lect 1.

A) $\dot{\theta} = -\sin\theta$

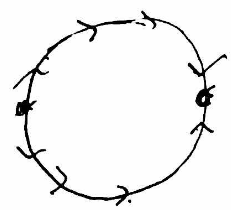
vs

B) $\ddot{\theta} = -\sin\theta$.

$\cdot = \frac{d}{dt}$ $\qquad \theta \in S^1 \simeq \mathbb{R}/2\pi\mathbb{Z}$

A.

$\theta = \pi$



$\theta = 0$

$\sin\theta = 0 \Leftrightarrow \theta = 0, \pi$.

Near $\theta = 0$

$\sin\theta \simeq \theta$

so eqn reads

$\dot{\theta} \simeq -\theta$.

To solve: $\dot{x} = -x$.

& : $\sin(\theta + \pi) \simeq$?

:

or: Now $(\cos\theta)' = -\sin\theta$.

so along solutions

$$\frac{d}{dt}\cos\theta = -\sin\theta\,\dot{\theta}$$

$$= -\sin^2\theta < 0$$

$$= 0 \Leftrightarrow \theta = 0, \pi$$

Now

$$\ddot{\theta} = -\sin\theta .$$

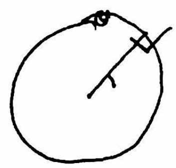
Trick: $\dot{\theta}\ddot{\theta} = -\sin\theta\,\dot{\theta}$

$$\Rightarrow \frac{d}{dt}\frac{1}{2}\dot{\theta}^2 = \frac{d}{dt}\cos\theta .$$

$$\Rightarrow \frac{d}{dt}\Big(\underbrace{\frac{1}{2}\dot{\theta}^2 - \cos\theta}_{\text{energy!}}\Big) = 0 .$$

K.E.: $\frac{1}{2}\dot{\theta}^2$; potl: $-\cos\theta = V(\theta)$

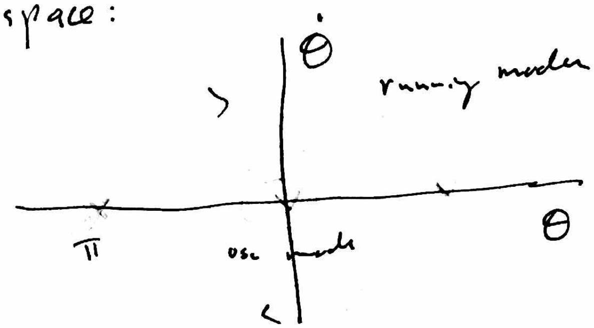
physics of



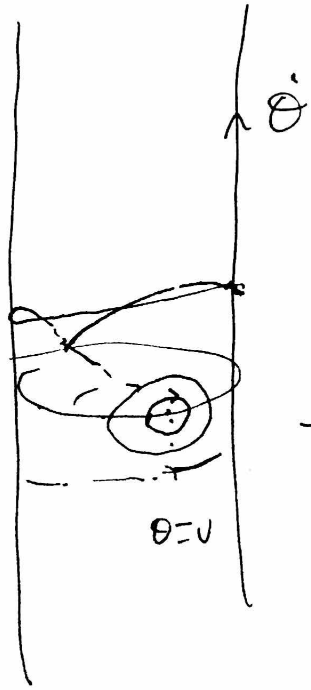
← top potl energy

bottom, potl energy

phase space:



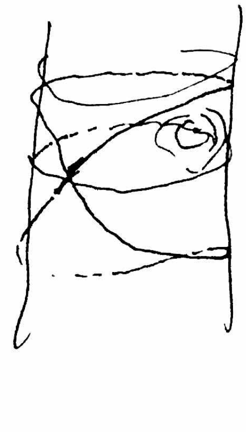
or



Osc modes: contractible in TS^1 = cylinder, θ - oscillation

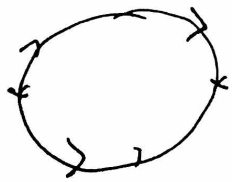
running modes: non contractible. θ, monotonic

$$TS^1 \simeq S^1 \times \mathbb{R}$$



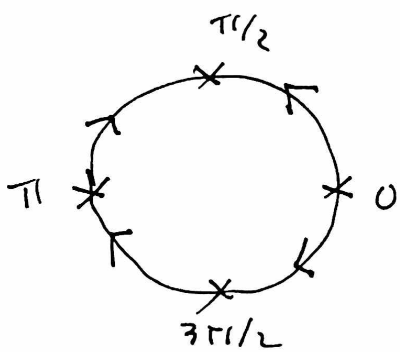
Ex Back to $\dot{\theta} = f(\theta)$

had

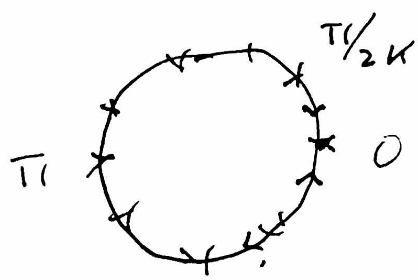


In Class

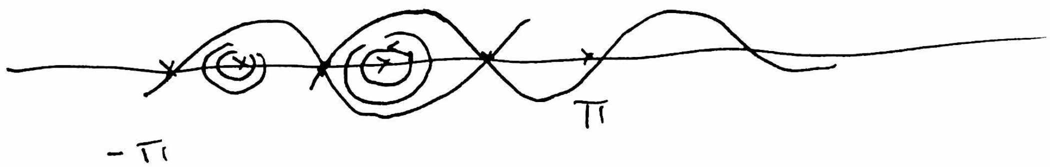
Design f so that:



Next: $2K$th roots of unity ?



Q: why $\rightarrow\times\leftarrow$ & $\leftarrow\times\rightarrow$ come in pairs?



$$\ddot{\theta} = f(\theta) \quad (= -V'(\theta))$$

eg $V = \frac{1}{n}\cos n\theta$

$V' = -\sin n\theta$.

---

replace $\theta \in \mathbb{R}$ or $\theta \in S^1$
By $q \in \mathbb{R}^n$. , $\cos\theta$ by $V(q)$
$-\sin\theta$ by "$V'(q)$" $= -\nabla V$

1) $\dot{q} = -\nabla V(q)$

vs

2) $\ddot{q} = -\nabla V(q)$.

1): Thm $V(q(t)) \downarrow$ along soln

pf. $\frac{d}{dt} V(q(t)) = \nabla V(q(t)) \cdot \dot{q}(t)$

$= -\langle \nabla V, \nabla V \rangle$

$= -\|\nabla V(q(t))\|^2$.

$\therefore \frac{d}{dt} V(q(t)) \leq 0$ w "=" iff $\nabla V(q(t)) = 0$

in which case: equilibrm.

<u>Def</u> An equilibrium ~~or "singular pt"~~ for a v-fld X is a pt q_0 s.t. $X(q_0) = 0$.

<u>ODEs</u> $\dot{q} = X(q)$. $q \in \mathbb{R}^n$ or $q \in M^n$.

<u>Synonyms</u> : "equilibrium"
= "singular point"
= "fixed point"
= zero of vector field.

<u>gradient flow</u> ; applications..

2). $H = \frac{1}{2}\langle \dot q, \dot q\rangle + V(q)$.

Then $\frac{dH}{dt} = 0$ along solns $q(t)$

Pf $\frac{d}{dt}H = \langle \dot q, \ddot q\rangle + \langle \nabla V(q), \dot q\rangle$

$= \langle \dot q, \ddot q + \nabla V(q)\rangle$

$= 0.$

~~So: $\mathbb{R}$~~ eqn as a v-fld.

$\dot q = v$ so $X(q,v) = (v, -\nabla V(q))$

$\dot v = -\nabla V(q)$

$\frac{d}{dt}(q, v) = X(q, v)$.

an ODE on $\mathbb{P} = \mathbb{R}^n \times \mathbb{R}^n$.

Invariant submfds: $H =$ const.

Harm osc, all frequencies equal

$V(q) = \langle q, Aq\rangle$ $A > 0$.

eg: $A = I$.

$\ddot q = -q$. solns (linear!)

$q(t) = \cos t\, A_0 + \sin t\, B_0$.

$=$ co

Break

talk about sign-ups?

HW

syllabus.

latest

~~Man~~ Bulletin Notices

Manifolds $\underline{P}$

"most general space on which it is "easy" to do calculus"

$\underline{Egs}$ $\mathbb{T}^n$, S^n, $\mathbb{R}P^n$,

$\{f = 0\} \subset \mathbb{R}^{n+1}$, $\nabla f(q) \neq 0$ wherever $f(q) = 0$.

$p \in \Sigma = \{f_1 = 0, \dots, f_k = 0\}$

$df_1 \wedge \dots \wedge df_k \neq 0$ where $p \in \Sigma$.

rigid body dyn.
v fld on $TSO(3) \simeq SO(3) \times \mathbb{R}^3$.

$$SO(3) = \{A \in Gl_3(\mathbb{R}) : AA^T = Id, \det A = 1\}$$

$$\simeq \mathbb{R}P^3,$$

$$\dot{\xi} = X(\xi)$$

Classical mech: $\underline{P} = TQ$

Q ~ Riem metric

$H = \frac{1}{2}\langle \dot{q}, \dot{q} \rangle + V(q)$

L8

(2'): $\nabla_{\dot{q}} \dot{q} = -\nabla V(q)$.

(1') : $\dot{q} = -\nabla V(q)$.

1st Basic Thms.
& HW preview

Flows, well-posedness.

1) if X is Lipshitz.
then $\forall \xi_0 \in P$ $\exists !$ curve $\xi(t)$
s.t. $\frac{d\xi}{dt} = X(\xi)$; $\xi(0) = \xi_0$.

- ~~Max domain of~~ $\xi : I \subset \mathbb{R} \to P$
Max domain?
open interval (a, b)
a or $b = \pm\infty$ possible.

Def The v-fld is complete if the max domain for each soln curve is $\mathbb{R}$.

"Well posedness" in PDE:
$\exists$ ✓
$!$ ✓
cts dep.: $\Phi_t(p)$
if $\xi_n \to \xi_0$
soln $\xi(t; \xi_n) \to \xi(t, \xi_0)$

If X is smooth, complete

Then $\phi_t(x)$ is a

diffeo $\forall t$

& $\phi : \mathbb{R} \times P \to P$

is an $\mathbb{R}$-action:

$$\phi_t \circ \phi_s = \phi_{t+s}.$$

If X analytic the flow

analytic.

Find counterex: X alg. $\Rightarrow$ flow alg.

Straightening lemma.

$$X(\xi_0) \neq 0.$$

Then $\exists$ smooth coord $x^1, \dots, x^n$

~~Near~~ centered at ξ_0 $\ni$: $X = \frac{\partial}{\partial x^1}$

Pf ? . . .

Cor: No local invariants near a nonzero pt of a v. field.

an M^n, Ym Z^n. 41

$) \neq 0$, $Y(q_0) \neq 0$.

- a local diffe.

$) \xrightarrow[\psi]{} (V, q_0)$

$X = Y$. in V.

notation