lect 2

General. X a smooth vector field on a manifold M.

ODE:
$$\frac{dm}{dt} = X(m), \quad \text{soln: } m(t; m_0) \quad \text{(i.c.)}$$
$$m(0, m_0) = m_0.$$

$\rightarrow$ $\phi_t(m_0)$ = unique soln w. i.c. $m(0) = m_0$.

~~If f~~ $:= m(t; m_0)$.

Flow complete : curves $m(t; m_0)$ defined for all time.

Thm Φ defines an $\mathbb{R}$-action on M:

$t \cdot m := \phi_t(m)$

& $(t+s)\cdot m := \phi_{t+s}(m) = \phi_t \circ \phi_s(m)$

~~case 1 $M = S^1$, $TM \simeq S^1 \times \mathbb{R}$~~

~~$X(\theta) = 1\theta$~~

~~If $M = S^1$, $\mathbb{R}^n$, or $\mathbb{T}^n \simeq \mathbb{R}^n/\mathbb{Z}^n$.~~

~~identify vector fields on M w.~~

maps $M \to 1$

case 1: $M = S^1$, $X(\theta) = -\sin\theta$.

ODE $\dot\theta = -\sin\theta$

$\mathbb{R}$-"Orbit" = solution

case 2: $M = TS^1 \simeq \overset{\theta}{S^1} \times \overset{\dot\theta}{\mathbb{R}}$

$$X(\theta, \dot\theta) = (\dot\theta, -\sin\theta)$$

so

$$\left.\begin{aligned} \frac{d\theta}{dt} &= \dot\theta \\ \frac{d\dot\theta}{dt} &= -\sin\theta \end{aligned}\right\} \Leftrightarrow \frac{d^2\theta}{dt^2} = -\sin\theta.$$

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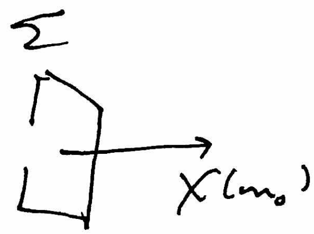
2 special types of orbits

- fixed pts m_0 then $X(m_0) = 0$.
  $\phi_t(m_0) = m_0$.

- periodic orbits: $\exists\, T > 0$
  $\phi_T(m_0) = m_0$
  But $\phi_t(m_0) \neq m_0$ for $0 < t < T$.

~~that~~ Then $\gamma(t) = \phi_t(m_0)$ factors thru $S^1_T = \mathbb{R}/T\mathbb{Z}$.

Poincare cross-section.



~~T~~ transverse disc:

$\Sigma \subset M$ embedded

$$T_{m_0}\Sigma \oplus \mathbb{R}X(m_0) = T_{m_0}M.$$

invertible

yields map:

$$(\Sigma, m_0) \xrightarrow{P} (\Sigma, m_0)$$

Describe notation

domain of P: open subset U of Σ
containing $m_0 \in U$

range " " V " "
-- $m_0 \in V$.

$P(m) =$ 1st time $t > 0$ s.t.
$\phi_t(m) \in \Sigma$.

why exists?

1) $\phi_{(-\varepsilon, \varepsilon)} \Sigma$ an open nbhd of Σ.

2) $\phi_{(-T-\varepsilon, T+\varepsilon)} \Sigma$ " " " "

3) Flow depends smoothly on (m

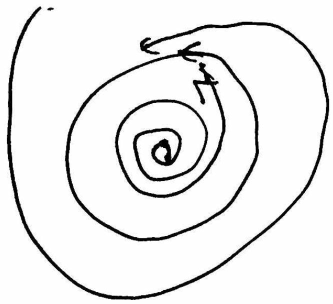
$\Rightarrow \exists!\ s(m) \sim T$

$\phi_{(s(m))}(m) \in \Sigma$

for m close to m_0.

NB: $P(m_0) = m_0$.

Egs. Attracting limit cycle



$\dot r = \cancel{r^2 +}\, r - r^3$

$\dot\theta = 1$

$\dot x = -p(r)\, y + p(r) x$

$\dot y = +p(r) x.$

Then $r \dot r = x \dot x + y \dot y = -p(r) xy + p(r) xy$

Pure θ motion $\dot x = -y$, $\dot y = +x$ $\leftarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Pure r: $\dot x = x$, $\dot y = y$ $\leftarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

mix $\dot x = -y + p(r) x$

$\dot y = x + p(r) y.$

$\Rightarrow x\dot x + y \dot y = p(r) r^2$ or $r \dot r = p(r) r^2$

$\dot r = p(r) r.$

$= r - r^3$

$\Rightarrow p(r) = 1 - r^2.$

$p(r) = r - r$

$p(r)$ even.

eg: $r^2 - 1$

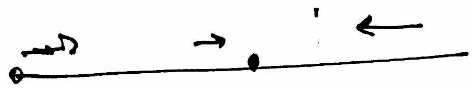
$p(r)r = r - 1$

$p(r) = \frac{r-1}{r}.$

$p(r)r = r^3 -$

$p(r) =$

r



$r^3 - r^2$

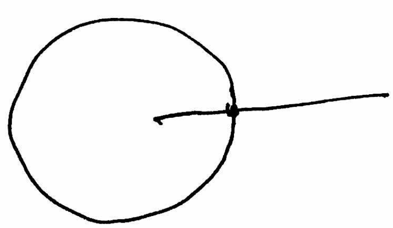
$r - r^3 = r(1-r^2)$

Alg syst.

$$\dot{x} = -y + (1-x^2-y^2)x$$
$$\dot{y} = x + (1-x^2-y^2)y.$$

cf "Hopf bi-furcation . . ."!

Assoc map:



$$\frac{dr}{dt} = r - r^3$$
$$= r(1-r^2)$$

$$\frac{dr}{r(1-r^2)} = dt.$$

S: $$\frac{1}{r(1-r^2)} = \frac{a}{r} + \frac{b}{(1-r)} + \frac{c}{1+r}!$$

$$\int \frac{dr}{r(1-r^2)} = \alpha \ln r + \beta \ln(1-r) + \gamma \ln(1+$$
$$= \ln(r^\alpha (1-r)^\beta (1+r)^\gamma. + c.$$
$$= t.$$

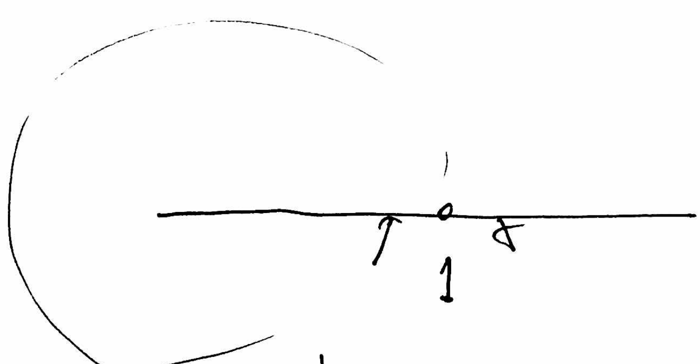
$$\Rightarrow u = e^t = C r^\alpha (1-r)^\beta (1+r)^\gamma.$$

⋮

Finish of like. set $r = 1+h;\ \theta = 0$

$F(h) = \lambda h + O(h^2)$

$0 < \lambda < 1.$



Slice : pos x-axis.

To HW.

$F(h) = -h$ impossible.
for a poinc section.
of an ODE in $\mathbb{R}^2$ having
unit circle as an orbit.

If have not gone to $\boxed{\text{Nathan}}$ yet

DO SO NOW.

Perhaps AFTER Break?

= Break! check: sign ups =

So: from

Flow w/ periodic orbit.

$\to$ germ of map in 1-lower dim.

To reverse: Map F

eg: $\mathbb{T}^2 \xrightarrow{F} \mathbb{T}^2$

$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = A$

$(x, y) \in \mathbb{T}^2 = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \simeq S^1 \times S^1$

so x, y mod 1.

$(x, y) \mapsto (x+y), (x+2y)$.

Find a matrix ξ

s.t. $e^{\xi} = \cancel{M}. A$.

Exer do it!

Hint: evaluating $\xi \in sl(2,\mathbb{R})$

i.e. $\operatorname{tr} \xi = 0$.

Introduce $z \in S^1 = \mathbb{R}/\mathbb{Z}$.

ode: $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \xi \begin{pmatrix} x \\ y \end{pmatrix}$

$\dot{z} = 1$.

Then if $x = 0, y = 0$ have periodic orbit $S^1 \subset \mathbb{T}^3_{x,y,z}$.

Global slice: $z = 1$.

Poinc. section is A.

since solving yields $z(t) = t$,

$\begin{pmatrix} x \\ y \end{pmatrix}(t) = e^{t\xi}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = M\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$.

1st Big surprise (Stability) [Anosov.]

Take the vector field ξ just described on $\mathbb{T}^3$ & look at a C^2-small perturb. $\tilde{\xi}$: $\|\tilde{\xi} - \xi\|_{C^2} < \varepsilon$

or look at the map $A : T^2 \to T^2$
& a C^1-perturb of it
$\|A_\varepsilon - A\| \leq \varepsilon$.

Then a) the flow of $\tilde{\xi}$ is orbit equivalent to the flow ξ,

& b) the Map A_ε is C^0-equiv. to the map A

b):
$$\begin{tikzcd} \mathbb{T}^2 \arrow[r, "A_\varepsilon"] \arrow[d, "h"'] & \mathbb{T}^2 \arrow[d, "h"] \\ \mathbb{T}^2 \arrow[r, "A"'] & \mathbb{T}^2 \end{tikzcd}$$
$\exists$ homeo h; s.t

a): $\exists\ \phi : \mathbb{T}^3 \to \mathbb{T}^3$ homeo s.t if ~~$\gamma(t)$ any orbit~~: ϕ takes solns to $\tilde{\xi}$ - viewed as oriented curves to curves of ξ.

<u>History</u> ; Smale,
visits Russia,
"Axiom A" ...