HW 4.

4.1 A Suppose that F is a discrete dynamical system, i.e a map which we iterate. Suppose that p is a point which has both period 2 and period 3 for F. Prove that F is a fixed point.

4.1. B. Generalize the previous problem by replacing $2, 3$ by relatively prime integers.

4.2. Consider the shift map on the bi-infinite sequence space of 0'a and 1's.

A. List the fixed points of F.

B. List the period 2 orbits which are not fixed points.

C. List the period 3 orbits which are not fixed points. Eg. $\ldots 011.011011011\ldots$ is one. Alternatively: you can use the notation $\bar{0}11$.

D. For p a prime, how many distinct period p orbits are there which are not prime?

4.3. Map the one-sided sequence space of 0's and 1's to the unit interval by thinking of a sequence as the binary expansion of a number. Thus: $0110000\ldots$ maps to the number $0 * \frac{1}{2} + 1 * \frac{1}{2^2} + 1 * \frac{1}{2^3}$. This map is onto.

A. *Prove* the map is not one-to-one by exhibiting two distinct sequences that map to the same number.

B. Recall the definition of "continuous" and "connected" . Look up or recall a basic theorem about the continuous image of a connected set. Use this theorem to prove abstractlly that the map of part A can have no continuous inverse. (Do not use that the map of A is not one-to-one.)

C. Show that the one-sided shift map on the space of one sided sequences of 0's and 1's is semi-conjugate to the doubling map $x \mapsto 2x$ on the circle $\mathbb{R}/\mathbb{Z}$ of numbers x mod 1. Hint: take as semi-conjugacy the map from 4.2 A, mod 1.

4.4. Look up the definition of "space-filling curve". Using such a curve, prove that we can map the Cantor set onto the closed unit square $0 \leq x \leq 1, 0 \leq y \leq 1$ in the place. Generalize your assertion to the unit cube in d-dimensional space.

ON TOPOLOGY, THE CANTOR SET, AND METRIC SPACES

YOU MAY need to consult Appendix 1, and Ch. 9, esp. p 101.

I. From Ch. 7, 80: 1, 2, 3,6; AND: Find the rational whose ternary expansion is a).10000...; b) .02222... Again from p. 80-81: 9-14.

II. Decide whether or not the following subsets of the real line are open, closed, or neither. EXPLAIN WHY, briefly.

a) The set of numbers formed from the sequence $1/2, 1/3, 1/4, \ldots$.

b) The same set as in a), but with the number 0 included.

c). The rational numbers.

d). The integers

e) The union of the intervals $(0, 1), (1, 1 + 1/2), (3, 3 + 1/3), (4, 4 + 1/4), \ldots$.

3. Remember that the Cantor set consists of all numbers in the unit interval whose ternary expansion contains ONLY 0's or 2's. For example .02222... $= 1/3$ is in ,and .01212222... is not. Define a map f from the Cantor set to the unit interval by taking all the 2s, changing them to 1's then thinking of the new decimal in BINARY. Thus, if $x = (2/3) + (0/9) + (2/27) + (2/81) + \ldots$, so that it is the

ternary expansion .2022..., then we map it to .1011... thought of in binary: $f(x) = (1/2) + (0/4) + (1/8) + (1/16) + ...$

a) Show that f is continuous.

b) Contrary to the text, show that f IS NOT one-to-one? HINT: consider points at the edge of deleted intervals, for example $x = 1/3 and y = 2/3$, or $x = 1/9$ and $y = 2/9$ and their images under f.

[It may help to consult the page of LC Young.]