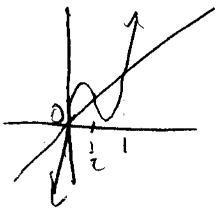


-1.100
2.120
$3,80$
$\Sigma .300$

NAME: George Narlesky

MIDTERM. Math 145. (aka Chaos aka Nonlinear dynamics). 2013.

PROBLEM 1. A cubic $p(x) = Ax^3 + Bx^2 + Cx + D$ defines a map of the line.

A. [20] Derive conditions on the coefficients A, B, C, D of $p(x)$ guaranteeing that the map has fixed points $x = 0, x = \frac{1}{2}$ and $x = 1$.

by def $p(0)=0 \quad p(\frac{1}{2}) = \frac{1}{2} \quad p(1) = 1$ $\qquad 1 = A+B+C$

$A(0)^3 + B(0)^2 + C(0) + D = 0$ $\qquad \frac{1}{2} = A(\frac{1}{2})^3 + B(\frac{1}{2})^2 + C(\frac{1}{2})$

$\boxed{D=0 \quad C = 1 - \frac{B}{3} \quad A = 1 - \frac{2}{3}B}$ $\qquad 1 = \frac{A}{4} + \frac{B}{2} + C$

$4 = A + 2B + 4C = (A+B+C) + B + 3C$
$= 1 + B + 3C = 4$
$B + 3C = 3$
$3C = 3 - B$
$C = 1 - \frac{B}{3}$

B. [20] Verify that $p_k(x) = kx^3 - \frac{3k}{2}x^2 + \frac{k}{2}x + x$ has fixed points at $x = 0, \frac{1}{2}, 1$.

$P_k(0) = 0 = k(0)^3 - \frac{3k}{2}(0)^2 + \frac{k}{2}(0) + (0) = 0$

$P_k(\frac{1}{2}) = k(\frac{1}{2})^3 - \frac{3}{2}k(\frac{1}{2})^2 + \frac{k}{2}(\frac{1}{2}) + \frac{1}{2} = \frac{k}{8} - \frac{3k}{8} + \frac{4k}{24} + \frac{1}{2} = -\frac{2k}{8} + \frac{2k}{8} + \frac{1}{2} = \frac{1}{2}$

FOR C, D, E use the $p_k(x)$ of problem B. $P_k(1) = k - \frac{3k}{2} + \frac{k}{2} + 1 = \frac{2k}{2} + \frac{k}{2} - \frac{3k}{2} + 1 = 1$

C. [20] Find the values of the parameter k guaranteeing that $x = 0$ and $x = 1$ are both attracting fixed points.

$P_k'(x) = 3kx^2 - 3kx + \frac{k}{2} + 1$

$P_k'(0) = \frac{k}{2} + 1 \qquad P_k'(1) = 3k - 3k + \frac{k}{2} + 1 = \frac{k}{2} + 1$

so for both $k \in (-4, 0)$

D. [20] At what value of k does $P_k(x)$ undergo a pitchfork bifurcation at the fixed point $x = \frac{1}{2}$? (This is the value of k at which a period 2 orbit is born out of the fixed point.)

$P_k'(\frac{1}{2}) = 3k(\frac{1}{2})^2 - 3k(\frac{1}{2}) + \frac{k}{2} + 1 = \frac{3k}{4} - \frac{3k}{2} + \frac{k}{2} + 1$
$= \frac{3k}{4} - \frac{2k}{2} + 1 = 1 -$

when $k = 8 \quad p_k'(\frac{1}{2}) = -1$
and a pitchfork bifurcation will occur

E. [20] Find some interval of values of the parameter k such that $p_k(x)$ maps the unit interval *onto* itself. (It is okay if your k- interval is smaller than the largest possible such interval consisting of all those values of k such that $p_k([0, 1]) = [0, 1]$.)

$P_k(x) = k(x^3 - \frac{3}{2}x^2 + \frac{1}{2}x) + x$

very good

$\forall k \in \mathbb{R} \; P_k(0) = 0, P_k(1) = 1 \quad P_k(x)$ is a polynomial, so it is continuous by IVT $P_k([0,1]) \supseteq [0,1]$

choose k s.t. $P_k(x)$ is monotonic increasing, so max/min are at the endpoints of the interval. Work on back

$k \in (0, 4)$

d. $\frac{3C}{3} = \frac{3-B}{3}$ $\quad$ $Ax^3 + Bx^2 + Cx$

$C = 1 - \frac{B}{3}$ $\quad$ $Ax^3 + Bx^2 + \left(1 - \frac{B}{3}\right)x$

by $P(1) = 1$

$$A + B + \cancel{1} - \frac{B}{3} = \cancel{1}$$

$$A + \frac{2B}{3} = 1$$

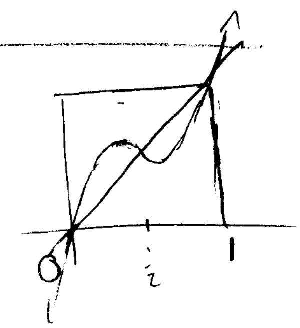
$$A = 1 - \frac{2B}{3}$$

$$\left(1 - \frac{2B}{3}\right)x^3 + Bx^2 + \left(1 - \frac{B}{3}\right)x$$

by $P\left(\frac{1}{2}\right) = \frac{1}{2}$

$$\frac{1}{8} - \frac{B}{12} + \frac{B}{4} + \frac{1}{2} - \frac{B}{6}$$

$P_k'(x) = 3k x^2 - 3kx + \frac{k}{2} + 1$



we want k s.t. P_k is monotonic increasing so k s.t. $P_k'([0,1]) > 0$

$$3kx^2 - 3kx + \frac{k}{2} + 1 > 0 \quad 1 \geq x \geq 0$$

so $3kx^2 + \frac{k}{2} + 1 - 3kx > 0$ $\quad$ $x^2 \leq x$

$$3kx^2 + \frac{k}{2} + 1 > 3kx$$

$$\frac{k}{2} + 1 > 3kx(1-x)$$

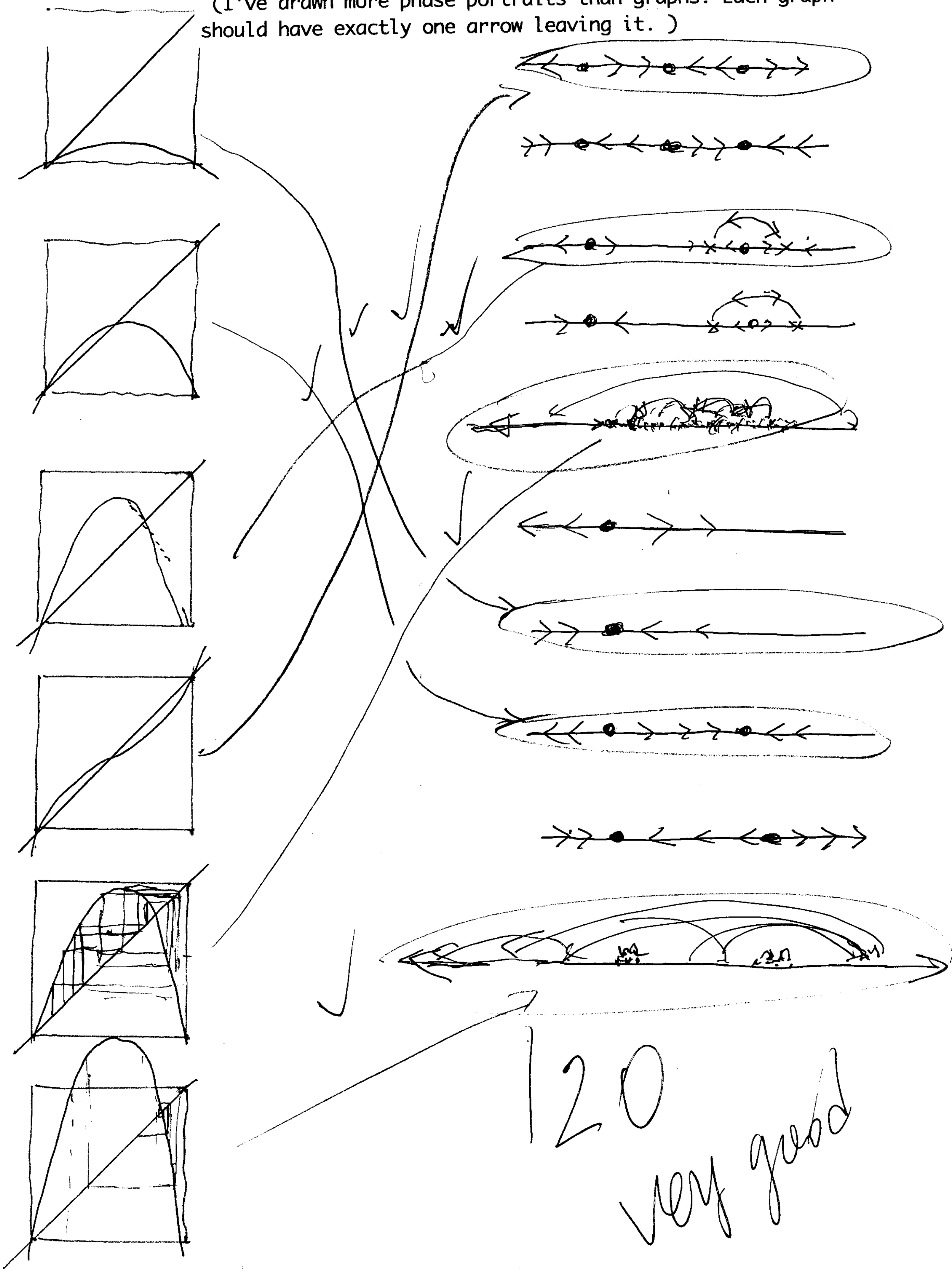
$$\frac{k+2}{2} > \frac{3}{4}k$$

$$k + 2 > \frac{3}{2}k \quad 0 < k < 4$$

$$2 > \frac{1}{2}k$$

for $k > 0$
$3kx(1-x) \leq \frac{3}{4}k$
$\frac{3}{4}k$ is the max
Which implies

on the left side of the paper to the phase portrait
describing that dynamics on the right side.
(I've drawn more phase portraits than graphs. Each graph should have exactly one arrow leaving it.)



120
very good

3. $\mathcal{A} = \{0, 1, 2\}$ is the alphabet with three letters, $0, 1, 2$ and $S = \mathcal{A}^{\mathbb{N}}$ is the set of infinite one-sided sequences in the letters $0, 1, 2$.

A. [20] Define the shift map on S.

for a sequence $\omega \in S$ s.t. $\omega = \omega_0 \omega_1 \omega_2 \omega_3 \dots$ $\omega_i \in A$

the shift map F is defined s.t.

$F(\omega) = \omega' = \omega_1 \omega_2 \omega_3 \dots$ so $F(\omega) = \left\{ \begin{array}{l} \omega'_i = \omega_{i+1} \\ \forall i \in \mathbb{N}_0 \end{array} \right.$

B. [40] Write down 5 *distinct* periodic orbits of period 3 for this shift map.
In your answer, use the over bars notation for indicating periodic sequences: so $\overline{1235} = 123512351235\dots$. Thus, the full list of distinct periodic orbits of period 2 is $\{\overline{01}, \overline{10}\}$, $\{\overline{02}, \overline{20}\}$, $\{\overline{12}, \overline{21}\}$.

all combos $\{$ 000, 001, 002, 010, 011, 012, 020, 021, 022, 100, 101, 102, 110, 111, 112, 120, 121, 122, 200, 201, 202, 210, 211, 212, 220, 221, 222 $\}$

1) $\{\overline{001}, \overline{010}, \overline{100}\}$ 4) $\{\overline{012}, \overline{120}, \overline{201}\}$ 7) $\{\overline{112}, \overline{121}, \overline{211}\}$

2) $\{\overline{002}, \overline{020}, \overline{200}\}$ 5) $\{\overline{021}, \overline{210}, \overline{102}\}$ 8) $\{\overline{122}, \overline{221}, \overline{212}\}$

3) $\{\overline{011}, \overline{110}, \overline{101}\}$ 6) $\{\overline{022}, \overline{220}, \overline{202}\}$

C. [20] Excluding the fixed points, is your list from B the full list of distinct periodic orbits of period 3? Why or why not?

Yes, there are 2 distinct periodic orbits of period 3 excluding fixed pts.
I made a list of all possible combos of length 3, and grouped each sequence of these repeated combos with equivalent sequences under the shift map.
This method is too tedious for higher numbers!