

# lect 2

General, smooth  
 $X$  a vector field on a manifold  $M$ .

ODE:  $\frac{dm}{dt} = X(m)$ , soln:  $m(t; m_0)$  i.c.  
 $m(0, m_0) = m_0$ .

$\rightarrow \phi_t(m_0) =$  unique soln w. i.c.  $m(0) = m_0$ .

~~if~~  $f := m(t; m_0)$ .

Flow complete: curves  $m(t; m_0)$  defined for all  $t$ .

$\mathbb{R}$  action defined on  $M$ :

$$t \cdot m := \phi_t(m)$$

$$\star (t+s) \cdot m := \phi_{t+s}(m) = \phi_t \circ \phi_s(m)$$

~~case 1 /  $M = S^1$ ,  $TM \cong S^1 \times \mathbb{R}$~~

~~if  $M = S^1$ ,  $\mathbb{R}^n$ , or  $\mathbb{T}^n \cong \mathbb{R}^n / \mathbb{Z}^n$ .~~

~~ident. by vector fields on  $M$  w.r. maps  $M \rightarrow \mathbb{R}$~~

case 1:  $M = S^1$ ,  $X(\theta) = -\sin \theta$ .

$$\text{and } \dot{\theta} = -\sin \theta$$

$\gamma$   
"Orbit" = solution  
 $\mathbb{R}$ -

case 2:  $M = T^2 \simeq S^1 \times \mathbb{R}$

$\frac{1}{2}$

$X(\theta, \dot{\theta}) = (\dot{\theta}, -\sin \theta)$

$$\left. \begin{aligned} \frac{d\theta}{dt} &= \dot{\theta} \\ \frac{d\dot{\theta}}{dt} &= -\sin \theta \end{aligned} \right\} \Leftrightarrow \frac{d^2\theta}{dt^2} = -\sin \theta.$$

2 special types of orbits

1. fixed pts  $m_0$  then  $X(m_0) = 0$   
 $\phi_t(m_0) = m_0$

• periodic orbits:  $\exists T > 0$

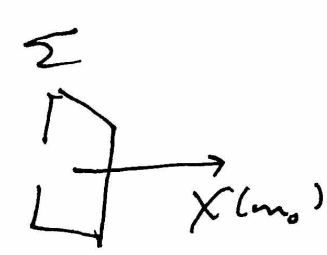
$\phi_T(m_0) = m_0$

But  $\phi_t(m_0) \neq m_0$  for  $0 < t < T$ .



then there  
 $\gamma(t) = \phi_t(m_0)$  factors  
 thru  $S^1_T = \mathbb{R}/T\mathbb{Z}$ .

Poincaré cross-section.



transverse disc:

$\Sigma \subset M$  embedd  
 $T_{m_0} \Sigma \oplus \mathbb{R} X(m_0) = T_{m_0} M.$

invertible  
yields map:  
 $(\Sigma, m_0) \xrightarrow{P} (\Sigma, m_0)$

Describe notation

domain of  $P$ : open subset of  $\Sigma$   
 with  $m_0 \in U$

range " "  $\checkmark$  " "  
 " "  $m_0 \in V$ .

$P(m) = \text{1st } t \text{ m.t. } t > 0 \text{ s.t.}$   
 $\phi_t(m) \in \Sigma$ .

why exists?

1)  $\phi_{(-\varepsilon, \varepsilon)} \Sigma$  an open nbhd of  $\Sigma$ .

2)  $\phi_{(T-\varepsilon, T+\varepsilon)} \Sigma$  " " " " " "

3)  $F(m)$  defn smoothly on  $(m)$

$\Rightarrow \exists! s(m) \sim T$

$\phi_{(s(m))}(m) \in \Sigma$

for  $m$  close to  $m_0$ .

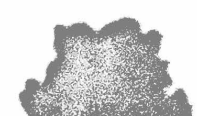
MB:  $P(m_0) = m_0$ .

The first part of the document discusses the importance of maintaining accurate records of all transactions. It emphasizes that every entry should be supported by a valid receipt or invoice. This ensures transparency and allows for easy verification of the data.

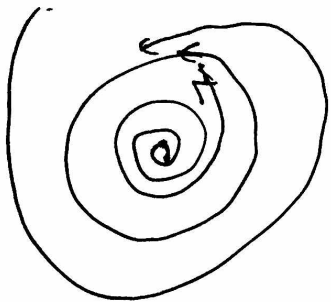
In the second section, the author details the various methods used to collect and analyze the data. This includes both manual entry and the use of specialized software tools. The goal is to ensure that the information is both precise and comprehensive.

The third part of the document focuses on the results of the analysis. It shows a clear upward trend in the data over the period covered, which is attributed to several key factors. These include improved operational efficiency and the implementation of new technologies.

Finally, the document concludes with a series of recommendations for future work. It suggests continuing to refine the data collection process and exploring new analytical techniques. The author believes that these steps will lead to even greater success and growth in the coming years.



Egs. Attracting limit cycle



$$\dot{r} = r^2 + r - r^3$$
$$\dot{\theta} = 1$$

$$\dot{x} = -p(r)y + p(r)x$$
$$\dot{y} = +p(r)x.$$

$$\text{Then } r \dot{r} = x \dot{x} + y \dot{y} = -p(r)xy + p(r)xy.$$

For  $\theta$  motion  $\begin{cases} \dot{x} = -y \\ \dot{y} = +x \end{cases} \leftarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

For  $r$  :  $\begin{cases} \dot{x} = x \\ \dot{y} = y \end{cases} \leftarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

with  $\begin{cases} \dot{x} = -y + p(r)x \\ \dot{y} = x + p(r)y \end{cases}$

$$\Rightarrow x \dot{x} + y \dot{y} = p(r)r^2$$

$$p(r) = r - r^3$$

$$\begin{aligned} r \dot{r} &= p(r)r^2 \\ \dot{r} &= p(r)r \\ &= r - r^3 \\ \Rightarrow p(r) &= 1 - r^2. \end{aligned}$$

$P(r)$

even.

eg:  $r^2 - 1$

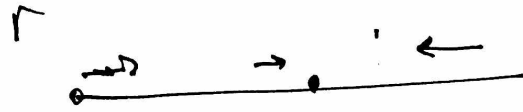
$$p(r)r = r^3 -$$

$$p(r)r = r - 1$$

$$p(r) = \frac{r-1}{r}$$

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$$p(r) =$$



$$r^3 - r^2$$

$$r - r^3 = r(1 - r^2)$$

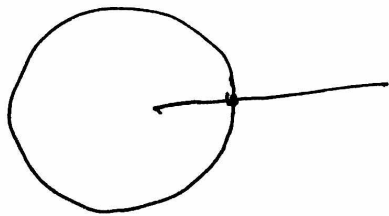
Alg syst.

$$\dot{x} = -y + (1 - x^2 - y^2)x$$

$$\dot{y} = x + (1 - x^2 - y^2)y$$

cf "Hopf bifurcation...!"

Assoc map:



$$\frac{dr}{dt} = r - r^3$$

$$= r(1-r^2)$$

$$\frac{dr}{r(1-r^2)} = dt.$$

$$\int: \quad \frac{1}{r(1-r^2)} = \frac{a}{r} + \frac{b}{1-r} + \frac{c}{1+r}$$

$$\int \frac{dr}{r(1-r^2)} = \alpha \ln r + \beta \ln(1-r) + \gamma \ln(1+r)$$

$$= \ln(r^\alpha (1-r)^\beta (1+r)^\gamma) + C.$$

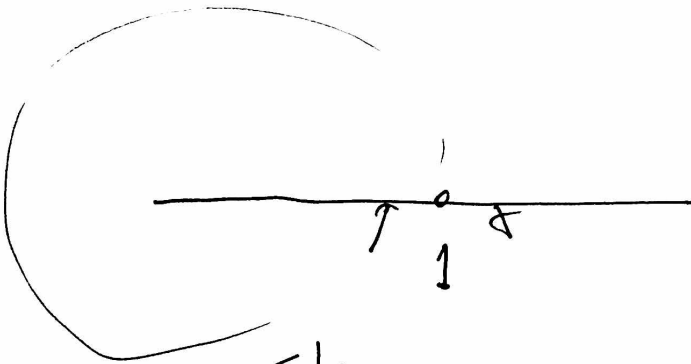
$$= t.$$

$$\Rightarrow u = e^t = (r^\alpha (1-r)^\beta (1+r)^\gamma).$$

Finish of like. set  $r = 1+h; \theta = 0$

$$F(h) = \lambda h + O(h^2)$$

$$0 < \lambda < 1.$$



Since : pos x-axis.

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To HW.

$F(h) = -h$  impossible.  
for a point section.  
of an ODE in  $\mathbb{R}^2$  having  
unit circle as an orbit.



If have not gone to Nathan yet L9  
DO SO NOW.

Perhaps AFTER Break?

Break! check: sign ups

So: from

Flow w/ periodic orbit.

→ germ of map is 1-linear  
dim.

To reverse:

$$\text{eg: } \mathbb{T}^2 \xrightarrow{F} \mathbb{T}^2$$
$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = A$$

Map F

$$(x, y) \in \mathbb{T}^2 = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \approx S^1 \times S^1$$

so  $x, y \pmod 1$ .

$$(x, y) \mapsto (x+y), (x+2y).$$

Find a matrix  $\mathfrak{F}$

$$\text{s.t. } e^{\mathfrak{F}} = A.$$

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Exer do it!

Hint: evaluate  $\mathfrak{F} \in \mathfrak{sl}(2, \mathbb{R})$

$$\text{tr } \mathfrak{F} = 0.$$

Introduce  $z \in S^1 = \mathbb{R}/2\pi$ .

$$\text{ode: } \frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \mathfrak{F} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\dot{z} = 1.$$

The  $\vec{v}$   $x=0, y=0$  have periodic  
orbit  $S^1 \subset \mathbb{T}^3_{x,y,z}$ .

Global slice:  $z=1$ .

Poincaré section is  $A$ .

same solving yields:  $z(t) = t$ .

$$\begin{pmatrix} x \\ y \end{pmatrix}(t) = e^{t\mathfrak{F}} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = M \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

1st Big surprise (Stability) [Answer.] ||

Take the vector field  $\xi$  just described on  $\mathbb{T}^3$  & look at a  $C^2$ -small perturbation

$$\tilde{\xi} ; \quad \|\tilde{\xi} - \xi\|_{C^2} < \varepsilon$$

or look at the map  $A: \mathbb{T}^2 \rightarrow \mathbb{T}^2$

& a  $C^1$ -perturbation of it  
 $\|A_\varepsilon - A\| \leq \varepsilon$ .

The a) the flow of  $\tilde{\xi}$  is whit equivalent to the flow  $\xi$ ,

& b) the map  $A_\varepsilon$  is  $C^0$ -equiv. to the map  $A$

$$b): \quad \begin{array}{ccc} \mathbb{T}^2 & \xrightarrow{A_\varepsilon} & \mathbb{T}^2 \\ \downarrow h & & \downarrow h \\ \mathbb{T}^2 & \xrightarrow{A} & \mathbb{T}^2 \end{array}$$

$\exists$  homeo  $h$ ;  
 s.t.

a):  $\exists \phi: \mathbb{T}^3 \rightarrow \mathbb{T}^3$  homeo s.t.  $\phi$

~~$\gamma(t)$  any whit~~  $\phi$  takes  
 solves to  $\tilde{\xi}$ -vector as wanted curves  
 to curves of  $\xi$ .

History

- Sample,

~ 3.73 Russia,

~ Axion A' - - -

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