SOME SOLUTIONS FOR HW 5

Solution to number 4.

We are to show that the operation of translation by a fixed $v \in V = \mathbb{Z}_2^n$ defines an automorphism of the graph.

STEP 1. The first step is to show that this map is a one-to-one onto map of V to itself.

Let us write $f(s) = s + v$ for this translation so that $f : V \to V$.

[S.C.] direct method. straight from the definitions.

f is one-to-one. Suppose $f(s_1) = f(s_2)$. Then $s_1 + v = s_2 + v$. Subtracting v from both sides yields $s_1 = s_2$.

f is onto. Let $s \in V$. Then $f(s - v) = (s - v) + v = s$. So f is onto.

[N.W.] Method 2. Use the fact that a map $f : V \to V$ is one-to-one and onto if and only if it has an inverse $g : V \to V$. (An inverse is a map such that $f(g(s)) = s$ for all $s \in V$.)

The inverse of adding is subtracting. That is, $g(s) = s - v$ is an inverse to f. So f is one-to-one and onto. [Note over $\mathbb{Z}_2$ adding is the same as subtracting: f is its own inverse!]

Here are Nick's words, the added words in brackets being mine: "By adding some v to all vertices s we see that $s - v$ would "undo" the translation, thus an inverse action of [adding of] v is possible, making this [adding of v] a one-to-one and onto operation."

[L.H.] $v + v = 0 \implies v = -v$ an inverse mapping exists.

STEP 2. Showing that translation preserves adjacency.

We show more. We show that translation preserves Hamming distance. Since two vectors are adjacent if and only if their Hamming distance is 1, this also proves that translation preserves adjacency.

The Hamming distance between $s, w \in V$ is

$$d(s, w) := \Sigma |s_i - w_i|.$$

Each summand $|s_i - w_i|$ is either 0 or 1, and is to be treated as a regular integer, not an integer mod 2, so that the Hamming distance itself is an integer.

Now $d(s + v, w + v) = \Sigma |(s_i + v_i) - (w_i + v_i)| = \Sigma |s_i - w_i| = d(s, w)$ which shows that translation by v preserves Hamming distance.