The point of this short note is to give most of the proof of:

Proposition 0.1. *If the monodromy matrix A associated to a periodic orbit has an eigenvalue with modulus greater than 1 then that orbit is* **not** *Lyapunov stable.*

This proposition is a special case of:

Proposition 0.2. *If a diffeomorphism of the disc has the origin as a fixed point and if the derivative of that diffeomorphism has an eigenvalue with modulus greater than 1 then that fixed point is* **not** *Lyapunov stable.*

What follows is a Lyapunov inspired proof of the 2nd proposition.

Write $\mathbb{D} \subset \mathbb{R}^n$ for the disc, $R : \mathbb{D} \to \mathbb{D}$ for the map and $A = dR_0$ for its differential at the fixed point 0. Let $\mathbb{E}_+$ be the direct sum of all generalized eigenspaces for eigenvalues of modulus greater than 1. Write $\mathbb{E}_0 \subset \mathbb{R}^n$ for the direct sum of all generalized eigenspaces for eigenvalues having modulus 1 or less. We then have

$$\mathbb{R}^n = \mathbb{E}_+ \oplus \mathbb{E}_0.$$

Write $\lambda_+ > 1$ for the minimum modulus of all eigenvalues having modulus greater than 1. For any inner product on $\mathbb{E}_+$ we have

$$|Av| \geq \lambda_+ |v|, v \in \mathbb{E}_+$$

where $|\cdot|$ is the associated norm. On the other hand, by choosing the basis of Jordan blocks arising in the factors associated to $\mathbb{E}_0$ to make the off-diagonal parts of the block arbitrarily small, we can find, for any $\epsilon > 0$, an inner product on $\mathbb{E}_0$ such that

$$|Av| \leq (1 + \epsilon)|v|, v \in \mathbb{E}_0.$$

Together, the direct sum of these two inner products, yields an inner product on $\mathbb{R}^n$ having the above properties on each factor. For later use we choose $\epsilon > 0$ so small that

this ϵ gets linked to the one below; necessitating a redesign of the proof .. - RM

$$\lambda_+ - 5\epsilon > 1 \tag{1}$$

We use this inner product for definitions and estimates below. We write $x \mapsto x_+$ and $x \mapsto x_-$ for the linear projections onto $\mathbb{E}_+$ and $\mathbb{E}_0$.

Consider the cone

$$C = \{x \in \mathbb{R}^n : |x_+| > |x_-|\}$$

We will describe below a small ball B about 0 with the following properties.

1) $R(C \cap B) \subset C$.

2) There is a constant $c > 1$ such that $|R(x)| > c|x|$ for all $x \in C \cap B$.

From these two properties, we get Liapunov instability. In order to see this, write $B(\delta) \subset \mathbb{D}$ for the ball of radius δ about 0 in this norm. Now suppose that $B_1 := B(\epsilon_1) \subset B$ and let $B_2 := B(\epsilon_2) \subset B(\epsilon_1)$ be any smaller ball. I claim there are points of B_2 which exit B_1 after sufficiently many iterations of R. Indeed, take N so that $c^N \epsilon_2 > \epsilon_1$. Then, for all $r = r_*$ sufficiently close to ϵ_2 we also have $c^N r_* > \epsilon_1$. Take any one of these radii r_* sufficiently close to ϵ_2 and any point $x_* \in C \cap B_2$ with $|x_*| = r_*$. Claim. After N iterations of R, the orbit of x_* has left B_1. That is $R^N x_* \notin B_1$.

Proof of claim. If, for some $j < N$ we have that $R^j(x_*)$ has left B then it has also left B_1 and we're done. Otherwise, for each $j < N$ we have that $R^j(x_*) \in B$. But in this case we also have $R^j(x_*) \in C$ by an inductive argument. For example:

$x_* \in C \cap B$ so that, by (1) above, we have that $R(x_*) \in C$. And we've assumed $R(x_*) \in B$ so now$R(x_*) \in C \cap B$ and we can repeat the argument based on (1) to see that $R(R(x_*)) \in C$. This process repeats so that iterating (2) above we get that $R^j(x_*) \geq c^j |x_*|$ for $j < N$ and finally we work all the way up to $j = N$ to get $R^N(x_*) \geq c^N r_* > \epsilon_1$, which is to say, $R^N x_*$ has left B_1.

Proof that we can find a ball B so that (1) and (2) hold.

We have

$$R(x) = Ax + g(x)$$

where $g(x)$ is smooth and $O(|x|^2)$. Thus, for any $\epsilon > 0$ we can choose a ball small enough so that $|g(x)| < \epsilon |x|$ on B. We choose the ϵ as per inequality (1).

(To fill in: THIS ϵ needs to be linked with the ϵ of nilpotency ...)

Now say $x \in B \cap C$. Then $|x_+| > |x_-|$ and $|g(x)| < \epsilon |x|$. I claim $|(Rx)_+| > |(Rx)_-|$. Indeed we can write

$$x = (x_+, x_-) \in \mathbb{E}_+ \oplus \mathbb{E}_0$$

and

$$g(x) = (g(x)_+, g(x)_-)$$

for the projections of x and $g(x)$ onto $\mathbb{E}_+$ and $\mathbb{E}_0$. Then

$$Rx = (A_+ x_+ + g(x)_+, A_- x_- + g(x)_-)$$

So that

$$|(Rx)_+| = |A_+ x_+ + g(x)_+| \geq |A_+ x_+| - |g(x)|$$

which yields

$$|(Rx)_+| \geq \lambda_+ |x_+| - \epsilon |x|$$

On the other hand $(Rx)_- = A_- x_- + g(x)_-$ and $|A_- x_-| \leq (1+\epsilon)|x_-|$ from which it follows that

$$|(Rx)_-| \leq (1+\epsilon)|x_-| + \epsilon |x|$$

It follows that

$$|(Rx)_+| - |(Rx)_-| \geq (\lambda_+)|x_+| - (1+\epsilon)|x_-| - 2\epsilon |x|.$$

Use now that

$$|x| \leq |x_+| + |x_-| < 2|x_+|, \text{ for } x \in C$$
$$|(Rx)_+| - |(Rx)_-| \geq (\lambda_+ - 4\epsilon)|x_+| - (1+\epsilon)|x_-|.$$

So, as long as $(\lambda_+ - 4\epsilon) \geq (1+\epsilon)$ or $\lambda_+ - 5\epsilon \geq 1$. But our choise of ϵ from inequality (1) is precisely this: $\lambda_+ - 5\epsilon \geq 1$. We have shown that

$$|(Rx)_+| - |(Rx)_-| \geq (1+\epsilon)(|x_+| - |x_-|)$$

and the latter is greater than zero for $x \in C$. We have shown that $R(x) \in C$ for $x \in C \cap B$. This establishes (1).

To establish (2), return to $|(Rx)_+| \geq \lambda_+ |x_+| - \epsilon |x|$. Use again $|x| \leq 2|x_+|$ to conclude that

$$|(Rx)_+| \geq \lambda_+ |x_+| - \epsilon |x| \geq \lambda_+ |x_+| - 2\epsilon |x_+|$$

from which it follows $|(Rx)_+| \geq c|x_+|$ with $c = \lambda_+ - 2\epsilon$ and $x \in C$. Note $\lambda_+ - 2\epsilon > \lambda_+ - 5\epsilon > 1$.

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