$\ell(\gamma) = \ell_{\mathbb{R}^2}(\pi \circ \gamma)$$

for any horizontal curve γ

Question [LeDonne]: are there any other metric lines besides those whose projections are straight lines ?

$\boxed{\text{`Martinet case': } B(x,y) = x.}$

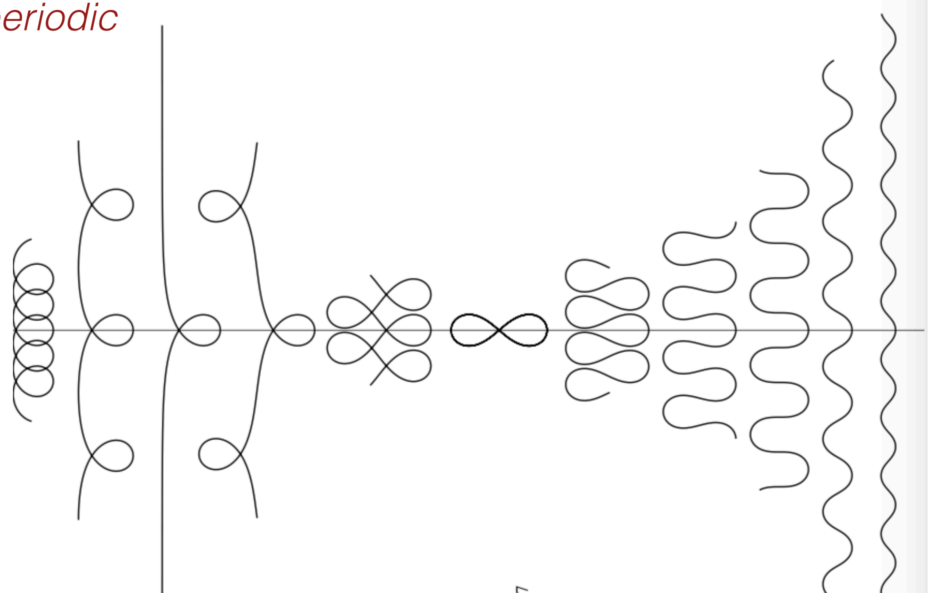
Normal geod eqns: $\kappa = \lambda x$

Theorem. **[Ardentov-Sachkov] Yes.**

The Euler kinks correspond to the other metric lines.

These are the full list of projected geodesics. They are the Euler elastica aligned to have y-axis (x = 0) as directrix.

All but the kink are periodic in the x direction



Their theorem actually concerns the **Engel group,**
a Carnot group of growth (2,3,4)

The Engel group admits a submetry onto $\pi : \mathbb{E} \to \mathbb{R}^3$

$$\mathbb{R}^3 \text{ with this magnetic-type sR structure}$$

consequently, the horizontal lifts to the Engel group of these elastica are sR geodesics,

and the lift of an Euler kink is a metric line,
since lifts of metric lines are metric lines

the projections of Engel geodesics to this 3-space are sR magnetic geodesics for magnetic fields B(x) = a x + b

$$\mathbb{E} = \mathbb{R}^2 \oplus \mathbb{R} \oplus \mathbb{R}$$

Engel group

$(2,3,4)$

$$\mathbb{E} = \mathbb{R}^2 \oplus \mathbb{R} \oplus \mathbb{R}$$

X, Y Z W

[X, Y] = Z [X,Z] = W

a nilpotent Lie algebra.

Exponentiate. Get a polynomial group structure on $\mathbb{E}$

View X, Y as left-invariant vector fields on this group.
They span a bracket generating distribution
and hence a left-invariant sR structure
such that the projection

$$\pi : \mathbb{E} \to \mathbb{R}^2 \cong \mathbb{E}/[\mathbb{E}, \mathbb{E}]$$

is a sR submersion

the plane

The projections of Engel geodesics to this ~~3 space are sR magnetic~~ geodesics for magnetic fields B(x) = a x + b

are sols for

parameter count: an n-1 dim family of sR geodesics through each point of a sR n manifold

Engel: n =4

geod eqns:

$$\kappa = \lambda(ax(s) + b) = Ax + B$$

Parameters: initial direction: 1
A, B: 2

3 parameters in all. **OK.**

Compare to Heisenberg group

$$\mathbb{H} = \mathbb{R}^2 \oplus \mathbb{R}$$

X, Y   Z

[X, Y] = Z

a nilpotent Lie algebra.

Exponentiate. Get the group structure on $\mathbb{H}$

View X, Y as left-invariant vector fields on this group.
They span a bracket generating distribution
and hence a left-invariant sR structure
such that the projection

$$\pi : \mathbb{H} \to \mathbb{R}^2 = \mathbb{H}/[\mathbb{H}, \mathbb{H}]$$

is a sR submersion

One big difference between Engel and Heisenberg.

Engel admits abnormal geodesics

These are the integral curves of the vector field Y.

What is special about Y:
write D = span{X, Y}, D^2 = D + [D, D] = Span{X, Y, Z}

then [Y, D^2] is contained in D^2
i.e. the flow of Y leaves D^2 invariant

Under the `Martinet projection' these integral curves project onto x = 0 , the zero locus of the Martinet magnetic field B(x) = x.

Theorem [M-; Sussmann-Liu] Take any inner product on the Engel distribution. Then the Engel lines are geodesics.

Take any magnetic field on the plane having a nondegenerate zero locus. Then the horizontal lifts of this zero locus are geodesics for 3-space with corresponding magnetic sR structure.

2. Carnot groups G =exp($\mathfrak{g}$) generally

$$\mathfrak{g} = V_1 \oplus V_2 \oplus \ldots V_s \quad Z(\mathfrak{g})$$

graded nilpotent: so $[V_i, V_j] \subset V_{i+j}$

$$V_r = 0, r > s$$

V_1 Lie generates $\dim(V_1) \geq 2$

r

D = V_1 , viewed as being left-translated about G
so as to form a left-inv. distribution

$$\pi : G \to V_1 \cong G/[G,G] \longrightarrow \mathbb{R}^2$$

is a group homo. **onto** V_1 , with V_1 now viewed as an
Abelian Lie group (vector space)

A choice of inner product on V_1 yields a
left-invariant sR struc (V_1, < , >)
on G, one for which \pi is a submetry onto a Euc space

F_r , 2, 3, 5, 8,

Goursat case: $\mathfrak{g} = \mathbb{R}^2 \oplus \mathbb{R} \oplus \ldots \oplus \mathbb{R}$

k factors of R

so: growth (2,1,1, … , 1) - k 1's

Canonical model : $G = J^k(\mathbb{R}, \mathbb{R})$

k =1: Heisenberg; k = 2: Engel.

$$\pi : J^k(\mathbb{R}, \mathbb{R}) \to \mathbb{R}^2$$

geod eqns?

Theorem. Anzaldo-Meneses and Monroy Perez [2004] ; Alejandro Doddoli [2019]. B(x,y) = p(x) = polynomial in x

The projections of Goursat geodesics to the plane are the trajectories of a particle in a polynomial magnetic field B = B(x) of degree k- 1 in x:

$$\kappa = \lambda(a_k x^{k-1} + a_{k-1} x^{k-2} + \ldots + a_0)$$
$$= A_k x^{k-1} + A_{k-1} x^{k-1} + \ldots A_0$$

does this dep on metric on $V_2 = \mathbb{R}^2$.

exercise: verify proper parameter count: n-1 = k+1 ..

Alejandro: isolating which solutions correspond to metric lines, i.e., the Euler kinks of the Engel / Martinet case

where from here?

Magnetic playground?

what does it (really) mean to be an abnormal extremal?

Hakavuouri’s surprise?

$$D \subset TQ$$

$$D_q^\perp = \{p : p(D_q) = 0\}$$

$$\begin{tikzcd}
D^\perp \arrow[dr] & \subset & T^*Q \arrow[dl] \\
& Q &
\end{tikzcd}$$

ω on T^*Q

$\omega_{D^\perp}$ on $D^\perp$.

abn. extremal: $\phi(0)$

of curve $z(t) = (q(t), p(t))$

$\subset D^\perp$, $\omega_{D^\perp}(\dot{z}, \cdot) \equiv 0$

Lip. ; $p(t) \neq 0$.

$\cdot\ p \in D_q^{\perp}$, $\tilde{p} \in \Gamma(D^{\perp})$

$$\dot{w}(p)(v,w) = -d\tilde{p}(\tilde{v},\tilde{w})$$

$v, w \in D_q$ $\quad = \pm p([\tilde{v},\tilde{w}]^{\perp})$

Prop. $\operatorname{Ker} \omega_{D^{\perp}}(q,p)$

maps linear isom

onto Ker $w(p)$

$w(p) \in \Lambda^2 D_q^*$

rank 2, case

$w(p) \in \Lambda^2 (\mathbb{R}^2)^*$.

if Abn passes thru

$(q, p) \in D^\perp$

$\Longrightarrow \quad p \in (D^2)$

$$D \subset D^2 \subset D^3$$

$$(D^3)^\perp \subset (D^2)^\perp \subset D^\perp$$

$\cup \qquad \not\supseteq$

(q, p)

then $\omega|_{(D^2)^\perp}$ is dh

q · constant

regular singular
curves

END