Lect ~~3~~4

HW back.

talks today F Faure
tomorrow Thibault] ~~hyperbolic~~ geodesic flow on neg curved surfaces

topics today:

doubling map → solenoid. ?
cat map (a la HW 3)
geod flows
conjugacy & equivalence notions.

DOUBLING

Doubling Map $f: S^1 \to S^1 = \mathbb{R}/\mathbb{Z}$.
$f(\theta) = 2\theta \mod 1$.

* $\boxed{\text{stretch}}$ * $\boxed{\text{fold back on self}}$

* ~~per~~? Fixed points? ?
? Periodic orbits ... ? $(\theta_0 = \frac{1}{3} \quad \theta_1 = \frac{2}{3})$.

To better understand. use binary:

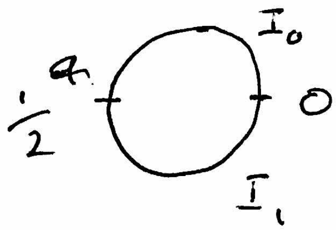
$\theta = .010110$
$= 0 \times \frac{1}{2} + 1 \cdot (\frac{1}{2})^2 + 0 \cdot (\frac{1}{2})^3 + 1 \cdot (\frac{1}{2})^4 + 1 \cdot (\frac{1}{2})^5 + 0 (\frac{1}{2})^6 \ldots$

$\theta = \sum a_i 2^{-i} \quad ; \quad f(\theta) = 2 \sum a_i 2^{-i} = \sum a_{i+1} 2^{-i}$

so $a_i \longrightarrow a_{i+1}$

Model of coin flip!

Meaning of "a_i" dynamically:



$a_0 = 0 \Leftrightarrow \theta \in I_0$

$a_1 = 1 \Leftrightarrow \theta \in I_1$

$$\theta \in I_0 \cap I_1 \Leftrightarrow \left.\begin{array}{ll} \theta = .100\cdots & \\ \text{or} & .0111\cdots \end{array}\right] \tfrac{1}{2}$$

$$\left.\begin{array}{ll} \text{or} & .00\cdots 0 \\ \text{or} & .111\cdots 1 \end{array}\right] \theta = 0$$

so $$a_i = \begin{cases} 0 \Leftrightarrow f^i(\theta) \in I_0. \\ 1 \Leftrightarrow f^i(\theta) \in I_1 \end{cases}$$

$$\begin{tikzcd} (\mathbb{Z}_2)^{\mathbb{N}} \arrow[r, "\text{shift}"] \arrow[d, "\text{binary expansion}"'] & (\mathbb{Z}_2)^{\mathbb{N}} \arrow[d, "\text{binary expansion}"] \\ S^1 \arrow[r, "f"', "\text{doubling}"] & S^1 \end{tikzcd}$$

Binary expansion map:

$$(a)_{i=1}^{\infty} \in (\mathbb{Z}_2)^{\mathbb{N}} \longrightarrow \sum a_i 2^{-i} \in \mathbb{R}/\mathbb{Z}$$

we say "doubling map" is
" semi-conjugate " to the
shift map.

Note : shift = standard
model for coin flipping!

Properties of the space $(\mathbb{Z}_2)^{\mathbb{N}}$

a) topologically a cantor set:
perfect (all pts limit pts of
sequences)

cpt
totally disconnect.

[isomorphism to usual cantor
$(a)_i \longrightarrow \sum 2a_i \left(\frac{1}{3}\right)^i$]

true for A^S $\# A < \infty$
$\# S = \# \mathbb{N}$.

Using the symbol sequence relook
at periodicity proble.
How to make .0110. periodic..

periodic points?
A point w/ dense orbit?

Devaney's criteria for "chaos" *
of a map $f: M \to M$. or "chaotic dynamical system"
M a metric space.

1) the periodic orbits are dense.
2) top. transitivity: $\exists$ a dense orbit
3) "sensitive dependence on initial conditions."

us: to show: $\forall x \in S^1$, $\forall \varepsilon \in \mathbb{R}$, $\varepsilon > 0$.

$\exists\ y \quad d(x,y) < \varepsilon$.

$d(f^n x, f^n y) \geq \frac{1}{4}$. indeed: $\geq \frac{1}{2} - \varepsilon$

($\frac{1}{2} = \operatorname{diam} S^1$)

easier! Devaney:

* cf any small arc eventually wraps all the way around S^1!

p 50

ONTO CAT MAP

A drawback of the doubling map:
is not invertible.

$\Rightarrow$ could not come from a Poincaré section.

To "make" invertible

try

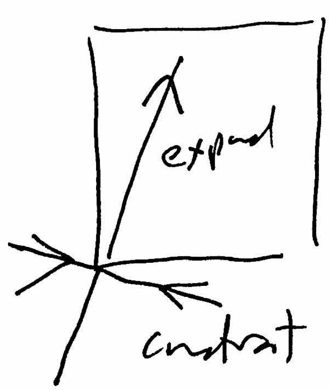
$$\theta \longrightarrow 2\theta \qquad \tfrac{1}{2}\cancel{\theta}$$

$+$

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} 2x \\ \frac{1}{2} y \end{pmatrix}$$ invertible

But cannot "unwrap back"

Instead



$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

on $\mathbb{T}^2$

Discuss now; cf HW.

Thu Oct 8.

Cat Map

Doubling Map.

Geod flows.

$F_A : \mathbb{T}^2 \to \mathbb{T}^2$.

$F_A \begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} \mod \mathbb{Z}^2$.

$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$.

indeed any integer matrix of determinant 1 is good.

HW. a) Find fixed pts *

b) Find periodic pts. *

a) ?

Notion of stable/unstable mfd. ← use eige/va

say p a fixed pt,
$\sigma\, dF_p$ hyperbolic.
$\lambda_1, \lambda_2 \notin S^1$

Thm $\exists$

Exer compute e values

λ_1, λ_2

show $\lambda_1 \lambda_2 = 1$

$\lambda_1, \lambda_2 \notin \mathbb{Q}$

compute e. vector ~~f~~

show eigenlines have irrational slope.

Extra HW. true $\forall M \in SL(2, \mathbb{Z})$

as long as M is not upper or lower triangular

$= -1 + \frac{5}{4} - \frac{1}{4} = 0$ ✓.

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eigendir'n for λ_+: ~~slope~~

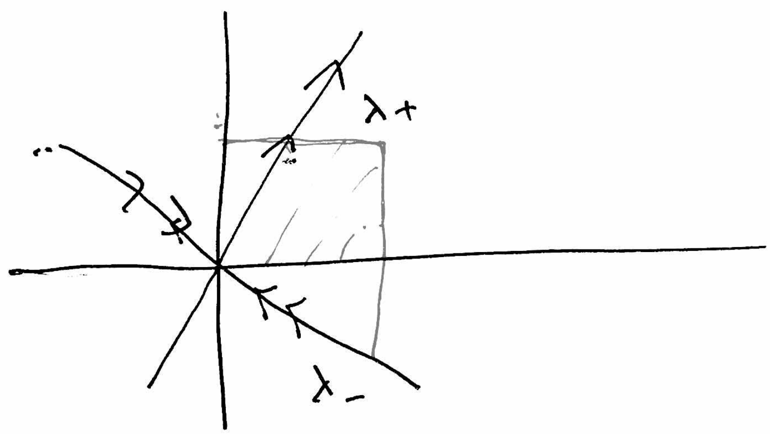
line slope: $\frac{1}{2} + \frac{\sqrt{5}}{2} = m$

" " for λ_-: $\perp$ line,

slope $\frac{1}{2} - \frac{\sqrt{5}}{2} = -\frac{1}{m}$.

scratch checking: $\frac{-1}{\frac{1}{2} + \frac{\sqrt{5}}{2}} = \frac{-1 \cdot \left(\frac{\sqrt{5}}{2} - \frac{1}{2}\right)}{\left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right)\left(\frac{\sqrt{5}}{2} - \frac{1}{2}\right)}$

$= \frac{\frac{1}{2} - \frac{\sqrt{5}}{2}}{1}$ ✓.



N.B.

$\lambda_+ \cdot \lambda_- = 1$

so $\lambda_- = \frac{1}{\lambda_+}$.

Stable / unstable mfld of 0,

Defns Abstract Thms

stable / unstable for flows —

Def.

Eg Pendulum.

Thm Re stable / unstable mfld.

Time permitting!

on to geodesic flows

1st for $\Sigma \subset \mathbb{R}^3$.

use $\ddot{q} \perp T_q\Sigma$. def.

Thm S^2, T^2

then general defns, $K<0$ Eg .. Fourier; Thibault

~~Thm pre~~ & billiards Jair

3 ways to write the ODEs for geodesic flow

$$\nabla_{\dot q} \dot q = 0$$

or

$$\ddot q^i + \Gamma^i_{jk}(q)\, \dot q^j \dot q^k = 0.$$

requires connection.

$$\delta \int \tfrac{1}{2} \langle \dot q(t), \dot q(t) \rangle \, dt = 0$$

Calc of variations

or

$$\frac{d}{dt} \frac{\partial L}{\partial \dot q^i} = \frac{\partial L}{\partial q^i}$$

Euler Lagrange.

$$L = \tfrac{1}{2} g_{ij}(q)\, \dot q^i \dot q^j$$

$$\dot \xi = \text{sgrad}\, H(\xi)$$

or

$$\dot q^i = \frac{\partial H}{\partial p_i}$$

$$\dot p_i = -\frac{\partial H}{\partial q^i}$$

Hamil's eqns

$$H = \tfrac{1}{2} g^{ij}(q)\, p_i p_j$$

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Geod flows

- Rotation of S^1 via oscillators.
- a geod flow on S^2, S^n as restricted oscillators!

$$\dot{q} = v$$
$$\dot{v} = -q.$$
$$\langle q, v\rangle = 0$$
$$|q|^2 = 1.$$

- geod flow of
  - $\Sigma \subset \mathbb{R}^3.$
  - $\Sigma^n \subset \mathbb{R}^{n-1}.$

No fixed pts.

Not chaotic.

$\Sigma, ds^2.$

case: $K = +1$ all geod periodic.
$K = 0.$ ← quasiperiodic & period.
$K = -1$ ← ergodic, mixing, chaotic

Anosov. Thm; state!

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set up for Jac.

Time permitting: conjugacys
semi-conjugacy ...

As special case of oscillator. 1

Odds & ends

Rotations of circle / transl of tori

$\Longleftrightarrow$ Harmonic oscillators.

Tu Oct 8 Lect 9

one oscillator: $\ddot{x} = -\omega^2 x$
$\uparrow$ freq.

To solve: $z = x + iy$

$$\dot{z} = -i\omega z$$
$$= -i\omega(x+iy)$$
$$\omega \quad \dot{x} + i\dot{y} = \omega y - i\omega x$$
$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \omega\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

soln: $z(t) = e^{i\omega t} z(0)$.

Note: $|z(t)| = |z(0)|$

2-oscillator

$$\ddot{x}_1 = -\omega_1^2 x_1$$
$$\ddot{x}_2 = -\omega_2^2 x_2$$

$$\Longleftrightarrow \frac{d}{dt}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} i\omega_1 z_1 \\ i\omega_2 z_2 \end{pmatrix} \text{ on } \mathbb{C}^2.$$

$z_j = x_j + i y_j \qquad y_j = \omega_j^{-1} \dot{x}_j$