Geodesic Flow

The flow assoc to the geodesic eqns on a Riemannian manifold.

Riem manifold.; review.

A manifold M, with a smoothly varying inner product on each tangent space. Inner product ~~No~~ called "the metric".

Notation. so $\forall q \in M$

$< \, , \, >_q \qquad < \, , \, >_q : T_q M \times T_q M \to \mathbb{R}$

$g \qquad = \quad g_{ij}(q) dq^i dq^j$

$ds^2 \qquad =$ $\quad q^1, \dots q^n$ coord on M.

length of curves.

$$\ell(c) = \int_a^b \sqrt{\langle \dot{c}(t), \dot{c}(t) \rangle}\, dt \quad \text{w/} \quad c : [a,b] \to M.$$

is indep of param.

~~Geodesic~~ Dist: ~~A, M~~ $q_0, q_1 \in M$.

$$d(q_0, q_1) = \inf_c \{ \ell(c) : c \text{ joins } q_0 \text{ to } q_1 \}$$

so $c(0) = q_0$, $c(1) = q_1$

Def ~~$c: I \to M$~~

~~is a geod~~ is

c is a min. geod if.

$c(a) = q_0$, $c(b) = q_1$

& $\ell(c) = d(q_0, q_1)$.

c is a geod if

$c: \mathbb{R} \to M$ if

$\forall\, t_0 \in \mathbb{R}$ $\exists\, \varepsilon$ s.t.

$c|_{[t_0 - \varepsilon,\, t_0 + \varepsilon]}$ is a min geodesic.

~~Calc~~ of
Arc length parameterization
of curves

Cauchy - Schwartz + Calculus of variation

$\Rightarrow$ c is a geod param by arc length

$\Leftrightarrow$ $(c, \dot{c})$ sat a certain ODE called the geodesic eqns

~~The domain of the geodes.~~

in coord.

Christoffel symbols

1) $\ddot{c}^i + \Gamma^i_{jk}(c)\dot{c}^j\dot{c}^k = 0.$

2) $\frac{d}{dt}\left(\frac{\partial}{\partial v^i}\frac{1}{2}g_{ij}(c)v^iv^j\right) = \frac{\partial}{\partial q^i}\frac{1}{2}g_{kl}(q)v^kv^l)$

+ substitute : $v^i = \dot{c}^i$

$\sim \frac{d}{dt}(g_{ij}\cancel{(v)}(c)\dot{c}^j) = \frac{\partial}{\partial q^i}(\frac{1}{2}g_{kl}(q))\dot{c}^k\dot{c}^l$

$\sim$

inverse matrices

3) $\dot{q}^i = \frac{\partial H}{\partial p_i}$ $\qquad H = \frac{1}{2}\sum g^{kl}(q)p_kp_l$

$\dot{p}_i = -\frac{\partial H}{\partial q^i}$

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coord free

Levi-Civita connec.

1) $\nabla_{\dot{c}}\dot{c} = 0$

2) EL eqn for $L(q,\dot{q}) = \frac{1}{2}\langle\dot{q},\dot{q}\rangle_q$.

3) $\dot{\rho} = JdH(\rho) := \text{sgrad}\, H.$

Some eqs.

$M = \mathbb{R}^n, \qquad g_{ij} = \delta_{ij}$

so : usual inner product

then $\Gamma_{ij}^{k} = 0.$

Geod eqn : $\ddot{q} = 0$

soln $q(t) = q_0 + t v_0$

if param by arc length : $\|v_0\| = 1.$

$S^n \subseteq \mathbb{R}^{n+1}$

metric : "induced metric"

Restrict std metric of $\mathbb{R}^{n+1}$

to TS : ~~so~~,

$$T_q S^n = q^{\perp}$$

Exercise in Riem geom.

X, Y v-fields on S^n.

$$\nabla_X Y = (D_{\tilde{X}} \tilde{Y})^T$$

tangential proj

~~tangential~~ any extension to $\mathbb{R}^{n+1}$

usual Jacobian

Cor. $\dot{q}(t) \in S^n$ a geod

$\Longleftrightarrow$ $\ddot{q}(t) \perp T_{q(t)} S^n$ at each t.

$\Longleftrightarrow$ $\exists\, \lambda(t) \in \mathbb{R}$

$$\boxed{\ddot{q}(t) = \lambda(t) q(t)}$$

HW: is to solve . . .

Where does flow happen?

$STM = \{(q, v) : |v|^2 = 1\}$

or $H^{-1}(\frac{1}{2}) \subset T^*Q$

= unit cosphere bundle

N.B. if $q(t)$ solves geod eqn
so does $q(\lambda t)$, $\lambda \in \mathbb{R}$.
just travelled at different speed

or TM or T^*M.

Back to egs:

$\mathbb{R}^n$; $\quad ST\mathbb{R}^n \simeq \mathbb{R}^n \times S^{n-1}$

$$\phi_t(q,v) = (q+tv, v).$$

S^n; $\quad TS^n = \{(q,v) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} :$
$\langle q,q\rangle = 1, \ \langle q,v\rangle = 0$
$\langle v,v\rangle = 1\}.$

$$\phi_t(q,v) = (\cos t\, q + \sin t\, v, -\sin t\, q + \cos t\, v)$$

$$\tilde{\phi}_t(e_0,e_1) = (\cos t\, e_0 + \sin t\, e_1, -\sin t\, e_0 + \cos t\, e_1)$$

rotation!

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geod: great circle

All geod closed of period 2π !

$\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ so: $q_i = q_i + 1$

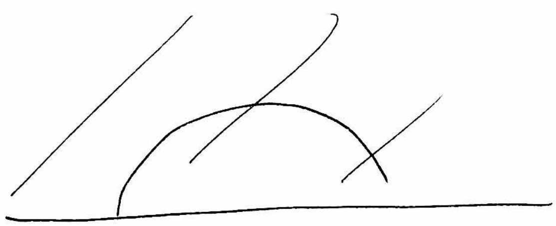
or think "angles"

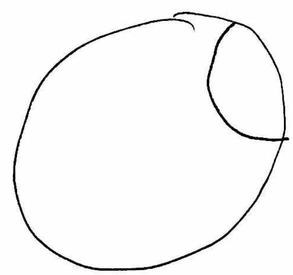
$\theta_i = \theta_i + 2\pi$

geod eqns : $\ddot{\theta}_i = 0$

Break;
see HW next,

& discuss $K=-1$
models





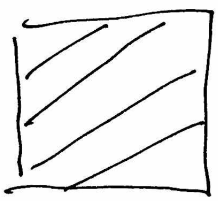
Has n "constants of the motion"

$\omega_1, \dots, \omega_n$

of $\dot{\theta}_i = \omega_i$

$\dot{\omega}_i = 0.$

$n = 2$



If (ω_1, ω_2)

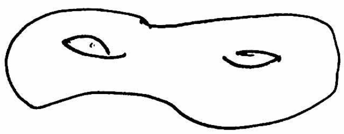
Irr or ratl

flow on torus!

Neg curved case.

"Recall"

closed surface of genus g, $g > 1$



Σ_2

Thm [uniformization thm] Σ_g, $g > 1$ admits a metric of const. curvature -1.

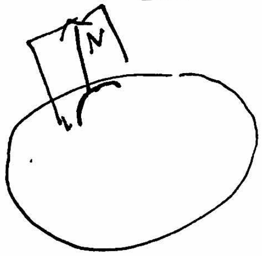
$\Sigma_1 =$  admits of $K = 0$

$\Sigma_0 =$  .. .. $K = +1$

Curvature??

Gaussian.

Sectional..



$\Sigma \subset \mathbb{R}^3$

compute curv of curves obtained by intersecting planes thru p containing N_p ~~m~~ at p.

$$\gamma''_{(p)} = \kappa(p)\, N(p)$$

~~Ma~~ $\kappa(p) = \mathrm{II}_p(v,v) \qquad v = \gamma'(0).$

$$\mathrm{II}_p(v,v) = \mathrm{II}_p(v, -dN_p(v))$$
$$:= \langle v, -dN_p(v)\rangle.$$

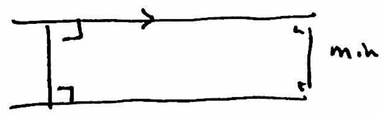
$$\det(-dN_p) = K$$
$$= K_1 K_2$$

where $K_1, K_2 = \mathrm{spec}(-dN_p(\cdot))$

$=$ max, & min. curvature

$=$ extrem values of $\dfrac{\mathrm{II}_p(v,v)}{\mathrm{I}_p(v,v)}$

<u>Meaning</u>

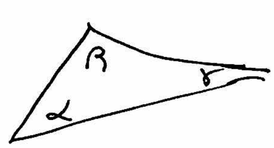


$K = 0$

Gauss Bonnet:

Take a geod polygon.
Let its angle deficit be
the difference between its angle sum
& that of Euc. polygon w/ the
same # of sides.

eg $N = 3$



$$\alpha + \beta + \gamma - \pi = \int K \, dA.$$

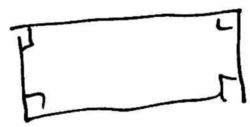
or $\alpha + \beta + \gamma > \pi$ $\quad K > 0$

$< \pi$ $\quad K < 0.$

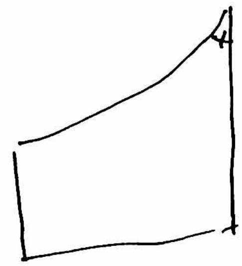
<u>$N = 4$</u>

obtuse : getting close

$K > 0$.



staying same: equidist.



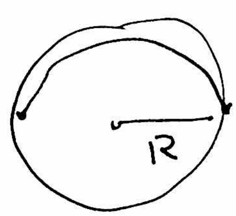
acute: getting farther away.

Thm if $K > \delta^2 > 0$

then $\operatorname{diam} \Sigma \le \dfrac{\pi}{\delta}$.

Eg $S^2(R)$; $K = \dfrac{1}{R^2}$; $\delta = \dfrac{1}{R}$

$\text{diam} = \pi R$



$\max\limits_{p,q \in \Sigma} d(p,q) = \text{diam}$

Exponential Map.

$$T_pM \longrightarrow M.$$

Geod flow: $\phi_t(p,v)$

$$\phi_t : TM \to TM.$$

$$\exp_p v = \pi\,\phi_t(p,v)$$
$$= \gamma(1)$$

where γ = unique geod w/ $\gamma(0)=p$, $\dot\gamma(0)=v$.

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aside: why? "exp"

Thm. Any compact matrix group ($SU(3)$, $SU(2)$, $U(n)$) ... admits a bi-inv. Rien metric. Rel this metrx

$$\exp_I(A) = e^A. = \text{Matrix exp'l.}$$

$$\& \; T_I G \cong \mathfrak{g} \subset \mathfrak{gl}(n)$$

<u>Thm</u> If $K \leq 0$ & M is complete then exp is a covering map.

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where are we going?

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Rigid body motion & $SU(3)$

sectional curv & Jacobi eqn

<u>Anosov flows</u>.