1. 1
2.
3.
4.
5.
total

Name. R Mont.

Final, 2013, winter. Chaos

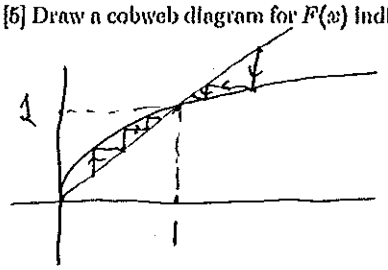
100 total points possible. 20 points per problem. Points divided equally among sub-problems of a given problem, unless stated otherwise.

1. [20 pts] Pick a positive number $N > 1$. Take its square root. Take the square root of that. . Continue, forming the sequence: $x_0 = N, x_1 = \sqrt{N}, x_2 = \sqrt{x_1}, \ldots, x_{n+1} = \sqrt{x_n}, \ldots$

a) [2 pts] What map $F(x)$ are you iterating to form the sequence?

$$F(x) = \sqrt{x}$$

b) [5] Draw a cobweb diagram for $F(x)$ indicating how the sequence is converging.



c) [5] To what number x_* does the sequence converge?

$F(x) = x$, for $x > 0 \iff \sqrt{x} = x \iff x^2 = x,\ x > 0$
$\iff x = 1.$

$x_* = 1.$

d) [8 pts] How fast does the sequence converge? That is: find a good asymptotic estimate (a positive constant) for the successive ratios $(x_{n+1} - x_*)/(x_n - x_*)$ of elements of this sequence, valid for large n.

$$F'(x) = \frac{1}{2} x^{-1/2}, \quad F'(1) = \frac{1}{2}.$$

$$\frac{x_{n+1} - x_*}{x_n - x_*} \simeq \frac{1}{2}.$$

$$\& \lim_{n \to \infty} \frac{x_{n+1} - x_*}{x_n - x_*} = \frac{1}{2}.$$

Name

2. [20] Consider the shift map $\sigma : \Sigma_2 \to \Sigma_2$ on the space Σ_2 of bi-infinite sequences of 0's and 1's. Describe the set of all points whose orbit under σ is heteroclinic connecting $\bar{0} = \ldots 00,00\ldots$ in the past to $\bar{1} = \ldots 11,11\ldots$ in the future. That is, describe the set $W^u(\bar{0}) \cap W^s(\bar{1})$

All sequences of the form

$$\ldots 0000\, s_{-N}\, s_{-N+1} \cdots s_{-1}\, s_0\, s_1 \cdots s_N\, 11111\ldots$$

s_i arbitrary

N finite, arbitrary.

Name

[3.] [20 pts] Drawn are phase portraits of five planar vector fields, labelled I, II, III, IV, and V. The origin is an equilibrium for each. Next to each are letters S and I followed by blanks. Fill in the blank for S according to the spectrum (=eigenvalues) of the linearization of that vector field at the origin. Fill in the blank for I according to the index of that vector field about the origin. Take your answers for S and I from the following lists:

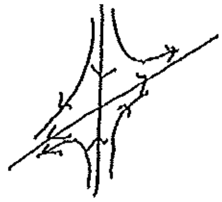
S: $\{0,0\}, \{1,1\}\{-1,1\}, \{-1,-1\},, \{2i,-2i\}, \{1+i,1-i\}, \{-1+i,-1-i\}$

I : $-4,-3,-2,-1,0,1,2,3,4, U =$ Undefined

I

S. $\{-1+i, -1-i\}$
I. 1

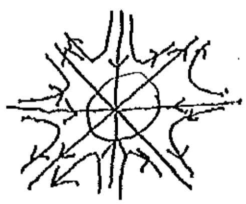
II



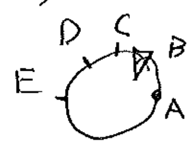
S. $\{-1, 1\}$
I. -1

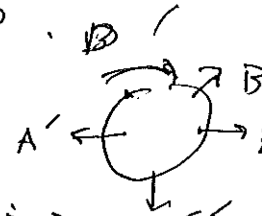
→ eigenlines: y-axis : −1
~ line like
y = x. : +1.

III



S. $\{0,0\}$ I. -3





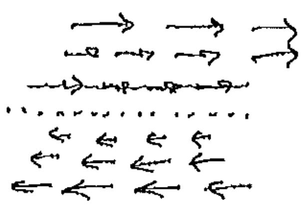
space of arrows.

$\frac{1}{2}$ way round circle ⟹ $-1\frac{1}{2}$ times around image: degree : −3

IV

S. $\{2i, -2i\}$ I. 1

V



S. $\{0,0\}$ I. U

→ v-field
$\dot{x} = y$
$\dot{y} = 0$

matrix: $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

Name

4. [20 pts] For phase portraits II and III of the previous problem please provide: A) a brief argument for your choice S of spectrum , and B) a formula for the ODE $\dot{x} = f(x,y), \dot{y} = g(x,y)$ which that phase portrait represents, and a brief argument for you choice of formula. Take your vector field (f,g) from the following list:
$(f,g) = (x,-y), (-x,-y), (x, 2x-y), (x-2y,-y), (y,0), (0,x),$
$(x^2-y^2, 2xy), (-xy, x^2-y^2), (-x^3+3x^2y, 3x^2y-y^3), (-x^3-3x^2y, 3x^2y+y^3)$.

II. A Saddles have real spectrum $\lambda_1 < 0 < \lambda_2$. $\{1,-1\}$ only thing on list of this type.

B.

on the list $(x,-y)$, $(x,2x-y)$, & $(x-2y,-y)$ all have spectrum $\{1,-1\}$. Their matrices are $\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$, $\begin{pmatrix} 1 & \\ 2 & -1 \end{pmatrix}$, $\begin{pmatrix} 1 & -2 \\ 0 & -1 \end{pmatrix}$.
only $\begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$ so $(x, 2x-y)$ has the depicted eigen<u>lines</u> ie stable unstable directions.

III.A

B.

If more than two lines come in to the origin the vector field has to be non linear with zero linear part. So spectrum is $\{0,0\}$.

Only the guys with squares & cubes are of this type.
Recall: $z^3 = (x+iy)^3 = x^3 + 3ix^2y + 3x(iy)^2 + 3(iy)^3$
$= x^3 - 3xy^2 + i(3x^2y - y^3)$.

Some algebra yields & process of elimination yields $-\bar{z}^3 \leftrightarrow (-x^3+3x^2y, 3x^2y-y^3)$ as our vector field.

5. [20 pts] Consider the system

$$(1) \qquad \begin{aligned} \dot{x} &= x(ax + by - e) \\ \dot{y} &= y(cx + dy - f) \end{aligned}$$

of differential equations with constant real parameters $a, b, c, d; e, f$ such that $ad \neq 0$ and $ef \neq 0$. This vector field has four equilibria, three of which are on the coordinate axes and one of which, say p_4, is not.

(i) (5 pts) (A) Find the x, y coordinates of the three equilibrium points which lie on the axes as functions of the parameters. And (B) write down the two equations which define the 4th point p_4.

p_1 ,,
$(0,0)$

$x = 0$
$cx + dy - f = 0$
w $(0, \frac{f}{d})$
,,
p_2

$y = 0$
$ax + by - e = 0$
$(e/a, 0)$,

$ax + by - e = 0$
$cx + dy - f = 0$
w
$ax + by = e$
$cx + dy = f$.

(ii) (5 pts) Suppose that $p_4 = (1, 1)$ and $e = f$. Can p_4 be a spiral source? Why or why not?

Observe eq. for p_4 reads:
$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e \\ f \end{pmatrix}$. set $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

if $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ solves & $f = e$ we have.

$A \begin{pmatrix} 1 \\ 1 \end{pmatrix} = e \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ so $\lambda = e \in \mathbb{R}$ an e-value!

Since A is real, both λ's must be real,
So: No! No spiral sourse.

(iii) (10 pts) Design a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

so that p_4 is as in Item (ii), that $e = 2$, and that p_4 is a saddle point whose stable manifold is tangent to the line $x = 1$.

write $f = x(ax + by - e)$

$g = y(cx + dy - f)$

The Jacobian $\begin{pmatrix} \partial f/\partial x & \frac{\partial f}{\partial y} \\ \partial g/\partial x & \frac{\partial g}{\partial y} \end{pmatrix}_{(1,1)} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = A$

by computation. We have $A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = e\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

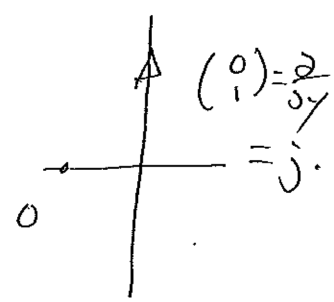
(iv) (EXTRA CREDIT) There is one free parameter for the answer to item (iii), according to the strength (eigenvalue) of attraction along the line $x = 1$ towards p_4. Make a choice for that number. Rewrite out the ODE (1), now with all parameters fixed numbers. Draw the phase portrait for your resulting vector field, indicating the stability type of all four equilibria in your diagram and any heteroclinic connections that might exist.

(iii) ctd.

& $A\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\lambda\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\lambda > 0$.

So: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -\lambda \end{pmatrix} \Rightarrow b = 0.$

Note: tangent direction to $x = 1$ is



eg $A = \begin{pmatrix} 2 & 0 \\ 3 & -1 \end{pmatrix}$ works:

check: $\begin{pmatrix} 2 & 0 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3-1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

$\begin{pmatrix} 2 & 0 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ ✓.

with this choice

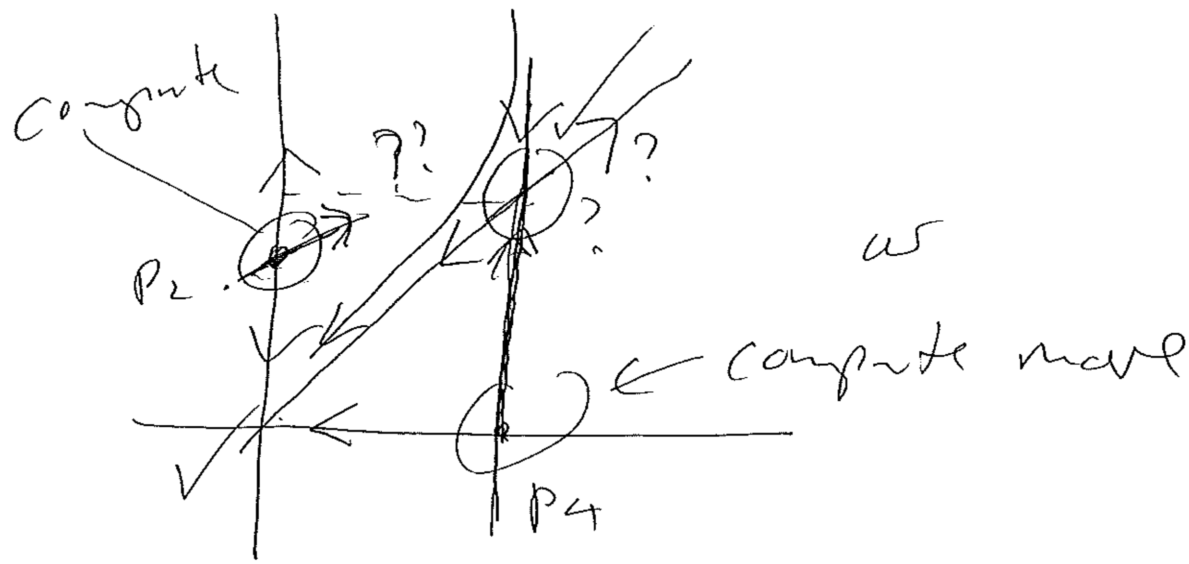
$$\dot{x} = x(2x-2)$$
$$\dot{y} = y(3x-y-2)$$

$p_1 = (0,0)$, $p_2 = (0, \frac{2}{3})$, $p_3 = (\frac{2}{2}, 0)$

$p_4 = (1, 1)$.

linearization structure.

$\frac{\partial f}{\partial x}$



Nope. keep trying Richard...

5/10 on EC.