THE GAVEAU AFFAIR: A MISSING APPENDIX TO 'A TOUR OF SUBRIEMANNIAN GEOMETRY'

I wrote this around 1998. I added the title and the italicized sentences on July 4, 2024. These pages complete the analysis of the "Gaveau affair" described telegraphically on p. 47 (section 3.9) of my book [Montgomery]. -Richard Montgomery

In 1977 Gaveau [Gaveau] claimed to have a counterexample to the assertion that all minimizers are normal. He claimed that in his example there were two points which could be made arbitrarily close but could not be joined by any normal extremal. If this were true it would follow that the points must be joined by a strictly abnormal minimizer since minimizers always exist by Arzela-Ascoli and the fact that the points can be made close. Brockett [Brocket] successfully refuted his claim in 1983. The essence of Gaveau's mistake is that he ignores "higher harmonics". Since Brockett does not fill in all the details, and since some controversy existed around the example and Brockett's refutation, *at least in the late 1990s, I wrote this up, meaning to include it as an appendix in my book. But I never did.*

Gaveau's example concerns the canonical 2-step free nilpotent Lie algebra $R^{2n} \oplus \Lambda^2 R^{2n}$ with its canonical subRiemannian structure. Write elements of Q as (x, ξ). The system is

$$\dot{x} = u$$

$$\dot{\xi} = x \wedge u$$

with cost $\int \|u\|^2 dt$. Gaveau claims that elements of the form $(0, \xi)$ cannot all be reached by solutions of the geodesic equations starting at the identity $(0, 0)$.

Here are the details of Brockett's refutation of Gaveau's claim. We focus on the case $n = 2$. Write $V = R^4$. For appropriate choice of orthonormal basis e_1, e_2, e_3, e_4 we can put ξ into the normal form

$$\xi = \alpha e_1 \wedge e_2 + \beta e_3 \wedge e_4$$

with α, β **nonnegative**. (This is Cartan's lemma. Note we have not chosen any orientation for V.) Let V_1 be the span of the first two vectors and V_2 be the span of the last two. Use the inner product on V to identify $\Lambda^2 V$ with the Lie algebra $o(V) = o(4)$ of orthogonal matrices. As an element of $o(V)$ we have $\xi = \alpha J \oplus \beta J$ where J denotes the operation of rotation of either two plane V_i by 90 degrees. Now the geodesic equations are

$$\ddot{x} = \mu \dot{x}$$

$$\dot{\mu} = 0$$

$$\dot{\xi} = x \wedge \dot{x}$$

Here $x \in V$, $\mu, \xi \in so(V)$. (μ is the Lagrange multiplier.)

The first equation can be solved as

$$\dot{x}(t) = e^{t\mu} \dot{x}(0).$$

Set $\mu = 2\pi k J \oplus 2\pi l J$ with k, l integers. Identify each two-plane V_i with the complex numbers so that J multiplication by i. Then $V = \mathbb{C}^2$ and we easily calculate that $x(t) = (z(t), w(t))$ where

$$z(t) = \frac{1}{i2\pi k}(e^{it2\pi kt}\dot{z}_0 - \dot{z}_0)$$
$$w(t) = \frac{1}{i2\pi l}(e^{it2\pi lt}\dot{w}_0 - \dot{w}_0)$$

and $\dot{x}(0) = (\dot{z}_0, \dot{w}_0)$. Clearly $x(t)$ is a closed path. Now

$$\xi(t) = \int_0^t x(s) \wedge \dot{x}(s) ds$$

Expand this out in terms of the $z's$ and $w's$. If $k \neq l$ then the $z - w$ cross terms integrate to zero since $e^{i2\pi kt}$ and $e^{i2\pi lt}$ are orthogonal on the unit interval, as are all there real and imaginary parts. One finds that

$$\xi := \xi(1) = \frac{1}{2\pi k}|\dot{z}_0|^2 e_1 \wedge e_2 + \frac{1}{2\pi l}|\dot{w}_0|^2 e_3 \wedge e_4.$$

Simply choose initial conditions $\dot{z}_0$ so that $\alpha = \frac{1}{2\pi k}|\dot{z}_0|^2$ and $\beta = \frac{1}{2\pi l}|\dot{w}_0|^2$. This completes Brockett's refutation.

As an aside, the corresponding action is $\pi(k\alpha + l\beta)$ so if $\alpha \geq \beta$ the optimal geodesic is $k = 1, l = 2$. The abnormal extremals for Gaveau's example consist of those curves whose projections $x(t)$ lie in some 2-plane.

References:

Gaveau, B. 1977. Principe de moindre action, propagation de la chaleur et estimées sous elliptiques sur certains groupes nilpotents. Acta Math. 139 (1-2): 95-153.

Brockett, R. W. 1984. Nonlinear control theory and differential geometry. In Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Warsaw, 1983), 1357-1368. PWN, Warsaw.

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