$$\|f^{-1}(y_1)-f^{-1}(y_2)-Df(x_2)^{-1}($$

$$= \|x_1 - x_2 -$$

$$= \|Df(x_2)^{-}$$

$$\leqslant M\|f(x_1)$$

This, together with (iii), shows that f^{-1} is differentiable with derivative $Df(x)^{-1}$ at $f(x)$; i.e., $D(f^{-1}) = \mathcal{J} \circ Df \circ f^{-1}$ on $V_0 = D_{r/2}(\mathbf{0})$. This formula, the chain rule, and Lemma 2.5.5 show inductively that if f^{-1} is C^{k-1} then f^{-1} is C^k for $1 \leqslant k \leqslant r$. ■

This argument also proves the following: If $f\colon U \to V$ is a C^r homeomorphism where $U \subset \mathbf{E}$, and $V \subset \mathbf{F}$ are open sets, and $Df(u) \in \mathrm{GL}(\mathbf{E}, \mathbf{F})$ for $u \in U$, then f is a C^r diffeomorphism.

BOX 2.5A THE SIZE OF THE NEIGHBORHOODS IN THE INVERSE MAPPING THEOREM

An analysis of the preceding proof also gives *explicit estimates* on the size of the ball on which $f(x) = y$ is solvable.† Such estimates are sometimes useful in applications. The easiest one to use in examples involves estimates on the second derivative.

2.5.6 Corollary. *Suppose $f\colon U \subset \mathbf{E} \to \mathbf{F}$ is of class C^r, $r \geqslant 2$, $x_0 \in U$ and $Df(x_0)$ is an isomorphism. Let*

$$L = \|Df(x_0)\| \quad \textit{and} \quad M = \|Df(x_0)^{-1}\|.$$

Assume

$$\|D^2 f(x)\| \leqslant K \quad \text{for} \quad \|x - x_0\| \leqslant R \quad \textit{and} \quad \bar{D}_R(x_0) \subset U.$$

Let

$$R_1 = \min\left\{\frac{1}{2KM}, R\right\},$$

$$R_2 = \min\left\{\frac{1}{R_1}, \frac{1}{2M(L + KR_1)}\right\} \quad \textit{and} \quad R_3 = \frac{R_2}{2L}.$$

---

†We thank M. Buchner for providing this formulation.

Then f maps the ball $\|x - x_0\| \leqslant R_2$ diffeomorphically onto an open set containing the ball $\|y - f(x_0)\| \leqslant R_3$. For $y_1, y_2 \in \bar{D}_{R_3}(f(x_0))$, we have $\|f^{-1}(y_1) - f^{-1}(y_2)\| \leqslant 2L\|y_1 - y_2\|$.

Proof. We can assume $x_0 = \mathbf{0}$ and $f(x_0) = \mathbf{0}$. From

$$Df(x) = Df(\mathbf{0}) + \int_0^1 D^2 f(tx)x\,dt$$

we get $\|Df(x)\| \leqslant L + K\|x\|$ for $x \in \bar{D}_R(\mathbf{0})$. From the identity

$$Df(x) = Df(\mathbf{0})\left\{ I + [Df(\mathbf{0})]^{-1} \int_0^1 D^2 f(tx)x\,dt \right\}$$

and the fact that

$$\|(I + A)^{-1}\| \leqslant 1 + \|A\| + \|A\|^2 + \cdots = \frac{1}{1 - \|A\|}$$

for $\|A\| < 1$ (see the proof of 2.5.5) we get

$$\|Df(x)^{-1}\| \leqslant 2L \quad \text{if} \quad \|x\| \leqslant R \quad \text{and} \quad \|x\| \leqslant \frac{1}{2MK},$$

i.e., if $\|x\| \leqslant R_1$.

As in the proof of the inverse function theorem, let $g_y(x) = [Df(\mathbf{0})]^{-1}(y + Df(\mathbf{0})x - f(x))$. Now

$$\|g_y(x)\| \leqslant M\left(\|y\| + \left\| \int_0^1 D^2 f(tx)x\,dt \right\| \right) \leqslant M(\|y\| + K\|x\|).$$

Hence for $\|y\| \leqslant R_1/2M$, g_y maps $\bar{D}_{R_1}(\mathbf{0})$ to $\bar{D}_R(\mathbf{0})$. We similarly get $\|g_y(x_1) - g_y(x_2)\| \leqslant \frac{1}{2}\|x_1 - x_2\|$ from the mean value inequality and the estimate

$$\|Dg_y(x)\| = \|Df(\mathbf{0})^{-1}\| \left(\left\| \int_0^1 D^2 f(tx)x\,dt \right\| \right) \leqslant M(K\|x\|) \leqslant \tfrac{1}{2},$$

if $\|x\| \leqslant R_1$. Thus, as in the previous proof, $f^{-1}\colon \bar{D}_{R_1/2M}(\mathbf{0}) \to \bar{D}_{R_1}$ is defined. Note that by the mean value inequality,

$$\|f(x)\| \leqslant (L + K\|x\|)\|x\|$$

so if $\|x\| \leqslant R_2$, then $\|f(x)\| \leqslant R_1/2M$. The rest now follows as in the proof of the inverse mapping theorem. ■