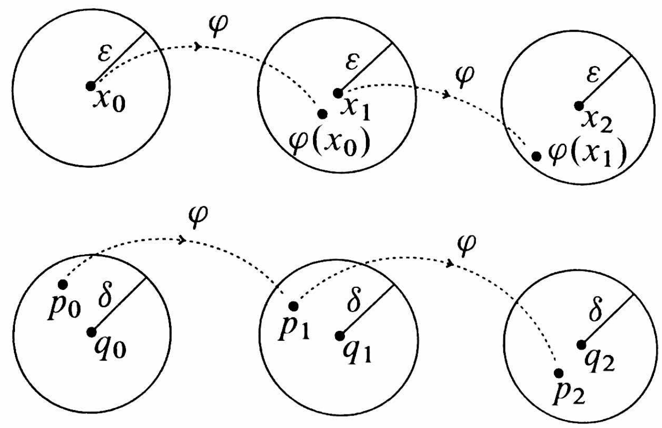

Figure III.4. An ε-pseudo orbit x (above) and a δ shadow orbit p for a sequence q (below).

III.2 The shadowing lemma

In the following, the shadowing lemma will be our main tool. If only an approximate orbit on a hyperbolic set is known, the shadowing lemma guarantees a real orbit nearby which shadows the approximate orbit. This way we shall construct orbits which are determined by their prescribed long-time behavior and not by their initial conditions. To formulate the shadowing lemma, we need some definitions.

Definition. Let $\varphi\colon \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism.

(i) The *sequence* $(x_j)_{j\in\mathbb{Z}}$ in $\mathbb{R}^n$ is an *orbit* of φ, if $x_{j+1} = \varphi(x_j)$ for $j \in \mathbb{Z}$.

(ii) For a given real number $\varepsilon > 0$, the sequence $(x_j)_{j\in\mathbb{Z}}$ is called an *ε-pseudo orbit* of φ, if $|x_{j+1} - \varphi(x_j)| \leq \varepsilon$ for all $j \in \mathbb{Z}$.

(iii) If $\delta > 0$ and $q = (q_j)_{j\in\mathbb{Z}}$ is a sequence in $\mathbb{R}^n$, then a *δ-shadowing orbit of q* is an orbit $p = (p_j)_{j\in\mathbb{Z}}$, satisfying $|p_j - q_j| \leq \delta$ for all $j \in \mathbb{Z}$.

The following theorem goes back to D. Anosov.

Theorem III.7 (Shadowing lemma). *Let Λ be a hyperbolic set of the diffeomorphism φ. Then, there exists a constant $\delta_0 > 0$ such that for every $0 < \delta \leq \delta_0$ there exists an $\varepsilon = \varepsilon(\delta) > 0$ having the following property.*

*For **every** ε-pseudo orbit $q = (q_j)_{j\in\mathbb{Z}}$ of φ **on** the set Λ,*

$$q_j \in \Lambda, \quad |q_{j+1} - \varphi(q_i)| \leq \varepsilon, \quad j \in \mathbb{Z},$$

*there exists **a unique** δ-shadowing orbit $p = (p_j)_{j\in\mathbb{Z}}$ of the pseudo, orbit q (for φ) in a neighborhood of Λ.*

Remark. (i) The bracket (for φ) can be replaced by the bracket (for ψ), if ψ is a diffeomorphism satisfying $|\varphi - \psi|_{C^1(U)} \leq \mu$ on an open neighborhood U of Λ and if μ is sufficiently small.

(ii) The ε-pseudo orbit q does not have to lie on Λ, it is enough to require that the pseudo orbit $q = (q_j)_{j\in\mathbb{Z}}$ belongs to a sufficiently small neighborhood $V(\Lambda)$ of the hyperbolic set Λ.

Proof of Theorem III.7 [Contraction principle]. We make use of the adapted norms guaranteed by Proposition III.6.

(1) *Formulation of the problem.* If the ε-pseudo orbit $q = (q_j)_{j\in\mathbb{Z}} \subset \Lambda$ is given, we look for an orbit $p = (p_j)_{j\in\mathbb{Z}}$ satisfying $|p_j - q_j| \le \delta$ for all $j \in \mathbb{Z}$. For this purpose, we look for *corrections* $x = (x_j)_{j\in\mathbb{Z}}$, so that the sequence

$$p = q + x$$

is an *orbit*, hence satisfies

$$q_{j+1} + x_{j+1} = \varphi(q_j + x_j), \quad j \in \mathbb{Z}.$$

Rewriting this equation we look for a sequence $x = (x_j)_{j\in\mathbb{Z}}$ solving the equation

$$x_{j+1} - d\varphi(q_j)x_j = \varphi(q_j + x_j) - q_{j+1} - d\varphi(q_j)x_j =: f_j(x_j).$$

The right-hand side is *small*, if ε is small, and if $\|x\| = \sup_{j\in\mathbb{Z}} |x_j|$ is small. Indeed, due to $f_j(0) = \varphi(q_j) - q_{j+1}$ we have, by assumption, $|f_j(0)| \le \varepsilon$. In addition, the derivative satisfies $df_j(0) = d\varphi(q_j) - d\varphi(q_j) = 0$ and $df_j(x_j) = d\varphi(q_j + x_j) - d\varphi(q_j)$.

We shall solve the equation $x_{j+1} - d\varphi(q_j)x_j = f_j(x_j)$ by means of the contraction principle.

(2) *The linear problem.* We abbreviate $A_j := d\varphi(q_j) \in \mathcal{L}(\mathbb{R}^n)$. Given a sequence $(g_j)_{j\in\mathbb{Z}}$ in $\mathbb{R}^n$ we look for the sequence $x = (x_j)_{j\in\mathbb{Z}}$ solving

$$x_{j+1} - A_j x_j = g_{j+1}, \quad j \in \mathbb{Z}.$$

For this purpose, we introduce a *sequence space*. Setting $E_j = T_{q_j}\mathbb{R}^n = \mathbb{R}^n$, we define the *Banach space* of bounded sequences by

$$E = \{x = (x_j)_{j\in\mathbb{Z}} \mid x_j \in E_j,\ \|x\| < \infty\}$$

equipped with the norm $\|x\| = \sup_{j\in\mathbb{Z}} |x_j|$. We define the linear map $A \in \mathcal{L}(E)$ by its restrictions $A|_{E_j} := A_j \colon E_j \to E_{j+1}$, as

$$(A(x))_{j+1} := A_j x_j.$$

We want to solve the operator equation $(\mathbb{1} - A)x = g$ in the Banach space E.

Lemma III.8. *If $(q_j)_{j\in\mathbb{Z}}$ is an ε-pseudo orbit on Λ, and if ε is sufficiently small, then the linear map $\mathbb{1} - A \in \mathcal{L}(E)$ is a continuous isomorphism whose inverse map $L := (\mathbb{1} - A)^{-1} \in \mathcal{L}(E)$ is also continuous and has the finite norm $\|L\| < \infty$.*

Proof. We introduce the notation $E_j = E_j^+ \oplus E_j^- = P_{q_j}^+ E_j \oplus P_{q_j}^- E_j$. Given the sequence $g = (g_j) \in E$ we look for a sequence $x = (x_j) \in E$ solving the equation $x_{j+1} - d\varphi(q_j)x_j = g_{j+1}$ for $j \in \mathbb{Z}$, or

$$x_{j+1} = d\varphi(q_j)x_j + g_{j+1}, \quad j \in \mathbb{Z}.$$

With respect to the above splitting we obtain the equivalent equations

$$(*) \quad \begin{cases} P_{q_{j+1}}^+ x_{j+1} = P_{q_{j+1}}^+ d\varphi(q_j)x_j + P_{q_{j+1}}^+ g_{j+1}, \\ P_{q_{j+1}}^- x_{j+1} = P_{q_{j+1}}^- d\varphi(q_j)x_j + P_{q_{j+1}}^- g_{j+1}. \end{cases}$$

The splitting $E_j^+ \oplus E_j^-$ is not invariant under the linearized map $d\varphi(q_j)$, since q is not an orbit. However, along the orbit we know from the definition of the hyperbolicity of the set Λ that

$$P_{\varphi(q_j)}^{\pm} d\varphi(q_j)x_j = d\varphi(q_j)P_{q_j}^{\pm} x_j, \quad j \in \mathbb{Z}.$$

Thus, the equation $(*)$ is equivalent to the following two equations of $(**)$ and $(***)$,

$$(**) \quad P_{q_{j+1}}^+ x_{j+1} = d\varphi(q_j)P_{q_j}^+ x_j + P_{q_{j+1}}^+ g_{j+1} + [P_{q_{j+1}}^+ - P_{\varphi(q_j)}^+]d\varphi(q_j)x_j,$$

$$(***) \quad P_{q_j}^- x_j = d\varphi(q_j)^{-1} P_{\varphi(q_j)}^- x_{j+1} - d\varphi(q_j)^{-1} P_{q_{j+1}}^- g_{j+1}$$
$$+ d\varphi(q_j)^{-1}[P_{q_{j+1}}^- - P_{\varphi(q_j)}^-](x_{j+1} - d\varphi(q_j)x_j).$$

We introduce the map

$$\Phi \colon E \to E, \quad x = (x_j) \mapsto (\Phi(x)_j), \quad \Phi(x)_j := P_{q_j}^+ \Phi(x)_j + P_{q_j}^- \Phi(x)_j,$$

where $P_{q_j}^+ \Phi(x)_j$ is defined by the right-hand side of the equation $(**)$ and $P_{q_j}^- \Phi(x)_j$ by the right-hand side of the equation $(***)$. By construction the desired sequence is a fixed point of this map,

$$\Phi(x) = x \iff x_{j+1} - d\varphi(q_j)x_j = g_{j+1}.$$

Since Λ is compact,

$$\sup_{q \in \Lambda} \|d\varphi(q)\| \leq K, \quad \sup_{q \in \Lambda} \|d\varphi(q)^{-1}\| \leq K$$

for a constant K and the mappings $q \mapsto P_q^{\pm} \colon \Lambda \to \mathcal{L}(\mathbb{R}^n)$ are not only continuous, but *uniformly continuous*. Hence, for every given $\varepsilon' > 0$ there exists an $\varepsilon > 0$ such that

$$\|P_{q_{j+1}}^{\pm} - P_{\varphi(q_j)}^{\pm}\| \leq \varepsilon' \text{ for all } j \in \mathbb{Z}, \quad \text{if } |q_{j+1} - \varphi(q_j)| \leq \varepsilon \text{ for all } j \in \mathbb{Z},$$

i.e., if the sequence q is an ε-pseudo orbit. Since Λ is hyperbolic, we have (in the adapted norms) the estimates

$$|d\varphi(q_j)P^+_{q_j}x_j| \le \vartheta|x_j|,$$
$$|d\varphi(q_j)^{-1}P^-_{q_j}x_{j+1}| \le \vartheta|x_{j+1}|,$$

with a constant $0 \le \vartheta < 1$. Using this, we shall estimate the Lipschitz constant of the map Φ. Recalling the definition of the norms and using the notation $a \vee b := \max\{a, b\}$, we have

$$\begin{aligned}\|\Phi(x) - \Phi(y)\| &= \sup_{j\in\mathbb{Z}} |\Phi(x)_j - \Phi(y)_j| \\ &= \sup_{j\in\mathbb{Z}} \big[|P^+_{q_j}\Phi(x)_j - P^+_{q_j}\Phi(y)_j| \vee |P^-_{q_j}\Phi(x)_j - P^-_{q_j}\Phi(y)_j| \big].\end{aligned}$$

The stable part is estimated as

$$\begin{aligned}|P^+_{q_j}\Phi(x)_j - P^+_{q_j}\Phi(y)_j| &= \big|d\varphi(q_j)P^+_{q_j}(x_j - y_j) \\ &\qquad + [P^+_{q_{j+1}} - P^+_{\varphi(q_j)}]d\varphi(q_j)(x_j - y_j)\big| \\ &\le \vartheta|x_j - y_j| + \varepsilon' K|x_j - y_j|.\end{aligned}$$

For the unstable part we get

$$\begin{aligned}|P^-_{q_j}\Phi(x)_j - P^-_{q_j}\Phi(y)_j| &= \Big|d\varphi(q_j)^{-1}P^-_{\varphi(q_j)}(x_{j+1} - y_{j+1}) \\ &\qquad + d\varphi(q_j)^{-1}[P^-_{q_{j+1}} - P^-_{\varphi(q_j)}] \\ &\qquad \cdot \big[x_{j+1} - y_{j+1} - d\varphi(q_j)(x_j - y_j)\big]\Big| \\ &\le \vartheta|x_{j+1} - y_{j+1}| + \varepsilon' K|x_j - y_j| + \varepsilon' K^2|x_j - y_j|.\end{aligned}$$

Taking the supremum over $j \in \mathbb{Z}$,

$$\|\Phi(x) - \Phi(y)\| \le (\vartheta + \varepsilon' K + \varepsilon' K^2)\|x - y\|$$

for all $x, y \in E$. If we choose $\varepsilon' > 0$ so small that $(\vartheta + \varepsilon' K + \varepsilon' K^2) =: \alpha^* < 1$, the map $\Phi\colon E \to E$ is a *contraction*. The unique fixed point $x = (x_j)_{j\in\mathbb{Z}} \in E$ of the map satisfies, in view of the equations $(**)$, $(***)$ and of Lemma III.3, the estimate

$$\|x\| = \|\Phi(x)\| \le \|\Phi(x) - \Phi(0)\| + \|\Phi(0)\| \le \alpha^*\|x\| + K'\|g\|,$$

with a constant $K' > 0$ and therefore,

$$\|x\| \le \frac{K'}{1 - \alpha^*}\|g\|.$$

In view of $x = (\mathbb{1} - A)^{-1}g$, we have verified the estimate

$$\|(\mathbb{1} - A)^{-1}\| \leq \frac{K'}{1 - \alpha^*},$$

and Lemma III.8 is proved. □

(3) *The nonlinear problem.* Let $r > 0$. We denote the closed balls of radius r in E_j and in E by $B_j(r) := \{x_j \in E_j \mid |x_j| \leq r\}$ and by $B(r) := \{x \in E \mid \|x\| \leq r\}$. We want to solve the equations

$$x_{j+1} - A_j x_j = f_j(x_j),$$

for a sequence $x = (x_j)_{j \in \mathbb{Z}}$ satisfying $x_j \in E_j$, while the sequence of maps $f_j \colon B_j(r) \subset E_j \to E_{j+1}$ is given. Introducing the mapping

$$F \colon B(r) \subset E \to E \quad \text{by} \quad F(x)_{j+1} = f_j(x_j),$$

our equation can be written as $(\mathbb{1} - A)x = F(x)$ or as

$$x = LF(x), \quad x \in B(r),$$

with the continuous linear map $L = (\mathbb{1} - A)^{-1}$. In the following, we write $|\cdot|$ instead of $\|\cdot\|$ for the norm on E and reserve the notation $\|\cdot\|$ for the operator norm.

Lemma III.9. *Let $F \colon B(r) \subset E \to E$ be a map. Assume that the real number $\alpha > 0$ is so small that $\alpha\|L\| \leq 1/2$. If $|F(0)| \leq \alpha r$ and $|F(x) - F(y)| \leq \alpha|x - y|$ for all $x, y \in B(r)$, then the equation $x = LF(x)$ has a* **unique** *solution $x \in B(r)$. This solution satisfies the estimate*

$$|x| \leq 2\|L\|\,|F(0)|.$$

Proof. Set $G(x) := LF(x)$. We claim that

(i) $G \colon B(r) \to B(r)$, and
(ii) $|G(x) - G(y)| \leq \frac{1}{2}|x - y|$ for all $x, y \in B(r)$.

In order to prove the claim we take $x, y \in B(r)$ and estimate, using the assumptions,

$$|G(x) - G(y)| \leq \|L\|\,|F(x) - F(y)| \leq \alpha\|L\|\,|x - y| \leq \frac{1}{2}|x - y|.$$

Observing that $|G(0)| = |LF(0)| \leq \|L\|\,|F(0)| \leq \|L\|\alpha r \leq r/2$, we obtain

$$|G(x)| \leq |G(x) - G(0)| + |G(0)| \leq \frac{1}{2}|x| + \frac{r}{2} \leq r,$$

and the claim is proved. Since the metric space $B(r)$ is complete, there exists a unique fixed point $x = G(x)$ satisfying $|x| \leq r$ and due to $|x| = |G(x)| \leq |G(x) - G(0)| + |G(0)| \leq \frac{1}{2}|x| + |G(0)|$, we arrive at the desired estimate

$$|x| \leq 2|G(0)| \leq 2\|L\|\,|F(0)|.$$

This concludes the proof of Lemma III.9.

Finally, we apply the lemma to our situation and complete the proof of the shadowing lemma. We recall that

$$|F(0)| = \sup_j |f_j(0)| = \sup_j |\varphi(q_j) - q_{j+1}| \leq \varepsilon.$$

We choose α so small that $\alpha\|L\| \leq \frac{1}{2}$. Since Λ is compact and $df_j(0) = 0$, we find a radius $r_0 = \delta_0$ such that $\|df_j(x_j)\| \leq \alpha$ for every $x_j \in B_j(r_0)$ and all $j \in \mathbb{Z}$. By the mean value theorem we conclude $|F(x) - F(y)| \leq \alpha|x - y|$ for $x, y \in B(r_0)$. If now $r \leq \delta_0$ and if $\varepsilon \leq \alpha r$, we conclude from Lemma III.9 that the statement of the shadowing lemma holds true with the constant $\delta = r$. This completes the proof of Theorem III.7.

Proof of the remark following the shadowing lemma. Let $|\varphi - \psi|_{C^1(U)} \leq \mu$ where U is a neighborhood of Λ. Replacing the maps $f_j(x_j)$ in the above proof by the maps

$$f_j'(x_j) = \psi(q_j - x_j) - q_{j+1} - d\psi(q_j)x_j,$$

we can argue as above, if μ is sufficiently small. As for the second part of the remark, we choose $\eta > 0$ such that the $\hat{\varepsilon}$-pseudo orbit $q = (q_j)_{j\in\mathbb{Z}}$ lies in the η-neighborhood of Λ. Choosing a sequence q' *on* Λ satisfying $|q_j - q_j'| \leq \eta$ for all $j \in \mathbb{Z}$, it follows that

$$\begin{aligned}|q_{j+1}' - \varphi(q_j')| &\leq |q_{j+1}' - q_{j+1}| + |q_{j+1} - \varphi(q_j)| + |\varphi(q_j) - \varphi(q_j')|\\ &\leq \eta + \hat{\varepsilon} + \eta \sup_{x\in\Lambda} \|d\varphi(x)\|\\ &=: \varepsilon,\end{aligned}$$

so that q' is an ε-pseudo orbit *on* Λ. If $\eta, \hat{\varepsilon}$ are sufficiently small, we can apply the first part of the theorem to the pseudo orbit $q' \subset \Lambda$ to obtain a $(\delta + \eta)$-shadowing orbit for the pseudo orbit q.

As a first application of the shadowing lemma, we shall prove the closing lemma of Anosov.

Theorem III.10 (Closing lemma of Anosov). *We consider the hyperbolic set Λ of the diffeomorphism φ and let ε, δ be as in the shadowing lemma. If there exists a point $x \in \Lambda$ and an integer $N \geq 1$ satisfying*

$$|\varphi^N(x) - x| \leq \varepsilon,$$

then there exists a point y in a δ neighborhood $U_\delta(\Lambda)$ of Λ satisfying

$$\varphi^N(y) = y.$$

Moreover, the periodic orbit $y, \varphi(y), \dots, \varphi^N(y) = y$ *lies in a δ-neighborhood of the set* $\{x, \varphi(x), \dots, \varphi^N(x)\}$.

Proof [Uniqueness of the δ-shadowing orbit]. We define the ε-pseudo orbit $q = (q_j)_{j\in\mathbb{Z}}$ by the N-periodic continuation of the finite piece of the orbit

$$\begin{array}{ccccc} x & \varphi(x) & \varphi^2(x) & \dots & \varphi^{N-1}(x) \\ \| & \| & \| & & \| \\ q_0 & q_1 & q_2 & \dots & q_{N-1}\,, \end{array}$$

so that $q_{j+N} = q_j$ for all $j \in \mathbb{Z}$. By the shadowing lemma there exists a unique δ-shadowing orbit $p = (p_j)_{j\in\mathbb{Z}}$ of the pseudo orbit q and we claim that

$$p_{j+N} = p_j, \quad j \in \mathbb{Z}.$$

To prove the claim, we introduce the shifted orbit sequence $\hat{p} = (\hat{p}_j)_{j\in\mathbb{Z}}$ by $\hat{p}_j = p_{j+N}$. Then, also $\hat{p}$ is a δ-shadowing orbit of the pseudo orbit q, since

$$|\hat{p}_j - q_j| = |p_{j+N} - q_j| = |p_{j+N} - q_{j+N}| \le \delta$$

holds true for all $j \in \mathbb{Z}$. From the uniqueness of the δ-shadowing orbit which shadows the pseudo orbit q, we conclude that $\hat{p} = p$, so that the orbit p is indeed the desired periodic orbit, as claimed in the theorem. □

III.3 Orbit structure near a homoclinic orbit, chaos

In the following we consider a transversal homoclinic point ν at which, by definition, the stable and unstable invariant manifolds issuing form a hyperbolic fixed point of the diffeomorphism φ intersect transversally. Assuming as before the fixed point to be the origin 0 we denote by

$$\Lambda = \overline{\mathcal{O}(\nu)} = \bigcup_{j\in\mathbb{Z}} \varphi^j(\nu) \cup \{0\} = \mathcal{O}(\nu) \cup \mathcal{O}(0)$$

the closure of the homoclinic orbit which consists of two orbits. The compact set Λ is a hyperbolic set of the diffeomorphism φ in view of Proposition III.5 and so we can use the shadowing lemma in order to prove first that the homoclinic point ν is a *cluster point* of other *homoclinic points* belonging to 0 and at the same time also a cluster point of *periodic points*.