L1 - linear symplectic category, composition of Lagr. corresp.

Note Title 2/7/2008

The linear symplectic category [Guillemin-Sternberg: The moment map revisited]

$(V_0,\omega_0), (V_1,\omega_1)$ symplectic vector spaces

$$\left(\begin{array}{l} \text{i.e. } \omega_j \text{ nondegenerate skew-symmetric bilinear form} \\ (V_j,\omega_j) \underset{\text{isom.}}{\cong} (\mathbb{R}^{2n_j}, \sum_{i=1}^{n_j} dx_i \wedge dy_i) \end{array} \right)$$

- Lagrangian subspace : $\Lambda \subset V_0$ subspace,

$$\omega_0|_\Lambda \equiv 0, \quad \dim\Lambda = \tfrac{1}{2}\dim V_0 \quad (\text{or } \omega_0(v,\Lambda)=\{0\} \Rightarrow v\in\Lambda)$$

- linear symplectomorphism: $\Phi : V_0 \xrightarrow{\sim} V_1$ isomorphism, $\Phi^*\omega_1 = \omega_0$

<u>Note</u>: $\text{graph}\,\Phi = \{(v_0,\Phi v_0) \mid v_0\in V_0\} \subset (V_0\times V_1, (-\omega_0)\oplus\omega_1) =: V_0^- \times V_1$ is a Lagrangian subspace

$$\left[\begin{array}{l} \bullet \dim \text{gr}\,\Phi = \dim V_0 = \tfrac{1}{2}(\dim V_0 + \underset{\dim V_0}{\underset{\shortparallel}{\dim V_1}}) \\ \bullet ((-\omega_0)\oplus\omega_1)\,((v_0,\Phi v_0),(v_0',\Phi v_0')) = -\omega_0(v_0,v_0') + \omega_1(\Phi v_0,\Phi v_0') \\ \qquad = -\omega_0(v_0,v_0') + \underset{\omega_0}{\underset{\shortparallel}{\Phi^*\omega_1}}(v_0,v_0') = 0 \end{array} \right]$$

<u>Def$^{\underline{n}}$</u>: A <u>canonical relation</u> (linear Lagrangian correspondence) from V_0 to V_1 is a Lagrangian subspace $\Lambda_{01} \subset V_0^-\times V_1$.

We write $V_0 \xrightarrow{\Lambda_{01}} V_1$ since it generalizes symplectomorphisms.

$\underline{\text{Def}^{\underline{n}}}$: The <u>composition</u> of $V_0 \xrightarrow{\Lambda_{01}} V_1$ and $V_1 \xrightarrow{\Lambda_{12}} V_2$ is

$$\Lambda_{01} \circ \Lambda_{12} := \{ (v_0, v_2) \mid \exists v_1 \in V_1 : (v_0, v_1) \in \Lambda_{01}, (v_1, v_2) \in \Lambda_{12} \}$$

$$= \pi_{02} (\underbrace{\Lambda_{01} \times_{\Delta_1} \Lambda_{12}}_{:= (\Lambda_{01} \times \Lambda_{12}) \cap (V_0 \times \Delta_1 \times V_2)}) \subset V_0 \times V_2$$

$\pi_{02} : V_0 \times V_1 \times V_1 \times V_2 \to V_0 \times V_2$ projection

$\Delta_1 = \{ (v_1, v_1) \mid v_1 \in V_1 \} \subset V_1 \times V_1$ diagonal

$\underline{\text{Ex.}}$:

$$\begin{tikzcd} V_0 \arrow[r, "\Phi"] \arrow[rr, bend right, "\Psi \circ \Phi"'] & V_1 \arrow[r, "\Psi"] & V_2 \end{tikzcd} \quad \text{symplectomorphisms}$$

$$\begin{tikzcd} V_0 \arrow[r, "gr\,\Phi"] \arrow[rr, bend right] & V_1 \arrow[r, "gr\,\Psi"] & V_2 \end{tikzcd}$$

$$gr\,\Phi \circ gr\,\Psi = gr(\Psi \circ \Phi)$$

<u>Lemma</u>: $\Lambda_{01} \circ \Lambda_{12} \subset V_0^- \times V_2$ is a canonical relation

<u>Proof</u>: • $(v_0, v_2), (v_0', v_2') \in \Lambda_{01} \circ \Lambda_{12}$ with corresponding $v_1, v_1' \in V_1$

$$((-\omega_0) \oplus \omega_2)((v_0, v_2), (v_0', v_2')) = \underbrace{-\omega_0(v_0, v_0') + \omega_1(v_1, v_1')}_{\Lambda_{01} \text{ Lagr.}\; = 0} \underbrace{+\, \omega_2(v_2, v_2') - \omega_1(v_1, v_1')}_{= 0\; \Lambda_{12} \text{ Lagr.}}$$

• $\Lambda_{01} \times_{\Delta_1} \Lambda_{12} \subset V_0 \times V_1 \times V_1 \times V_2$ is a subspace of dimension

$$\underbrace{\dim \Lambda_{01} + \dim \Lambda_{12} - \dim V_1}_{= \frac{1}{2} \dim V_0 \times V_2} + \dim \frac{V_0 \times V_1 \times V_1 \times V_2}{(\Lambda_{01} \times \Lambda_{12}) + (V_0 \times \Delta_1 \times V_2)}$$

• $\pi_{02}(\Lambda_{01} \times_{\Delta_1} \Lambda_{12})$ is a subspace of dimension

$$\dim(\Lambda_{01} \times_{\Delta_1} \Lambda_{12}) - \dim(\Lambda_{01} \times_{\Delta_1} \Lambda_{12} \cap \overbrace{\{0\} \times V_1 \times V_1 \times \{0\}}^{= \ker \pi_{02}})$$

$$\frac{V_0\times V_1\times V_1\times V_2}{(\Lambda_{01}\times\Lambda_{12})+(V_0\times\Delta_1\times V_2)} \cong {V_1}\Big/{\pi_1(\Lambda_{01})+\pi_1(\Lambda_{12})} \cong \pi_1(\Lambda_{01})^{\perp\omega_1}\cap\pi_1(\Lambda_{12})^{\perp\omega_2}$$

$$\| *$$

$$\Lambda_{01}\times_{\Delta_1}\Lambda_{12}\cap\{0\}\times V_1\times V_1\times\{0\} \cong \{v\in V_1 \mid (0,v)\in\Lambda_{01},\ (v,0)\in\Lambda_{12}\}$$

$$* \quad \pi_1(\Lambda_{01})^{\perp\omega_1} = \{v\in V_1 \mid \underbrace{\omega_1(v,v_1) = 0 \quad \forall (v_0,v_1)\in\Lambda_{01}}_{\overset{-\omega_0(0,v_0)}{}} \}$$

$$\Leftrightarrow \quad (0,v)\in\Lambda_{01}^{\perp_{-\omega_0+\omega_1}} = \Lambda_{01}$$

$$= \{v\in V_1 \mid (0,v)\in\Lambda_{01}\} \qquad ∎$$

We define the linear symplectic category <u>Symp</u> by

- objects : (V,ω) symplectic vector space
- morphisms $\mathrm{Mor}(V_0,V_1)$: canonical relations $V_0 \xrightarrow{\Lambda_{01}} V_1$
- composition – as above
- identity $\mathrm{Mor}(V,V)\ni 1_V := \Delta_V \subset V^-\times V$ diagonal

TO CHECK: composition is associative, $1_V\circ\Lambda = \Lambda = \Lambda\circ 1_V$

Lagrangian correspondences and geometric composition

$(M_0,\omega_0), (M_1,\omega_1)$ symplectic manifolds

$$\left[\begin{array}{l} \omega_j \text{ nondegenerate 2-form}, \quad d\omega_j = 0 \\ \text{Darboux}: \quad (M_j,\omega_j) \underset{\substack{\text{locally}\\ \text{diffeom.}}}{\cong} (\mathbb{R}^{2n_j}, \sum dx_i \wedge dy_i) \end{array}\right]$$

- Lagrangian submanifold : $L \subset M_0$ submanifold

  $\omega_0|_L \equiv 0$, $\dim L = \frac{1}{2}\dim M_0$ $\left(\text{or } \omega_0(v, T_xL) \equiv 0 \Rightarrow v \in T_xL \ \forall x \in L\right)$

- symplectomorphism : $\varphi : M_0 \xrightarrow{\sim} M_1$ diffeom., $\varphi^*\omega_1 = \omega_0$

<u>Note</u>: $\operatorname{graph}\varphi = \{(m_0, \varphi(m_0)) \mid m_0 \in M_0\} \subset M_0^- \times M_1 := (M_0 \times M_1, -\pi_0^*\omega_0 + \pi_1^*\omega_1)$

$$\begin{array}{ccc} & M_0 \times M_1 & \\ {\scriptstyle \pi_0}\swarrow & & \searrow{\scriptstyle \pi_1} \\ M_0 & & M_1 \end{array}$$

is a Lagrangian submanifold

$\rightsquigarrow$ —"— correspondence from M_0 to M_1

<u>Def$^{\underline{n}}$</u>: A <u>Lagrangian correspondence</u> from M_0 to M_1 is a Lagrangian submanifold $L_{01} \subset M_0^- \times M_1$; short $M_0 \xrightarrow{L_{01}} M_1$.

<u>Ex.</u>: Lagr. corresp. $pt \to M$ $\;\hat{=}\;$ Lagr. submanifolds of M

Lagr. corresp. $M \to pt$ $\;\hat{=}\;$ Lagr. submanifolds of M^-

⚠ A Lagr. submfld $L \subset M_0^- \times M_1 \times M_2^-$ can be a correspondence
$M_0 \to M_1 \times M_2^-$
$M_0 \times M_1^- \to M_2^-$
$pt \to M_0^- \times M_1 \times M_2^-$
$M_0 \times M_1^- \times M_2 \to pt$

Def$^{\underline{n}}$: The <u>dual</u> of $M_0 \xrightarrow{L_{01}} M_1$ is $M_1 \xrightarrow{L_{01}^t} M_0$

$$L_{01}^t = \{(v_1, v_0) \mid (v_0, v_1) \in L_{01}\} \subset M_1^- \times M_0 .$$

Ex.: $(gr\,\varphi)^t = gr(\varphi^{-1})$

Exercise: When is $L_{01} \circ L_{01}^t = \Delta_{M_0}$, $L_{01}^t \circ L_{01} = \Delta_{M_1}$, or both ?

Def$^{\underline{n}}$: The geometric <u>composition</u> of $M_0 \xrightarrow{L_{01}} M_1$ and $M_1 \xrightarrow{L_{12}} M_2$ is

$$L_{01} \circ L_{12} := \{(x_0, x_2) \mid \exists x_1 \in M_1 : (x_0, x_1) \in L_{01} , (x_1, x_2) \in L_{12}\}$$

$$= \pi_{02}(L_{01} \underset{\Delta_1}{\times} L_{12}) \qquad \subset M_0^- \times M_2$$

Ex: $gr\,\varphi \circ gr\,\psi = gr(\psi \circ \varphi)$

$$gr\,\varphi_0 \circ L_{01} = (\varphi_0 \times \mathcal{U}_{M_1})(L_{01}) \quad , \quad L_{01} \circ gr\,\varphi_1 = (\mathcal{U}_{M_0} \times \varphi_1)(L_{01})$$

(IDENTITY) $\qquad \Delta_{M_0} \circ L_{01} = L_{01} \quad , \quad L_{01} \circ \Delta_{M_1} = L_{01}$

(ASSOCIATIVITY) $\qquad (L_{01} \circ L_{12}) \circ L_{23} = L_{01} \circ (L_{12} \circ L_{23})$

(DUALITY) $\qquad (L_{01} \circ L_{12})^t = L_{12}^t \circ L_{01}^t$

<u>Def$^{\underline{n}}$</u>: $L_{01} \circ L_{12}$ is

- <u>transverse</u> if $(L_{01} \times L_{12}) \pitchfork (M_0 \times \Delta_{M_1} \times M_2)$

  i.e. $\left(T_{(x_0,x_1)} L_{01} \times T_{(x_1,x_2)} L_{12}\right) + \left(T_{x_0} M_0 \times T_{(x_1,x_1)} \Delta_{M_1} \times T_{x_2} M_2\right) = T_{(x_0,x_1,x_1,x_2)} M_0 \times M_1 \times M_1 \times M_2$

  for all $(x_0,x_1,x_1,x_2) \in (L_{01} \times L_{12}) \cap (M_0 \times \Delta_{M_1} \times M_2)$.

- <u>embedded</u> if $\forall (x_0,x_2) \in L_{01} \circ L_{12} \ \exists!\, x_1 \in M_1 : \begin{array}{l} (x_0,x_1) \in L_{01} \\ (x_1,x_2) \in L_{12} \end{array}$

<u>Lemma</u>: (i) transverse $\Rightarrow L_{01} \times_{M_1} L_{12} \subset M_0 \times M_1 \times M_1 \times M_2$ is a submanifold

and $\pi_{02} : L_{01} \times_{M_1} L_{12} \longrightarrow M_0 \times M_2$ is an immersion

(ii) transverse & embedded $\Rightarrow L_{01} \circ L_{12} \subset M_0^- \times M_2$ is a Lagr. correspondence

(and π_{02} an embedding)

<u>Proof</u>: (i) implicit function theorem for $L_{01} \times L_{12} \longrightarrow M_1 \times M_1 \xrightarrow{\text{local coordinates}} \mathbb{R}^{2n_1}$

$(x_0,x_1,x_1',x_2) \mapsto (x_1,x_1') \longmapsto x_1 - x_1'$

$$\underbrace{\ker d\pi_{02} \cong}_{\text{as in linear Lemma}} T(M_0 \times M_1 \times M_1 \times M_2) \big/ (TL_{01} \times TL_{12}) + (TM_0 \times T\Delta_1 \times TM_2) \underbrace{=}_{\text{transversality}} \{0\}$$

(ii) $\pi_{02} : L_{01} \times_{M_1} L_{12} \longrightarrow M_0^- \times M_2$ is injective ■

L2 - Examples of Lagrangian correspondences and composition

Note Title 2/7/2008

Def$^{\underline{n}}$: $L_{01} \circ L_{12}$ is

- **transverse** if $(L_{01} \times L_{12}) \pitchfork (M_0 \times \Delta_{M_1} \times M_2)$ i.e. $\forall (x_0, x_1, x_1, x_2) \in (\ldots) \cap (\ldots)$

  i.e. $\left(T_{(x_0,x_1)} L_{01} \times T_{(x_1,x_2)} L_{12}\right) + \left(T_{x_0} M_0 \times T_{(x_1,x_1)} \Delta_{M_1} \times T_{x_2} M_2\right) = T_{(x_0,x_1,x_1,x_2)} M_0 \times M_1 \times M_1 \times M_2$

- **embedded** if $\forall (x_0, x_2) \in L_{01} \circ L_{12} \quad \exists! x_1 \in M_1 : \begin{array}{l} (x_0, x_1) \in L_{01} \\ (x_1, x_2) \in L_{12} \end{array}$

Lemma: (i) transverse $\Rightarrow$ $L_{01} \times_{M_1} L_{12} \subset M_0 \times M_1 \times M_1 \times M_2$ is a submanifold

and $\pi_{02} : L_{01} \times_{M_1} L_{12} \longrightarrow M_0 \times M_2$ is an immersion

(ii) transverse & embedded $\Rightarrow$ $L_{01} \circ L_{12} \subset M_0^- \times M_2$ is a Lagr. correspondence

(and π_{02} an embedding)

Proof: (i) implicit function theorem for $L_{01} \times L_{12} \longrightarrow M_1 \times M_1 \xrightarrow{\text{local coordinates}} \mathbb{R}^{2n_1}$

$(x_0, x_1, x_1', x_2) \longmapsto (x_1, x_1') \longmapsto x_1 - x_1'$

- $TL_{01} \times TL_{12} \longrightarrow TM_1$ surjective

  $(v_0, v_1, v_1', v_2) \longmapsto v_1 - v_1'$

$\Leftrightarrow$ $(TL_{01} \times TL_{12}) \times \underset{TM_0}{\vee} \, T\Delta_{M_1} \underset{TM_2}{\vee} \longrightarrow TM_0 \times TM_1 \times TM_1 \times TM_2$ surjective

$(v_0, v_1, v_1', v_2 \,; w_0, w_1, w_1, w_2) \longmapsto (w_0, w_1 + v_1, w_1 + v_1', w_2)$

$\Updownarrow$ transversality

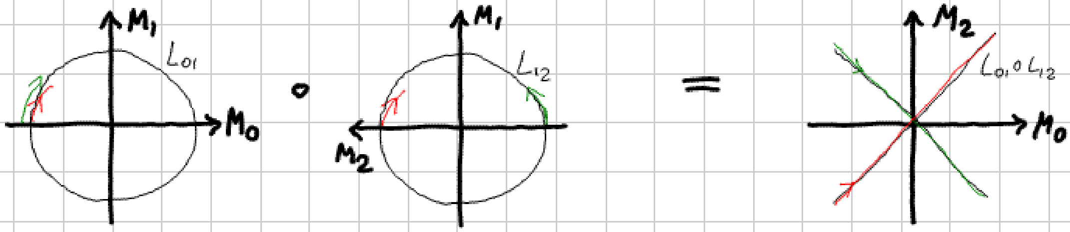
- immersion:

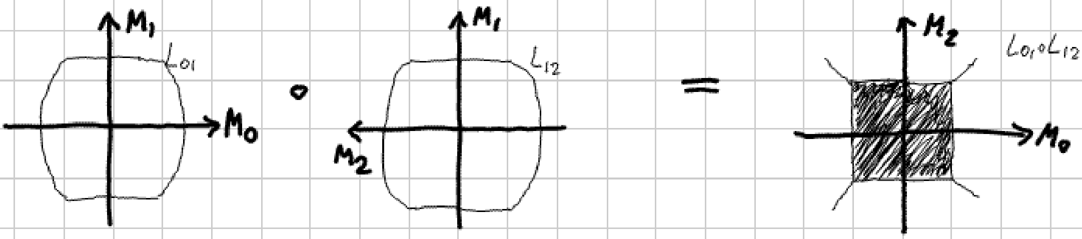
$\ker d\pi_{02} \cong T(M_0 \times M_1 \times M_1 \times M_2) \big/ (TL_{01} \times TL_{12}) + (TM_0 \times T\Delta_1 \times TM_2) = \{0\}$

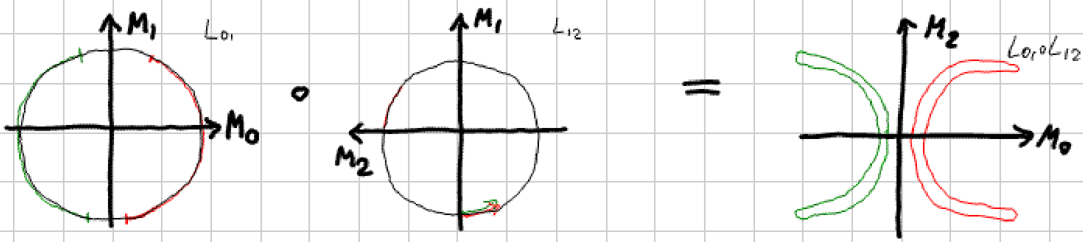
(the $\cong$ as in linear Lemma)

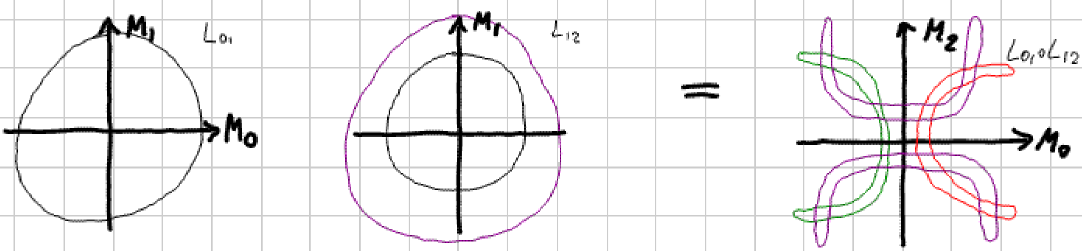
(ii) $\pi_{02} : L_{01} \times_{M_1} L_{12} \longrightarrow M_0^- \times M_2$ is injective immersion $\blacksquare$

Examples $\quad M_i = \mathbb{R}$ (not symplectic)









<u>Question</u>: Can transversality be achieved by shifting L_{01}, L_{12} with Hamiltonian diffeomorphisms on M_0, M_2?

NO?! $\rightsquigarrow$ no easy definition of $L_{01} \circ L_{12}$ as immersed Lagrangian / Ham. diffeom.

Lagrangian correspondences arising from fibered coisotropics

[McDuff-Salamon §5.3]

(M,ω) symplectic mfd. , $\dim M = 2n$

- $C \subset M$ coisotropic submanifold

$$\left[\forall x \in C \quad T_xC^{\omega} := \{ v \in T_xM \mid \omega(v, T_xC) \equiv 0 \} \subset T_xC \right]$$

$$\left(\begin{array}{l} \Rightarrow \dim C = n+k \quad ; k \geq 0 \quad , \quad \dim T_xC^{\omega} = n-k \\ \Rightarrow \dim {}^{TC}/_{TC^{\omega}} = 2k \qquad \text{and} \quad \left({}^{TC}/_{TC^{\omega}}, \omega \right) \text{ is symplectic} \end{array} \right)$$

<u>Lemma:</u> The "null foliation" $(TC)^{\omega} \subset TC$ is integrable,

i.e. locally $(TC)^{\omega} = TN$; $N \subset C$ submanifold ("isotropic leaf").

<u>Proof:</u> $X, Y \in \Gamma(TC)$ vector fields

need to check: $X, Y \in TC^{\omega}$ in nbhd of $p \in C$ $\Rightarrow [X,Y](p) \in T_pC^{\omega}$

$\forall Z \in \Gamma(TC) \qquad \omega([X,Y],Z)$

$$= \omega([X,Y],Z) + \omega(\underset{TC}{[Y,Z]}, \underset{TC^{\omega}}{X}) + \omega(\underset{TC}{[Z,X]}, \underset{TC^{\omega}}{Y}) + \mathcal{L}_X\omega(\underset{TC}{Z}, \underset{TC^{\omega}}{Y}) + \mathcal{L}_Y\omega(\underset{TC^{\omega}}{X}, \underset{TC}{Z}) + \mathcal{L}_Z\omega(\underset{\substack{TC^{\omega}\\TC}}{Y}, \underset{\substack{TC\\TC^{\omega}}}{X})$$

$= d\omega(X,Y,Z) = 0$

$\Rightarrow [X,Y] \in TC^{\omega}$ ∎

- Suppose $(TC)^{\omega}$ is <u>regular</u> : all leafs are compact submanifolds.

Then $B := C/(p \sim q$ if p,q on same leaf$)$ is a symplectic manifold with ω_B induced by ω on $TB = TC/TC^{\omega}$.

So we have a fibration $\pi : C \twoheadrightarrow B$ with $\pi^* \omega_B = \omega|_C$.

We also have the embedding $\iota : C \hookrightarrow M$ and

$(\iota \times \pi)(C) \subset M^- \times B$ is a Lagrangian correspondence.

<u>CHECK</u>: $\iota \times \pi$ embeds, $\dim C = n + k = \frac{1}{2}(\dim M + \dim B)$

$$(\iota \times \pi)^* (-\omega \oplus \omega_B) = -\omega|_C + \pi^* \omega_B = 0$$

Lagrangian correspondences arising from moment maps

[McD-Sal. 5.2, 5.3]

(M,ω) symplectic mfd

G Lie group ; $\mathfrak{g} := T_{\mathbb{1}}G$ Lie algebra , fix G-invariant inner product $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$

$\psi : G \to \mathrm{Symp}(M)$ Hamiltonian group action with moment map $\mu : M \to \mathfrak{g}$

- $\psi(\mathbb{1}) = \mathrm{Id}_M$, $\psi(gh) = \psi(g)\circ\psi(h)$
- $d_{\mathbb{1}}\psi : \mathfrak{g} \longrightarrow \Gamma(TM)$, $\xi \longmapsto X_\xi(p) = \frac{d}{dt}\big|_{t=0}\psi(\exp(t\xi))p$ — maps to Hamiltonian vector fields: $\omega(X_\xi,\cdot)$ exact 1-form
- $\omega(X_\xi,\cdot) = d\big(\underbrace{\langle\mu,\xi\rangle_{\mathfrak{g}}}\big)$ $\quad\forall \xi\in\mathfrak{g}$
  $H_\xi : M \to \mathbb{R}$; $p \mapsto \langle\mu(p),\xi\rangle_{\mathfrak{g}}$
- ($\xi \mapsto H_\xi$ is a Lie algebra homomorphism)
  $\Updownarrow$
  μ is equivariant : $\mu(\psi(g)x) = g\,\mu(x)\,g^{-1}$

Ex.: Any Hamiltonian S^1-action has a moment map.

$\frac{d}{dt}\big|_{t=0}\psi_{e^{it}}(p) = X_H$; $\omega(X_H,\cdot) = dH$

($\lambda\in\mathbb{R}$, λ, λ)

$\rightsquigarrow$ $\mu : M \to \mathfrak{g} \cong \mathbb{R}$ given by $\mu(p) = H(p)$

<u>Note</u>: $\mu^{-1}(0) \subset M$ is G-invariant ($\psi(g): \mu^{-1}(0) \to \mu^{-1}(0) \quad \forall g$)

* Suppose G acts freely on $\mu^{-1}(0)$ $\quad [\ \psi_g(p) = p \in \mu^{-1}(0) \Rightarrow g = \mathbb{1}\ ]$
and G is compact, connected

<u>Claim:</u>

$\mu^{-1}(0) \subset M$ is a coisotropic submanifold

orbits $\{\psi_g(p) \mid g \in G\}$ are the isotropic leaves

$M/\!\!/_G := \mu^{-1}(0)/G$ is a symplectic manifold "symplectic quotient"

$\iota \times \pi : \mu^{-1}(0) \longrightarrow M^- \times M/\!\!/_G$ embeds to a Lagrangian correspondance

L3 - Lagrangian correspondences arising from moment maps

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<u>Ex.</u>: $M = \mathbb{C}^{n+1}$, $S^1 = \mathbb{R}/\mathbb{Z} \circlearrowright M$ by rotation $\psi_{[t]}(z_0, \dots z_n) = (e^{2\pi i t} z_0, \dots, e^{2\pi i t} z_n)$

$\forall \xi \in \mathbb{R} = \mathfrak{g} \quad X_\xi(z_0 \dots z_n) = 2\pi i \xi \cdot (z_0, \dots, z_n)$

$$\omega(X_\xi(\underline{z}), \underline{v}) = (i X_\xi(\underline{z})) \underset{\mathbb{R}^{2n}}{\cdot} \underline{v} = -2\pi \xi \; \underline{z} \cdot \underline{v}$$

$$= \xi \cdot d\mu(\underline{z}) \underline{v}$$

with $\mu(\underline{z}) = -\pi |\underline{z}|^2 \quad \rightsquigarrow \mu^{-1}(0)/S^1 = \{0\}/S^1 = \{0\}$

or $\mu(\underline{z}) = \pi - \pi |\underline{z}|^2 \quad \rightsquigarrow \mu^{-1}(0)/S^1 = \{|\underline{z}| = 1\}/S^1 = S^{2n+1} \subset \mathbb{C}^{n+1}/\text{rotation} = \mathbb{CP}^n$

(or any other constant)

<u>Note</u>: $\mu^{-1}(0) \subset M$ is G-invariant $\quad (\psi(g) : \mu^{-1}(0) \to \mu^{-1}(0) \quad \forall g)$

* Suppose G acts freely on $\mu^{-1}(0)$ $\quad [\psi_g(p) = p \in \mu^{-1}(0) \Rightarrow g = \mathbb{1}]$
  and G is compact, connected

$\Rightarrow$ infinitesimal action is injective : $\quad X_\xi(p) = 0 \Rightarrow \xi = 0$

$$\left[\frac{d}{dt} \psi_{\exp(t\xi)}(p) = d\psi_{\exp(t\xi)}(X_\xi(p)) = 0 \quad \forall t \Rightarrow \exp(t\xi) = \mathbb{1} \Rightarrow \xi = 0 \right]$$

$\Rightarrow$ 0 regular value of $\mu : M \to \mathfrak{g}$

$$\left[\begin{array}{l} d\mu(p) : T_pM \to \mathfrak{g} \quad \text{surj.} \; \forall p \in \mu^{-1}(0) \quad \text{since} \\ \xi \in (\mathrm{im}\, d\mu(p))^\perp \Leftrightarrow \langle d\mu(p) v, \xi \rangle_{\mathfrak{g}} = 0 \quad \forall v \in T_pM \\ \qquad X_\xi(p) = 0 \Leftrightarrow \omega(X_\xi(p), v) = 0 \text{ —— '' ——} \end{array} \right]$$

$\Rightarrow \mu^{-1}(0) \subset M$ is a coisotropic submanifold of $\dim = 2n - \dim G$

[Let $\mathcal{O}(p) := \{\psi_g(p) \mid g \in G\} \cong G$ be the orbit through $p \in \mu^{-1}(0)$

$T_p\mathcal{O} = \{X_\xi(p) \mid \xi \in \mathfrak{g}\} \subset T_p\,\mu^{-1}(0)$

$$T_p\mu^{-1}(0) = \ker d\mu(p) = \{v \in T_pM \mid \underbrace{\langle d\mu(p)v, \xi\rangle}_{\omega(X_\xi(p), v)} = 0 \;\; \forall \xi \in \mathfrak{g}\}$$

$$= (T_p\mathcal{O})^\omega$$

$\Rightarrow (T_p\mu^{-1}(0))^\omega = T_p\mathcal{O} \subset T_p\mu^{-1}(0)$ coisotropic]

The isotropic leaves are the orbits $\mathcal{O}(p)$, so $(T_p\mu^{-1}(0))^\omega$ is a regular foliation

$\Rightarrow \mu^{-1}(0)/\sim \;= \mu^{-1}(0)/G$ is a symplectic manifold of $\dim = \dim M - 2\dim G$

$\mu^{-1}(0)/G =: M/\!\!/G$ "symplectic quotient"

(Notation: We can shift μ by any central constant $\tau \in \mathfrak{g}$, $g^{-1}\tau g = \tau \;\; \forall g$

or, equivalently, take the quotient $M/\!\!/G[\tau] := \mu^{-1}(\tau)/G$)

$\iota \times \pi : \mu^{-1}(0) \longrightarrow M^- \times M/\!\!/G$ embeds to a Lagrangian correspondance

$$\widetilde{\mu^{-1}(0)} := \{(p, \mathcal{O}(q)) \in M \times M/\!\!/G \mid \mu(p) = 0 = \mu(q),\; \mathcal{O}(p) = \mathcal{O}(q)\}$$

previous Example: $\mu^{-1}(\pi) \cong S^{2n+1} \hookrightarrow \mathbb{C}^{n+1} \times \mathbb{CP}^n$

$$\underline{z} \longmapsto (\underline{z}, [z_0 : \ldots : z_n])$$

Composition with $\widetilde{\mu^{-1}(0)} \subset M^- \times M/\!\!/G$

$\pi : \mu^{-1}(0) \to \mu^{-1}(0)/_G = M/\!\!/G$

- $\underset{\underset{pt^- \times M}{\cong}}{L \subset M}$ Lagrangian $\rightsquigarrow$ $L \circ \widetilde{\mu^{-1}(0)} = \pi(\underbrace{L \cap \mu^{-1}(0)}_{\subset M}) \subset \underset{\underset{pt^- \times M/\!\!/G}{\cong}}{M/\!\!/G}$

  - transverse if $L \pitchfork \mu^{-1}(0) \subset M$

  - embedded if $L \cap \mathcal{O}(p) = \text{point or } \emptyset$ for all orbits $\mathcal{O}(p)$

  Ex.: $\mathbb{R}^n \subset \mathbb{C}^n \rightsquigarrow \mathbb{R}^n \circ \widetilde{\mu^{-1}(\pi)} = \pi(\underbrace{\mathbb{R}^n \cap S^{2n+1}}_{S^{n-1} \subset \mathbb{R}^n}) = \mathbb{RP}^n \subset \mathbb{CP}^n$

  composition is transverse but not embedded ($S^{n-1} \to \mathbb{RP}^n$ is a double cover)

- $\widetilde{\mu^{-1}(0)} \circ \widetilde{\mu^{-1}(0)}^t = \{(x,y) \in M \times M \mid \mu(x) = \mu(y) = 0,\ \pi(x) = \pi(y) \in M/\!\!/G\}$

  is always "transverse" (since $\pi : \mu^{-1}(0) \to M/\!\!/G$ surjective)

  and "embedded" (since $\pi(x) \in M/\!\!/G$ uniquely determined by x).

- $\widetilde{\mu^{-1}(0)}^t \circ \widetilde{\mu^{-1}(0)} = \{(p,q) \in M/\!\!/G \times M/\!\!/G \mid \exists x \in M :\ p = \pi(x) = q\} = \Delta_{M/\!\!/G}$

  is smooth but neither transverse nor embedded :

$$\begin{array}{l} \widetilde{\mu^{-1}(0)}^t \underset{\Delta_M}{\times} \widetilde{\mu^{-1}(0)} = \{(\pi(x), x, x, \pi(x)) \mid x \in M\} \cong M \hookleftarrow G \text{ fiber} \\ \quad \big\downarrow \pi_{02} \qquad\qquad\qquad \pi_{02}^{-1}(\pi(x),\pi(x)) = \{(\pi(x), \psi_g(x), \psi_g(x), \pi(x)) \mid g \in G\} \\ \widetilde{\mu^{-1}(0)}^t \circ \widetilde{\mu^{-1}(0)} = \Delta_{M/\!\!/G} \qquad \ni (\pi(x), \pi(x)) \end{array}$$

- $\underset{\underset{pt \times M/\!\!/G}{\cong}}{\ell \subset M/\!\!/G} \rightsquigarrow \ell \circ \widetilde{\mu^{-1}(0)}^t = \pi^{-1}(\ell) \subset \underset{\underset{pt \times M}{\cong}}{M}$

  always transverse, embedded

Note: Any Lagrangian $\ell \subset M/\!\!/G$ is the composition with $\widetilde{\mu^{-1}(0)}$ of

a Lagrangian $L \subset M$.

E.g. $L = \ell \circ \widetilde{\mu^{-1}(0)}^t \rightsquigarrow L \circ \widetilde{\mu^{-1}(0)} = \ell \circ \widetilde{\mu^{-1}(0)}^t \circ \widetilde{\mu^{-1}(0)} = \ell \circ \Delta_{M/\!\!/G} = \ell$

Question: Which ℓ are transverse (& embedded) compositions?

(Is there $L \subset M$ s.t. $L \pitchfork \mu^{-1}(0)$ and $\pi(L \cap \mu^{-1}(0)) = \ell$?)

L4 - generalized Lagrangian correspondences

Note Title 2/13/2008

PREVIEW $\ell, \ell' \subset M/\!\!/G$ Lagrangian

Suppose $\ell = L \circ \widetilde{\mu^{-1}(0)}$ is the transverse & embedded composition with $L \subset M$ Lagrangian.

Then (under various compactness and monotonicity assumptions)

$$HF(\ell, \ell') \cong HF(L, \pi^{-1}(\ell'))$$

CF: $M/\!\!/G \supset \ell \cap \ell' \xleftrightarrow{1-1} L \cap \pi^{-1}(\ell')$

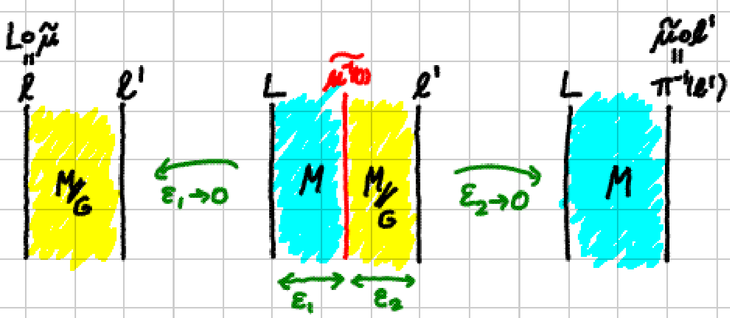
$$\qquad\qquad\qquad\qquad\qquad \| $$
$$\{\underbrace{L \cap \mathcal{O}(p)}_{\text{none or unique point}} \mid \mathcal{O}(p) \in \ell'\}$$

$CF \circlearrowleft \partial$: holomorphic curves in $M/\!\!/G$ $\longleftrightarrow$ holomorphic curves in M

"Proof" $(\ell, \ell') = (L \circ \widetilde{\mu^{-1}(0)}, \ell')$

$\updownarrow$ equivalence of generalized Lagrangian correspondences

$$(L, \widetilde{\mu^{-1}(0)} \circ \ell') = (L, \pi^{-1}(\ell'))$$



Composition and intersections:

1.) $L_0 \subset M_0$, $L_1 \subset M_1$, $L_{01} \subset M_0^- \times M_1$ Lagrangian

$$\Rightarrow (L_0 \circ L_{01}) \cap L_1 \cong L_0 \cap (L_{01} \circ L_1)$$

$$L_0 \circ L_{01} = \{x_1 \in L_1 \mid \exists x_0 \in L_0 : (x_0, x_1) \in L_{01}\} \qquad L_{01} \circ L_1 = \{x_0 \in L_0 \mid \exists x_1 \in L_1 : (x_0, x_1) \in L_{01}\}$$

$\rightsquigarrow$ unique $\qquad$ $\rightsquigarrow$ unique

$x_1 \longmapsto x_0$

$x_1 \longleftarrow\!\shortmid x_0$

if $L_0 \circ L_{01}$ and $L_{01} \circ L_1$ are embedded.

2.) $L \subset M$, $L' \subset M$ Lagrangian

$L = pt \times M$, $L' = M \times pt$

- $L \circ L'$ is transverse if $L \times L' \pitchfork \Delta_M$ i.e. $L \pitchfork L'$

  $\Rightarrow L \times_\Delta L' = \{(x, x) \mid x \in L \cap L'\} \cong L \cap L'$ finite set

- $L \circ L'$ is embedded if $L \cap L' = \emptyset$ or point

Def$^{\underline{n}}$: M_0, M_1 symplectic manifolds

- A <u>generalized Lagrangian correspondence</u> from M_0 to M_1, $M_0 \xrightarrow{\underline{L}} M_1$ is a finite sequence $\underline{L} = (L_{01}, L_{12}, \dots, L_{k-1\,k})$ of Lagrangian correspondences $L_{(j-1)j} \subset N_{j-1}^- \times N_j$ between an underlying sequence $M_0 = N_0, N_1, \dots, N_{k-1}, N_k = M_1$ of symplectic manifolds.

$$M_0 = N_0 \xrightarrow{L_{01}} N_1 \xrightarrow{L_{12}} N_2 \longrightarrow \dots \longrightarrow N_{k-1} \xrightarrow{L_{(k-1)k}} N_k = M_1$$

- Its <u>dual</u> is the reversed sequence $\underline{L}^t := (L^t_{k-1\,k}, \dots, L^t_{01})$.
- The <u>algebraic composition</u> of gen. Lagr. corr. $M_0 \xrightarrow{\underline{L}} M_1$, $M_1 \xrightarrow{\underline{L}'} M_2$ is the concatenation $M_0 \xrightarrow{\underline{L} \# \underline{L}'} M_2$, given by

$$\underline{L} \# \underline{L}' := (L_{01}, \dots, L_{k-1\,k}, L'_{01}, \dots, L'_{k'-1\,k'})$$

with underlying $M_0 = N_0, N_1, \dots N_{k-1}, N_k = M_1 = N'_0, N'_1, \dots, N'_{k'-1}, N'_{k'} = M_2$.

$$\underbrace{M_0 = N_0 \xrightarrow{L_{01}} N_1 \longrightarrow \dots \longrightarrow N_{k-1} \xrightarrow{L_{k-1\,k}} N_k = M_1}_{\underline{L}} = \underbrace{N'_0 \xrightarrow{L'_{01}} N'_1 \longrightarrow \dots \longrightarrow N'_{k'-1} \xrightarrow{L'_{k'-1\,k'}} N'_{k'} = M_2}_{\underline{L}'}$$

Def^{n}: Two generalized Lagr. corresp. $M_0 \xrightarrow{\underline{L}} M_1$ and $M_0 \xrightarrow{\underline{L}'} M_1$ are <u>equivalent</u> if they are connected by a sequence of "good moves". A "good move" takes

$$M_0 \longrightarrow \dots \longrightarrow N_{\ell-1} \xrightarrow{L_{\ell-1\,\ell}} N_\ell \xrightarrow{L_{\ell\,\ell+1}} N_{\ell+1} \longrightarrow \dots \longrightarrow M_1$$

to

$$M_0 \longrightarrow \dots \longrightarrow N_{\ell-1} \xrightarrow{L_{\ell-1\,\ell} \circ L_{\ell\,\ell+1}} N_{\ell+1} \longrightarrow \dots \longrightarrow M_1$$

(or vice versa), where <u>$L_{\ell-1\,\ell} \circ L_{\ell\,\ell+1}$ is transverse & embedded</u>.

<u>Example</u>:

$$M_0 \xrightarrow{L_{01}} N_1 \xrightarrow{L_{12}} N_2 \xrightarrow{L_{23}} M_1$$

$$\wr\wr$$

$$M_0 \xrightarrow{L_{01} \circ L_{12}} N_2 \xrightarrow{L_{23}} M_1 \quad \underset{\substack{\text{if} \\ \pitchfork, \hookrightarrow}}{\approx} \quad M_0 \xrightarrow{L_{01} \circ L_{12} \circ L_{23}} M_1$$

$$\wr\wr$$

$$M_0 \xrightarrow{L_{01} \circ L_{12}} N_2 \xrightarrow{L'_{23}} N_3 \xrightarrow{L'_{34}} M_1$$

$$\wr\wr$$

$$M_0 \xrightarrow{L_{01} \circ L_{12} \circ L'_{23}} N_3 \xrightarrow{L'_{34}} M_1$$

<u>Homework</u>: Let $L, L' \subset M_0^- \times M_1$ be (simple) Lagrangian correspondences. If $L \neq L'$ show that $L \not\approx L'$ as generalized Lagr. corresp.

We define the <u>symplectic category</u> $\mathcal{Symp}$ by

- objects : (M,ω) symplectic manifold (finite dimensional, could specify to e.g. compact)
- morphisms $\mathcal{Mor}(M_0,M_1)$: $\left(\text{generalized Lagrangian correspondences } M_0 \xrightarrow{\underline{L}} M_1\right)$ modulo equivalence

- composition – algebraic as above
- identity $\mathcal{Mor}(M,M) \ni 1_M := \Delta_M \subset M^- \times M$ diagonal

TO CHECK : • composition is associative,

$$\bullet \; 1_M \circ \underline{L} = (\Delta_M, L_{01}, \dots, L_{k-1\,k}) \sim (\underbrace{\Delta_M \circ L_{01}}_{= L_{01}}, \dots, L_{k-1\,k}) = \underline{L}$$

$$\underline{L} \circ 1_M = \dots \sim \underline{L}$$

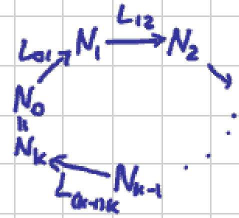
<u>Next</u>: extend $\mathcal{Symp}$ to a 2-category, i.e. make

morphism space $\mathcal{Mor}(M_0,M_1)$ a category

composition $\mathcal{Mor}(M_0,M_1) \times \mathcal{Mor}(M_1,M_2) \to \mathcal{Mor}(M_0,M_2)$ a functor

PREVIEW : $\quad {}^2\mathcal{Mor}(\underset{\substack{\cap \\ \mathcal{Mor}(M_0,M_1)}}{\underline{L}}, \underline{L}') := HF(\underline{L}, \underline{L}')$

L5 – generalized Floer homology

We will define Floer homology $HF(\underline{\mathcal{L}})$ for a 

cyclic correspondence $\underline{\mathcal{L}} = (L_{01}, L_{12}, \dots, L_{(k-1)k})$

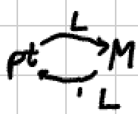
of Lagrangian correspondences $L_{(j-1)j} \subset N_{j-1}^- \times N_j$

with underlying symplectic manifolds $N_0, N_1, \dots, N_{k-1}, N_k = N_0$.

Ex.: (0) $L \subset M^- \times M$ Lagrangian (e.g. graph of symplectomorphism φ)

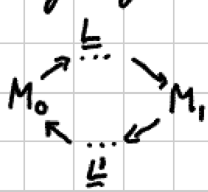
$\rightsquigarrow \underline{\mathcal{L}} = (L)$ with underlying $N_0 = N_1 = M$

(i) $L, L' \subset M$ Lagrangian submanifolds

$\rightsquigarrow \mathcal{L} = (L, L')$ 

(underlying pt, M, pt (or M, pt, M))

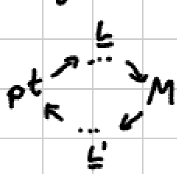
(ii) $\underline{L}, \underline{L}'$ generalized Lagrangian correspondences from M_0 to M_1

$\rightsquigarrow \underline{\mathcal{L}} = (\underline{L}, \underline{L}'^t)$ 

$\left(\begin{array}{l}\text{underlying} \\ M_0 = N_0, N_1 \dots N_k = M_1 = N'_{k'} \dots N'_1, N'_0 = M_0\end{array}\right)$

(iii) $\underline{L}, \underline{L}'$ generalized Lagrangian submanifolds of M

(i.e. generalized Lagrangian correspondences from pt to M)

$\rightsquigarrow \underline{\mathcal{L}} = (\underline{L}, \underline{L}'^t)$ 

"Def$^{\underline{n}}$": $HF(\underline{\mathcal{L}}) := \frac{\ker\partial}{\operatorname{im}\partial}$ is the "Morse homology" (a la Witten, Floer)

on the (generalized) path space $\mathcal{P}$

of the (generalized) symplectic action functional $\mathcal{A}: \mathcal{P} \to \mathbb{R}/_{\ldots}$

Floer complex CF generated by critical points of $\mathcal{A}: \mathcal{P} \to \mathbb{R}/_{\ldots}$

Floer differential $\partial \circlearrowright CF$ defined by "counting" ~~gradient flow lines~~ Floer trajectories of $\mathcal{A}$

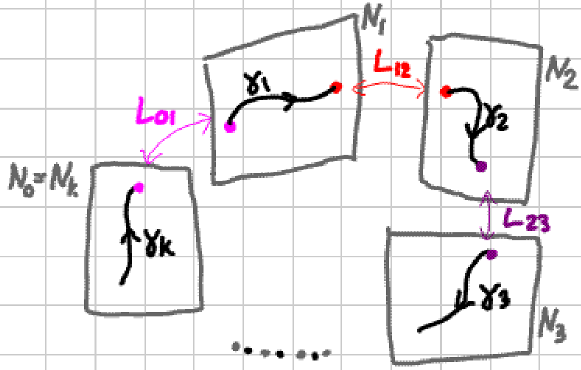
<u>path space</u>

$$\mathcal{P} = \left\{ \underline{\gamma} := (\gamma_1, \gamma_2, \ldots, \gamma_{k-1}, \overset{\gamma_0}{\overset{\shortparallel}{\gamma_k}}) \;\middle|\; \gamma_j : [0,1] \to N_j \,,\, (\gamma_{j-1}(1), \gamma_j(0)) \in L_{(j-1)j} \quad \forall j = 1..k \right\}$$

<u>Ex (i)</u>: $\underline{\gamma} = \gamma_1 : [0,1] \to M \quad , \quad \underset{\underset{(\gamma_0(1),\gamma_1(0))}{\shortparallel}}{\gamma_1(0) \in L} \quad , \quad \underset{\underset{(\gamma_1(1),\gamma_2(0))}{\shortparallel}}{\gamma_1(1) \in L'}$

$\gamma_0 = \gamma_2 \equiv pt$

<u>general</u>:



Symplectic action

$\mathcal{A} : \mathcal{P} \longrightarrow \mathbb{R}/_{***}$ 　　 fix $\gamma^0 \in \mathcal{P}$

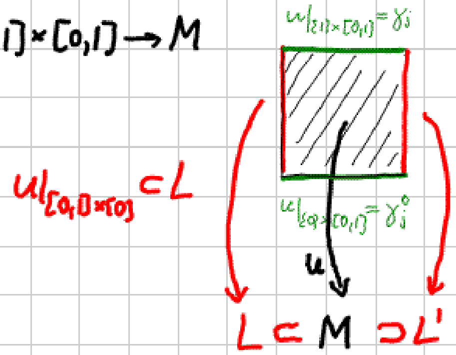
$$\gamma \longmapsto -\sum_{j=1}^{k} \int_{[0,1]^2} u_j^* \omega_{N_j}$$

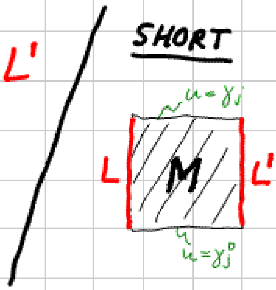
$u_j : [0,1]\times[0,1] \longrightarrow N_j$

$u_j(0,\cdot) = \gamma_j^0, \quad u_j(1,\cdot) = \gamma_j$

$(u_j(s,\cdot))_{j=1..k} \in \mathcal{P} \quad \forall s \in [0,1]$

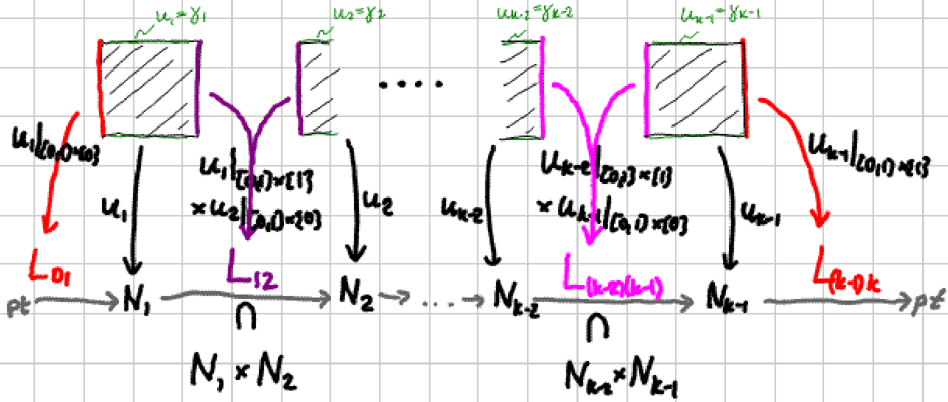
Ex. (i): $u_1 : [0,1]\times[0,1] \longrightarrow M$



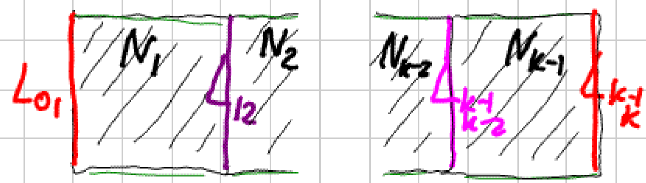


*** $\mathcal{A}$ well defined up to $\left\{ \int w^*\omega \;\middle|\; \begin{array}{l} w : S^1\times[0,1] \longrightarrow M \\ S^1\times\{0\} \longrightarrow L \\ S^1\times\{1\} \longrightarrow L' \end{array} \right\}$

Ex (iii): $\underline{L} = (L_{01}, \dots, L_{k-1\,k})$ $\quad N_0 = N_k = pt$



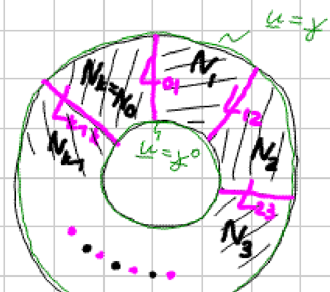
SHORT:



general case: picture $\underline{u} = (u_1, \dots, u_k)$ as "quilt"

*** $A =$ "$-\int \underline{u}^* \underline{\omega}$" is well defined up to

$$\left\{ \sum_{j=1}^{k} w_j^* \omega_{N_j} \;\middle|\; w_j : S^1 \times [0,1] \to N_j \,,\ (w_j(s,\cdot))_{j=1\dots k} \in P \ \forall s \in S^1 \right\}$$



critical points:

$$dA(\underline{\gamma}) : T_{\underline{\gamma}}\mathcal{P} \longrightarrow \mathbb{R} \qquad \begin{pmatrix} \xi_j \in \Gamma(\gamma_j^* TN_j) \\ (\xi_{j-1}(1), \xi_j(0)) \in T_{(\gamma_{j-1}(1),\gamma_j(0))} L_{(j-1)j} \end{pmatrix}$$

$$(\underline{\xi}) \longmapsto -\int_0^1 \sum_{j=1}^k \omega_j(\xi_j(t), \partial_t\gamma_j(t))\, dt$$

$$\left(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \big(A(\exp_{\underline{\gamma}}(\underline{\xi})) - A(\underline{\gamma})\big) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} -\int_{[0,\varepsilon]\times[0,1]} \underbrace{\overbrace{\exp_{\underline{\gamma}}(s\,\underline{\xi})}^{\underline{u}}{}^* \underline{\omega}}_{\sum_{j=1}^k \omega_j(\partial_s u_j,\, \partial_t u_j)\, ds\, dt} \right)$$

$$\partial_s u_j = \xi_j, \quad \partial_t u_j = \partial_t \gamma_j \quad \text{at } s=\varepsilon=0$$

$$\underline{\gamma} \in \mathrm{Crit}\, A \iff dA(\underline{\gamma})\,\underline{\xi} = 0 \ \ \forall \underline{\xi} \iff \partial_t \gamma_j = 0 \ \ \forall j$$

$$\Rightarrow \mathrm{Crit}\, A = \{ \underline{p} = (p_1, \dots, p_k \overset{=p_0}{}) \in N_1 \times \dots \times N_k \mid (p_{j-1}, p_j) \in L_{(j-1)j} \ \ \forall j = 1..k \}$$

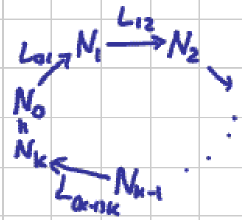
$$= \cap\, \underline{\mathcal{L}} \quad \text{generalized intersection}$$

<u>Ex.(i)</u>: $\mathrm{Crit}\, A = \{ (pt, p, pt) \in pt \times M \times pt \mid \underset{\text{i.e. } p \in L}{(pt, p) \in L},\ \underset{\text{i.e. } p \in L'}{(p, pt) \in L'} \} = L \cap L'$

L6 – generalized Floer homology – trajectories

$\underline{\mathcal{L}} = (L_{01}, L_{12}, \dots, L_{(k-1)k})$ cyclic correspondence

$\left(\begin{array}{l} L_{(j-1)j} \subset N_{j-1}^- \times N_j \text{ Lagrangian correspondences} \\ N_0, N_1, \dots, N_{k-1}, N_k = N_0 \text{ symplectic manifolds} \end{array} \right)$



<u>critical points</u> of $\mathcal{A}$: $\mathcal{P} = \left\{ (\gamma_j : [0,1] \to N_j)_{j=1..k} \,\middle|\, \begin{smallmatrix} L_{j-1\,j} \\ \text{conditions} \end{smallmatrix} \right\} \to \mathbb{R}/_{\cdots}$

$$\mathrm{Crit}\,\mathcal{A} = \left\{ \gamma \in \mathcal{P} \;\middle|\; d\mathcal{A}(\gamma)\xi = -\sum_{j=1}^{k} \int_0^1 \omega_j(\xi_j(t), \partial_t\gamma_j(t))\, dt = 0 \quad \forall \xi \in T_\gamma \mathcal{P} \right\}$$

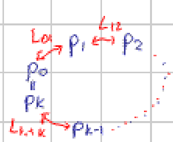
$$\cong \left\{ \underline{p} = (p_1, \dots, \overset{=p_0}{p_k}) \in N_1 \times \dots \times N_k \;\middle|\; (p_{j-1}, p_j) \in L_{(j-1)j} \;\; \forall j = 1..k \right\}$$

$=: \cap\, \underline{\mathcal{L}}$ generalized intersection

<u>Ex.(i)</u>: $\mathrm{Crit}\,\mathcal{A} = \{ (pt, p, pt) \in pt \times M \times pt \mid \underset{\text{i.e. } p \in L}{(pt, p) \in L},\; \underset{\text{i.e. } p \in L'}{(p, pt) \in L'} \} \cong L \cap L'$

<u>Ex.(o)</u>: $\underline{\mathcal{L}} = (\mathrm{graph}\,\varphi)$, $\varphi \in \mathrm{Symp}(M)$ $\rightsquigarrow$ $\mathrm{Crit}\,\mathcal{A} = \{ (p_0 = p_1) \in \mathrm{gr}\,\varphi \} \cong \mathrm{Fix}\,\varphi$

<u>Ex (iii) / general</u>: "quilt" with each component $u_j \equiv p_j \in N_j$ constant



"<u>Def$^{\underline{n}}$</u>:" Floer homology $HF(\underline{\mathcal{L}}) := \ker / \mathrm{im}\,\partial$

$$CF(\underline{\mathcal{L}}) := \bigoplus_{\underline{p} \in \mathrm{Crit}\,\mathcal{A}} \mathbb{Z}\langle \underline{p} \rangle \qquad (\text{assuming } \cap \underline{\mathcal{L}} \text{ is finite})$$

$\partial : CF \to CF$ defined by "counting" Floer trajectories
to be defined

<u>gradient</u>: To define $\nabla\mathcal{A}$, fix a metric on $\mathcal{P}$

For $j=1..k$ pick J_j $\quad\omega_j$-compatible almost complex structure

$$\left[\begin{array}{l} J : M \to \mathrm{End}(TM) \quad \text{smooth} \quad (\text{but not neccessarily } \nabla J = 0) \\ J^2 = -\mathrm{Id} \quad , \quad g_J(X,Y) := \omega(X, JY) \quad \text{is a metric on } M \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\text{i.e. symmetric, positive definite}) \\ \underline{\text{Thm}}: \text{ The space of such } J \text{ is nonempty, contractible} \end{array}\right]$$

L^2-metric on $\mathcal{P}$: $\xi, \eta \in T_\gamma\mathcal{P}$ $\left(\text{i.e. } \xi_j, \eta_j \in \Gamma(\gamma_j^* TN_j) \text{ with } TL_{(j-1)j} \text{ conditions}\right)$

$$\langle \xi, \eta \rangle := \sum_{j=1}^{k} \int_0^1 g_{J_j}(\xi_j(t), \eta_j(t))\, dt$$

$$\forall \xi \in T_\gamma\mathcal{P}$$

$$\langle \xi, \nabla\mathcal{A}(\gamma)\rangle = d\mathcal{A}(\gamma)\,\xi = \sum_{j=1}^{k} -\int_0^1 \omega_j(\xi_j(t), \partial_t\gamma_j(t))\, dt = \sum_{j=1}^{k}\int_0^1 g_j(\xi_j(t), J(\gamma_j(t))\,\partial_t\gamma_j(t))\, dt$$

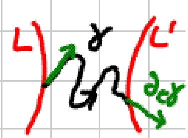
$$\Rightarrow \nabla\mathcal{A}(\gamma) = \underline{J}(\gamma)\,\partial_t\gamma = \left(J_j(\gamma_j)\,\partial_t\gamma_j\right)_{j=1..k}$$

<u>Note</u>: $\nabla\mathcal{A}$ cannot really be viewed as vector field on $\mathcal{P}$.

For $\nabla\mathcal{A}(\gamma) \in T_\gamma\mathcal{P}$ the linearized conditions in Ex. (i) are

$J(\gamma(0))\partial_t\gamma(0) \in T_{\gamma(0)}L$, $J(\gamma(1))\partial_t\gamma(1) \in T_{\gamma(1)}L'$ (i.e. $\partial_t\gamma(0) \perp TL$, $\partial_t\gamma(1) \perp TL'$)

but a general $\gamma \in \mathcal{P}$ only satisfies $\gamma(0) \in L$, $\gamma(1) \in L'$.



We can still try to study the flow lines of $\nabla\mathcal{A}$ on the subset

$\{\gamma \in \mathcal{P} \mid \nabla\mathcal{A}(\gamma) \in T_\gamma\mathcal{P}\}$.

<u>negative gradient flow lines:</u> $\eta : \mathbb{R} \to \mathcal{P}, \quad \frac{d}{ds}\eta = -\nabla A(\gamma)$

i.e. $\eta_j : \mathbb{R} \to \mathcal{C}^\infty([0,1], N_j), \begin{cases} \frac{d}{ds}\eta_j(s) = -J_j(\eta_j(s))\frac{d}{dt}\eta_j(s) & \text{on } [0,1]\ \forall s \in \mathbb{R} \\ \left(\eta_{j-1}(s)\big|_{t=1}, \eta_j(s)\big|_{t=0}\right) \in L_{(j-1)j} & \forall j = 1..k, \forall s \in \mathbb{R} \end{cases}$

<u>Note</u>: If $\gamma = \eta(s_0)$ is a point on a neg. gradient flow line $\eta : (0, s_0] \to \mathcal{P}$ (in Ex. (i)) then $J(\gamma(0))\frac{d}{dt}\gamma(0) = \frac{d}{ds}\eta(s_0)\big|_{t=0} \in T_{\gamma(0)}L$

since $s \mapsto \eta(s_0)|_{t=0}$ is a path in L

However, the neg. gradient flow equation is still not well posed.

For existence would need $\mathcal{P}' := \{\gamma \in \mathcal{P} \mid \nabla A(\gamma) \in T_\gamma \mathcal{P}\}$ at least complete.

If we take a completion $\mathcal{P}^\ell := \overline{\mathcal{P}'}^{W^{\ell,2}} \subset \bigoplus_{j=1}^k W^{\ell,2}([0,1], N_j)$ then for $\gamma \in \mathcal{P}^\ell$ the gradient again is not necessarily a tangent vector – for analytic reasons:

$\nabla A(\gamma) = \underline{J}\partial_t \gamma \subset \bigoplus_{j=1}^k W^{\ell-1,2}([0,1], \gamma_j^* TN_j) \quad$ whereas $T_\gamma \mathcal{P}^\ell \subset \oplus\, W^{\ell,2}(\ldots)$.

Conley-Zehnder solved this by using a different metric to define ∇A.

Floer got inspired by Gromov and noticed that the L^2-gradient flow lines are holomorphic curves.

<u>Floer trajectories</u> : view $\frac{d}{ds}\underline{\gamma} + \nabla\mathcal{A}(\underline{\gamma}) = 0$

as PDE $\partial_s \underline{u} + \underline{J}(\underline{u})\partial_t \underline{u} = 0$ for $\underline{u}(s,t) = \underline{\gamma}(s)(t)$

$$\boxed{\begin{array}{ll} \underline{u} = (u_j)_{j=1..k} & \\ & (*) \begin{cases} \partial_s u_j + J_j(u_j)\,\partial_t u_j = 0 & \text{on } \mathbb{R}\times[0,1] \quad \forall j \\ (u_{j-1}(s,1), u_j(s,0)) \in L_{(j-1)j} & \forall s\in\mathbb{R} \quad \forall j=1..k \end{cases} \\ u_j : \mathbb{R}\times[0,1] \to N_j & \end{array}}$$

- trivial solutions : $\underline{u}(s,t) = \underline{p} \in \cap\underline{\mathcal{L}}$
- $\mathbb{R}$-symmetry : if $\underline{u}$ is a solution then so is $(\sigma * \underline{u})(s,t) := \underline{u}(\sigma + s, t)$ for any $\sigma\in\mathbb{R}$.

<u>'Def$^{\underline{n}}$'</u>:

$$CF(\underline{\mathcal{L}}) := \bigoplus_{\substack{\underline{p}\in \mathrm{Crit}\,\mathcal{A} \\ = \cap\underline{\mathcal{L}}}} \mathbb{Z}\langle \underline{p}\rangle \qquad \left(\text{assuming } \cap\underline{\mathcal{L}} \text{ is finite}\right)$$

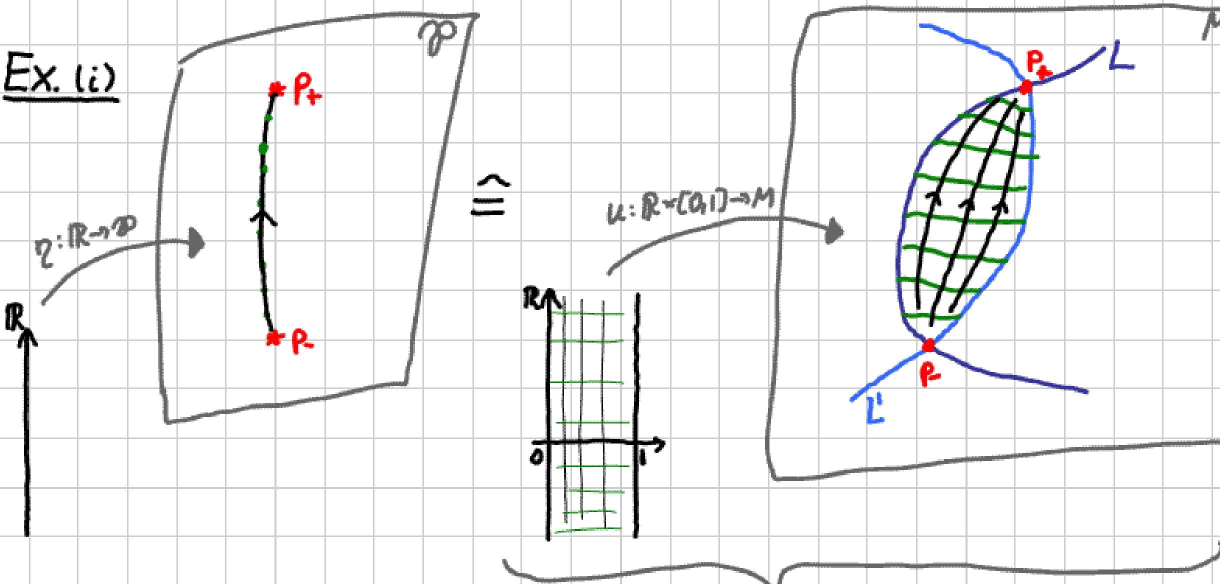
$\circlearrowleft$

∂ linear and

$$\partial\langle \underline{p}_+\rangle := \sum_{\underline{p}_-\in\mathrm{Crit}\,\mathcal{A}} \#\left\{ \underline{u} \in \bigoplus_{j=1}^{k} \mathcal{C}^\infty(\mathbb{R}\times[0,1], N_j) \;\middle|\; (*),\ \lim_{s\to\pm\infty} \underline{u}(s,\cdot) = \underline{p}_\pm \right\} \Big/ \mathbb{R}\text{-translation}$$

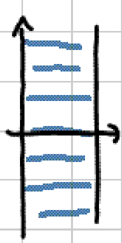
(signed count of isolated trajectories from $\underline{p}_-$ to $\underline{p}_+$
(assuming transversality, compactness, etc.)
$=0$ if moduli space $\{\dots\}/_{\mathbb{R}}$ has dimension >0
To Be Defined)

Ex. (i)



picture this as holomorphic strip

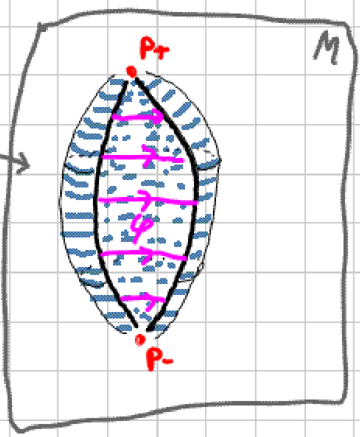
Ex. (0):



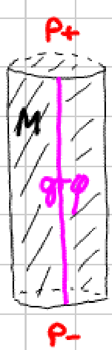
$u:\mathbb{R}\times[0,1]\to M$

$u(s,1)=\varphi(u(s,0))$

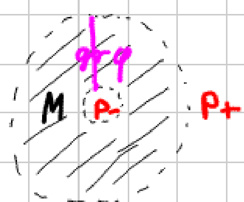
$\lim_{s\to\pm\infty} u(s,\cdot)=p_\pm\in \mathrm{Fix}\,\varphi$



SHORT:

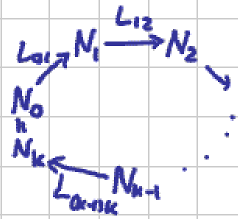


or



L7 – generalized Floer homology – quilts

$\underline{\mathcal{L}} = (L_{01}, L_{12}, \dots, L_{(k-1)k})$ cyclic correspondence



$\left(\begin{array}{l} L_{(j-1)j} \subset N_{j-1}^- \times N_j \text{ Lagrangian correspondences} \\ N_0, N_1, \dots, N_{k-1}, N_k = N_0 \text{ symplectic manifolds} \end{array} \right)$

<u>Floer complex</u> $CF(\underline{\mathcal{L}})$ is generated by

Crit $\mathcal{A} \cong \cap \underline{\mathcal{L}} := \{ \underline{p} = (p_1, \dots, p_k^{=p_0}) \in N_1 \times \dots \times N_k \mid (p_{j-1}, p_j) \in L_{(j-1)j} \ \forall j = 1..k \}$

<u>Floer differential</u> $\partial : CF(\underline{\mathcal{L}}) \to CF(\underline{\mathcal{L}})$ is defined by "counting"

(mod $\mathbb{R}$ isolated) <u>Floer trajectories</u> :

$\boxed{\begin{array}{ll} \underline{u} = (u_j)_{j=1..k} & \circledast \left\{ \begin{array}{ll} \partial_s u_j + J_j(u_j) \partial_t u_j = 0 & \text{on } \mathbb{R} \times [0,1] \quad \forall j \\ (u_{j-1}(s,1), u_j(s,0)) \in L_{(j-1)j} & \forall s \in \mathbb{R} \quad \forall j = 1..k \end{array} \right. \\ u_j : \mathbb{R} \times [0,1] \to N_j & \end{array}}$

<u>Prop$^{\underline{n}}$</u>: Any Floer trajectory $\underline{u}$ with finite energy

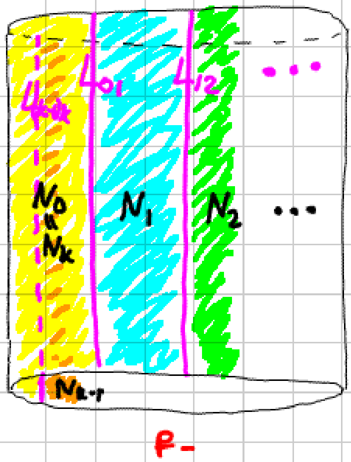
$$\mathcal{E}(\underline{u}) := \sum_{j=1}^{k} \int_{\mathbb{R} \times [0,1]} u_j^* \omega_j < \infty \qquad \text{converges (exponentially)}$$

to some $\underline{p}^{\pm} \in \cap \underline{\mathcal{L}}$, i.e. $u_j(s,t) \xrightarrow[s \to \pm\infty]{} p_j^{\pm}$ uniformly in t.

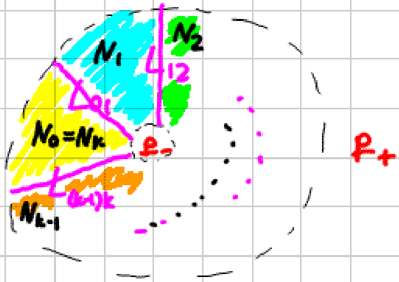
<u>"Proof"</u>: $\mathcal{E}(\underline{u}) = \sum \int \omega_j(\partial_s u_j, \underbrace{\partial_t u_j}_{= J \partial_s u_j}) = \sum \int |\partial_s u_j|^2 = \sum \int |\partial_t u_j|^2$

general picture of $\underline{u}$ as holomorphic quilt for $HF(\underline{L})$

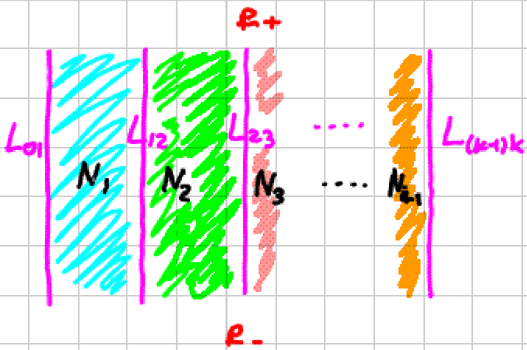
$\underline{L}: \; N_0 \xrightarrow{L_{01}} N_1 \xrightarrow{L_{12}} N_2 \to \cdots \to N_{k-1} \xrightarrow{L_{(k-1)k}} N_k = N_0$



or



<u>Ex. (iii)</u> $N_0 = N_k = pt$



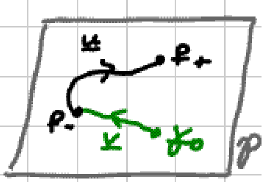
→ $CF(\underline{L})$ is generated by "constant quilts" $\underline{u} \equiv \underline{p} \in \cap \underline{L}$

(each patch $u_j \equiv p_j \in N_j$ is a constant map)

→ $\partial \circlearrowright CF(\underline{L})$ counts nonconstant holomorphic quilts (mod $\mathbb{R}$)

connecting (different) intersection points $\underline{p}_+, \underline{p}_- \in \cap \underline{L}$

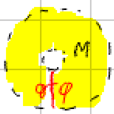
<u>Note</u>: A Floer trajectory with $\lim_{s \to \pm\infty} \underline{u}(s, \cdot) = \underline{p}_\pm$ has energy



$E(\underline{u}) = \int \underline{u}^* \underline{\omega} = \left(-\int \underline{v}^* \underline{\omega} + \int (\underline{u} \# \underline{v})^* \underline{\omega} \right) = \mathcal{A}(\underline{p}_-) - \mathcal{A}(\underline{p}_+)$.

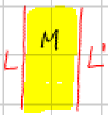
Alternative / "classical" definition

Ex. (0) For $\varphi \in \mathrm{Symp}\,\varphi$ $\quad HF(\underline{L} = (\mathrm{gr}\,\varphi)) = HF(\varphi)$



is the symplectic Floer homology by [Floer]

Ex. (i) For $L, L' \subset M$ $\quad HF(\underline{L} = (L, L')) = HF(L, L')$

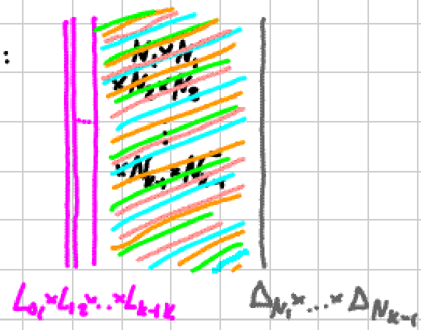


is the Lagrangian Floer homology by [Floer, Oh]

Ex. (iii) For $\underline{L} = (L_{01}, \dots, L_{(k-1)k})$ with $N_0 = N_k = pt$

- $HF(\underline{L}) = HF(L_{01} \times L_{12} \times \dots \times L_{(k-1)k}\,,\ \Delta_{N_1} \times \Delta_{N_2} \times \dots \times \Delta_{N_{k-1}})$

by folding in middle of each strip:



$u_j : \mathbb{R} \times [0,1] \to (N_j, J_j) \quad \rightsquigarrow \quad v_j(s,t) := u_j(\tfrac{s}{2}, \tfrac{t}{2}) : \mathbb{R} \times [0,1] \to (N_j, J_j)$

$\qquad v_j'(s,t) := u_j(\tfrac{s}{2}, 1 - \tfrac{t}{2}) : \mathbb{R} \times [0,1] \to (N_j, -J_j)$

$\Rightarrow (v_1, v_1', \dots, v_k, v_k')(s, 0) = (u_1(\tfrac{s}{2}, 0), u_1(\tfrac{s}{2}, 1), u_2(\tfrac{s}{2}, 0), \dots) \in L_{01} \times L_{12} \times \dots$

$\quad (v_1, v_1', \dots, v_k, v_k')(s, 1) = (u_1(\tfrac{s}{2}, \tfrac{1}{2}), u_1(\tfrac{s}{2}, \tfrac{1}{2}), \dots) \in \Delta \times \dots$

- by folding at every seam:

$$HF(\underline{L}) = \begin{cases} HF(L_{01} \times L_{23} \times \dots \times L_{(k-2)(k-1)}\,,\ L_{12} \times L_{34} \times \dots \times L_{(k-1)k}) & ;\ k \text{ even} \\ HF(L_{01} \times L_{23} \times \dots \times L_{(k-1)k}\,,\ L_{12} \times L_{34} \times \dots \times L_{(k-2)(k-1)}) & ;\ k \text{ odd} \end{cases}$$

all Lagrangian submanifolds of $N_1 \times N_2 \times \dots \times N_{k-1}$

$\qquad \underline{v}(s,0) = (u_1(s,0), u_2(s,1), u_3(s,0), \dots) \in L_{01} \times L_{23} \times \dots$

$v_{2j+1} = u_{2j+1}, \quad v_{2j}(s,t) := u_{2j}(s, 1-t) : \mathbb{R} \times [0,1] \to (N_{2j}, -J_{2j}) \quad \rightsquigarrow \quad \underline{v}(s,1) = (u_1(s,1), u_2(s,0), \dots) \in L_{12} \times \dots$

in general

- $HF(\underline{\mathcal{L}}) = HF(L_{01}\times L_{12}\times\ldots\times L_{(k-1)k},\ \widetilde{\Delta})$ for

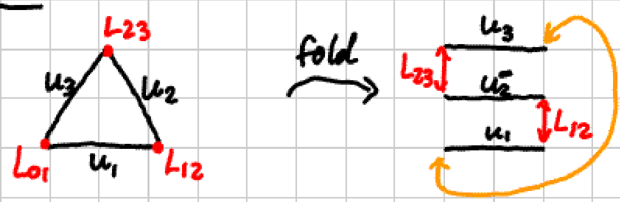
$$\widetilde{\Delta} = \tau(\Delta_{N_1}\times\Delta_{N_2}\times\ldots\times\Delta_{N_{k-1}}\times\Delta_{N_k})\ ;\ \tau: N_1\times N_1\times\ldots\times N_k\times N_k \to N_0\times N_1\times N_1\times\ldots\times N_k$$
$$(p_1,q_1,\ldots,p_k,q_k)\mapsto(q_k,p_1,q_1,\ldots,p_k)$$

- $$HF(\underline{\mathcal{L}}) = \begin{cases} HF(L_{01}\times L_{23}\times\ldots\times L_{(k-2)(k-1)},\ \tau(L_{12}\times L_{34}\times\ldots\times L_{(k-1)k})) ; & k \text{ even} \\ HF(L_{01}\times L_{23}\times\ldots\times L_{(k-1)k},\ \tau(L_{12}\times L_{34}\times\ldots\times L_{(k-2)(k-1)}\times\Delta_{N_k})) ; & k \text{ odd}\end{cases}$$

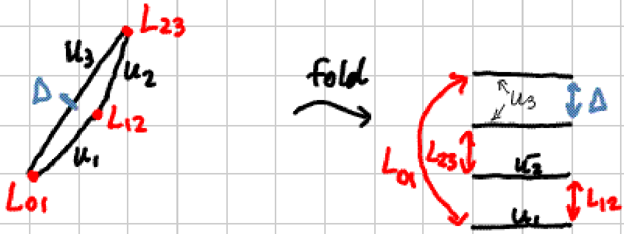
⚠ For odd k we fold the N_k-strip in the middle, but not the others. So in order to obtain one strip (of width 1) in the product manifold, we need to start from a quilt containing strips of different widths.

$u_j : \mathbb{R}\times[0,1]\to N_j \quad ; j=1..k-1$
$u_k : \mathbb{R}\times[0,2]\to N_k$

Ex. $k=3$



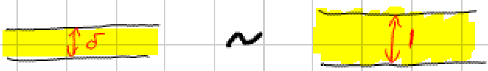
L_{01} → not a strip in $N_1\times N_2\times N_3$ with simple boundary conditions



→ strip in $N_1\times N_2\times N_3\times N_3$ with boundary conditions in $\tau(L_{01}\times L_{23})$
$L_{12}\times\Delta_{N_3}$

We will hence define the "quilted Floer homology" $HF(\underline{\mathcal{L}})$ by allowing any widths $\underline{\delta} = (\delta_j)_{j=1..k} \in (0,\infty)^k$ and counting the Floer trajectories

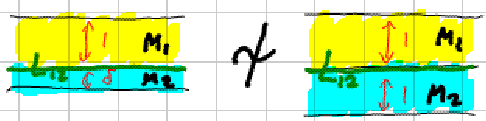
$$\boxed{\begin{array}{ll} \underline{u} = (u_j)_{j=1..k} & \begin{cases} \partial_s u_j + J_j(u_j)\partial_t u_j = 0 & \text{on } \mathbb{R}\times[0,\delta_j] \ \forall j \\ (u_{j-1}(s,\delta_{j-1}), u_j(s,0)) \in L_{(j-1)j} & \forall s\in\mathbb{R} \ \ \forall j=1..k \end{cases} \\ u_j : \mathbb{R}\times[0,\delta_j] \to N_j & \end{array}}$$



<u>Note</u>: $(\mathbb{R}\times[0,\delta], j_{std}) \sim (\mathbb{R}\times[0,1], j_{std})$ is biholomorphic,
$$(s,t) \longmapsto (\delta^{-1}s, \delta^{-1}t)$$

hence the moduli spaces of holomorphic discs of different widths can be identified.

The moduli space of holomorphic quilted strips with different widths cannot be identified, since separate rescaling of strips destroys the seam condition:



$$\Phi : (s_1,t_1,s_2,t_2) \longmapsto (s_1,t_1,\delta^{-1}s_2,\delta^{-1}t_2)$$

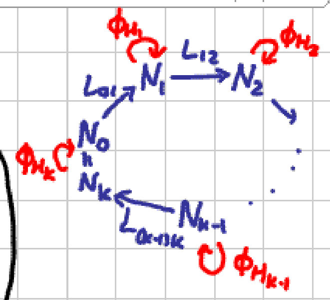
If (u_1,u_2) satisfies $(u_1(s,1), u_2(s,0)) \in L_{12}$ $\forall s$ then $\Phi^*(u_1,u_2) =: (v_1,v_2)$ satisfies $(v_1(s,1), v_2(\delta s,0)) \in L_{12}$ $\forall s$, but the seam condition is $(v_1(s,1), v_2(s,0)) \in L_{12}$ $\forall s$. This rescaling preserves the seam condition only for correspondences $L_{12} = L_1 \times L_2$ of split form; $L_1 \subset M_1$, $L_2 \subset M_2$.

L8 - quilted Floer homology

Note Title 2/25/2008

$\underline{\mathcal{L}} = (L_{01}, L_{12}, \dots, L_{(k-1)k})$ cyclic correspondence

$$\left(\begin{array}{l} L_{(j-1)j} \subset N_{j-1}^{-} \times N_j \text{ Lagrangian correspondences} \\ N_0, N_1, \dots, N_{k-1}, N_k = N_0 \text{ symplectic manifolds} \end{array} \right)$$



Construction of quilted Floer homology $HF(\underline{\mathcal{L}})$

① Choose Hamiltonians $\underline{H} = (H_j) \in \bigoplus_{j=1}^{k} \mathcal{C}^\infty(N_j, \mathbb{R})$ such that

$$(L_{01} \times L_{12} \times \dots \times L_{(k-1)k}) \pitchfork \tau(gr\,\phi_{H_1} \times \dots \times gr\,\phi_{H_k}) \subset N_0 \times N_1 \times N_1 \times \dots \times N_k$$

where $gr\,\phi_H \subset N \times N$ is the time 1-flow of the Hamiltonian vector field $X_H = J\nabla H$

and $\tau : N_1 \times N_1 \times \dots \times N_k \times N_k \to N_0 \times N_1 \times N_1 \times \dots \times N_k$; $(x_1, y_1, \dots, x_k, y_k) \mapsto (y_k, x_1, y_1, \dots x_k)$.

<u>Prop$^{\underline{n}}$</u>: Such $\underline{H}$ exist and make the <u>perturbed generalized intersection</u> a finite set.

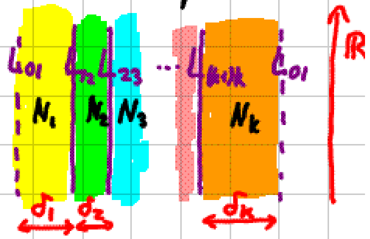
$$\cap_{\underline{H}} \underline{\mathcal{L}} := \{ \underline{p} = (p_1, \dots, \overset{p_0}{p_k}) \in N_1 \times .. \times N_k \mid (\phi_{H_{j-1}}(p_{j-1}), p_j) \in L_{(j-1)j} \ \forall j \}$$

$$\cong \{ \underline{\gamma} = (\gamma_j : [0,1] \to N_j)_{j=1..k} \mid \dot{\gamma}_j = X_{H_j}(\gamma_j),\ (\gamma_{j-1}(1), \gamma_j(0)) \in L_{(j-1)j} \ \forall j \} \subset \mathcal{P}$$

$$\cong (L_{01} \times L_{12} \times \dots \times L_{(k-1)k}) \pitchfork \tau(gr\,\phi_{H_1} \times \dots \times gr\,\phi_{H_k}) \cong \underbrace{\cap \left((\phi_{H_{j-1}}^{-1} \times Id_{N_j})(L_{(j-1)j}) \right)_{j=1..k}}_{\text{unperturbed intersection of perturbed Lagrangians}}$$

$$\to CF(\underline{\mathcal{L}}, \underline{H}) := \bigoplus_{\underline{p} \in \cap_{\underline{H}} \underline{\mathcal{L}}} \underset{\text{or } \mathbb{Z} \text{ (with orientations for } \mathcal{M}^0, \mathcal{M}^1)}{\underbrace{\mathbb{Z}_2}} \langle \underline{p} \rangle \quad \text{is a finitely generated complex}$$

② Choose strip widths $\underline{\delta} = (\delta_1, \ldots, \delta_k) \in (0,\infty)^k$ and $p > 2$.



For $\underline{p}^\pm \in \cap_H \underline{L}$ define a Banach manifold

$$\mathcal{B}(\underline{L}, \underline{p}^-, \underline{p}^+) := \{\underline{u} \mid (i), (ii)\} \subset \bigoplus_{j=1}^{k} W^{1,p}_{loc}(\mathbb{R} \times [0,\delta_j], N_j)$$

(i) $u_j(s,t) \xrightarrow[s \to \pm\infty]{} \gamma_j^\pm(t/\delta_j) = \phi_{H_j}^{t/\delta_j}(p_j)$ uniformly $\forall t \in [0,\delta_j]$ $\forall j = 1..k$

(ii) $(u_{j-1}(s,1), u_j(s,0)) \in L_{(j-1)j}$ $\forall s \in \mathbb{R}$ $\forall j = 0..k$ (with $u_0 := u_k$)

and a Banach bundle $\mathcal{E} \to \mathcal{B}(\underline{L}, \underline{p}^-, \underline{p}^+)$ with fibers

$$\mathcal{E}_{\underline{u}} = \bigoplus_{j=1}^{k} L^p(\mathbb{R} \times [0,\delta_j], u_j^* TN_j).$$

③ Choose a "t-dependent split almost complex structure" $\underline{J} = (J_j)_{j=1..k}$

$J_j \in \mathcal{C}^\infty([0,\delta_j], \mathrm{End}(TN_j))$, where each $J_j(t)$ is an ω_j-compatible almost complex structure.

<u>Prop$^{\underline{n}}$</u>: $\bar\partial := \bar\partial_{\underline{J},\underline{H},\underline{\delta}} : \mathcal{B}(\underline{L}, \underline{p}^-, \underline{p}^+) \to \mathcal{E}$, $\underline{u} \mapsto \left(\partial_s u_j + J_j(\partial_t u_j - \delta_j^{-1} X_{H_j})\right)_{j=1..k}$

is a Fredholm section $\left(\mathcal{C}^1\text{-map}, (T_{\underline{u}}\bar\partial)^{ver} : T_{\underline{u}}\mathcal{B} \to \mathcal{E}_{\underline{u}} \text{ Fredholm } \forall \underline{u}\right)$.

using a connection $T_\xi \mathcal{E} \cong T_{\underline{u}}\mathcal{B} \times \mathcal{E}_{\underline{u}}$ $\forall \xi \in \mathcal{E}_{\underline{u}}$

There exists $\underline{J}$ such that $\bar\partial$ is transverse to the 0-section $\forall \underline{p}^\pm \in \cap_H \underline{L}$,

i.e. $D_{\underline{u}} := (T_{\underline{u}}\bar\partial)^{ver}$ surjective $\forall \underline{u} \in \bar\partial^{-1}(0)$.

$\underset{\text{implicit function thm}}{\Longrightarrow}$ $\mathcal{M}(\underline{L}, \underline{p}^-, \underline{p}^+, \underline{H}, \underline{\delta}, \underline{J}) := \bar\partial^{-1}(0) \subset \mathcal{B}(\underline{L}, \underline{p}^-, \underline{p}^+)$

is a smooth manifold of local dimension

$$\dim_{\underline{u}} \mathcal{M} := \dim(\text{neighborhood of } \underline{u} \text{ in } \mathcal{M}) = \mathrm{index}\, D_{\underline{u}} = \dim\ker D_{\underline{u}}$$

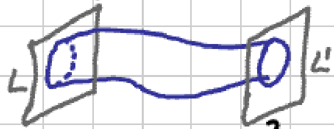
Index & Energy

<u>energy</u> : $\mathcal{M}(\underline{L}, \underline{p}^-, \underline{p}^+, \underline{H}, \underline{\delta}, \underline{J}) \longrightarrow [0,\infty)$

- $\mathcal{E}(\underline{u}) := \sum_{j=1}^{k} \int |\partial_s u_j|^2 = \sum \int \omega_j(\partial_s u_j, \partial_t u_j - \delta_j^{-1} X_H)$ $\quad (\omega(\cdot, X_H) = -dH)$

$$= \sum_{j=1}^{k} \int u_j^*\omega_j + d(u_j^* H_j \cdot \delta_j^{-1} dt) = \sum_{j=1}^{k} \int u_j^*\omega_j + \sum_{j=1}^{k} \left(H_j(p_j^+) - H_j(p_j^-) \right)$$

is determined by $\underline{p}^\pm$ up to



$$\left\{ \int \underline{v}^*\underline{\omega} = \sum_{j=1}^{k} \int v_j^*\omega_j \;\middle|\; v_j : S^1\times[0,1] \to N_j,\ (v_{j-1}(s,1), v_j(s,0)) \in L_{(j-1)j} \right\}$$

- <u>index</u> : $\mathcal{M}(\underline{L}, \underline{p}^-, \underline{p}^+, \underline{H}, \underline{\delta}, \underline{J}) \longrightarrow \mathbb{N}_0$ is determined by $\underline{p}^\pm$ up to

$$\left\{ I_{Maslov}(\underline{v}) \;\middle|\; v_j : S^1\times[0,1] \to N_j,\ (v_{j-1}(s,1), v_j(s,0)) \in L_{(j-1)j} \right\} = N_{\underline{L}} \cdot \mathbb{Z}$$

$$\Vert$$

$$\sum_{j=1}^{k} I_{Maslov}\Big(\underbrace{(v_{j-1}|_{t=1} \times v_j|_{t=0})^* TL_{(j-1)j}}_{S^1 \longrightarrow \mathcal{L}ag(\underbrace{TN_{j-1}\times TN_j}_{\cong \mathbb{C}^N})} \Big)$$

using trivializations of $v_j^* TN_j \;\; \forall j$

($I_{Maslov} : \pi_1(\mathcal{L}ag(\mathbb{C}^N)) \xrightarrow{\sim} \mathbb{Z}$ is the Maslov index [McDuff-Salamon, Intr. to Symp. Top., §2.3])

Monotonicity Assumption

$\exists\, \tau > 0$ s.t. $\int \underline{v}^*\underline{\omega} = \tau \cdot I_{Maslov}(\underline{v}) \quad \forall\, \underline{v}$ as above

(For experts this means in particular each N_j and each $L_{(j-1)j}$ is τ-monotone.)

L9 - quilted Floer homology

<u>Construction of $HF(\underline{\mathcal{L}})$ for $\underline{\mathcal{L}}$ cyclic correspondence</u>: Choose regular Hamiltonians $\underline{H}$, strip widths $\underline{\delta}$, regular almost complex structures $\underline{J}$.

We defined $CF(\underline{\mathcal{L}},\underline{H}) := \bigoplus_{\underline{p}\in\cap_{\underline{H}}\underline{\mathcal{L}}} \mathbb{Z}_2\langle \underline{p}\rangle$

and will define ∂ from the moduli spaces $\mathcal{M}(\underline{\mathcal{L}},\underline{p}^-,\underline{p}^+,\underline{H},\underline{\delta},\underline{J})$

$$= \left\{ \underline{u}\in\bigoplus_{j=1}^k W^{1,p}_{loc}(\mathbb{R}\times[0,\delta_j],N_j) \;\middle|\; \begin{array}{l} \lim_{s\to\pm\infty} u_j(s,t) = \phi_{H_j}^{t/\delta_j}(p_j^\pm),\quad E(\underline{u})<\infty \\ (u_{j-1}(s,\delta_{j-1}),u_j(s,0))\in L_{(j-1)j} \\ \partial_s u_j + J_j(\partial_t u_j - \delta_j^{-1}X_{H_j}) = 0 \end{array} \right\}$$

with energy $E(\underline{u}) = \frac{1}{2}\sum_{j=1}^k \int_{\mathbb{R}\times[0,\delta_j]} |\partial_s u_j|^2 + |\partial_t u_j - \delta_j^{-1}X_{H_j}|^2$

<u>Remark</u>: Rescaling $w_j(s,\tau) := u_j(s,\delta_j\tau)$ identifies $\mathcal{M}$ with a moduli space of "$(\delta_j^{-1}J_j)$-holomorphic" quilts

$$\left\{ \underline{w}\in\bigoplus_{j=1}^k W^{1,p}_{loc}(\mathbb{R}\times[0,1],N_j) \;\middle|\; \begin{array}{l} \lim_{s\to\pm\infty} w_j(s,\tau) = \phi_{H_j}^{\tau}(p_j^\pm),\quad E_{\underline{\delta}}(\underline{w})<\infty \\ (w_{j-1}(s,1),w_j(s,0))\in L_{(j-1)j} \\ \partial_s w_j + \delta_j^{-1} J_j(\partial_\tau w_j - X_{H_j}) = 0 \end{array} \right\}$$

with energy $E_{\underline{\delta}}(\underline{w}) = \frac{1}{2}\sum_{j=1}^k \int_{\mathbb{R}\times[0,1]} \delta_j|\partial_s w_j|^2 + \delta_j^{-1}|\partial_\tau w_j - X_{H_j}|^2 \, ds\,d\tau$

For a single strip $\underline{u} = (u:\mathbb{R}\times[0,\delta]\to N)$ as in "classical" $HF(L,L')$ or $HF(\varphi)$ the moduli spaces for width δ and width 1 are identified by $w(s,t) := u(\delta s,\delta t)$.

Energy & Index

Assume monotonicity: $\int \underline{v}^*\underline{\omega} = \tau \cdot I_{Maslov}(\underline{v}) \quad \forall \underline{v}: S^1 \to \mathcal{P}$
(with $\tau \geq 0$)

Let $N_{\underline{\mathcal{L}}} \in \mathbb{N}$ be the generator of $\{I_{Maslov}(\underline{v})\} \subset \mathbb{Z}$.

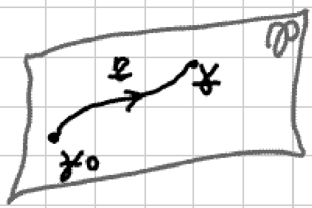
Fix a base point $\underline{\gamma}_0 \in \mathrm{Crit}\, \mathcal{A}_{\underline{H}} \cong \pitchfork_{\underline{H}} \underline{\mathcal{L}}$ in each connected component of

$\mathcal{P} = \{\underline{\gamma} = (\gamma_j : [0,1] \to N_j) \mid L_{(j-1)j}\text{-conditions}\}$ that has a crit. pt.. Then we have

(I) <u>S^1-valued action</u> $\mathcal{A}_{\underline{H}} : \mathcal{P} \longrightarrow \mathbb{R}/_{\tau N_{\underline{\mathcal{L}}}\mathbb{Z}}$, $\underline{\gamma} \longmapsto -\int \underline{\eta}^*\underline{\omega} - \underline{H}(\underline{\gamma})$

- $\underline{H}(\underline{\gamma}) := \sum_{j=1}^{k} \int_0^1 H_j(\gamma_j(t))\, dt \quad \left(\text{if } \dot{\gamma}_j = X_{H_j} \text{ then } \frac{d}{dt} H_j(\gamma_j) = dH_j(X_j) = \omega_j(X_j, X_j) = 0 \right)$
- $\underline{\eta} : [0,1] \to \mathcal{P}$ path from $\underline{\gamma}_0$ to $\underline{\gamma}$

  i.e. $\underline{\eta} = (\eta_j : [0,1] \times [0,1] \to N_j)$

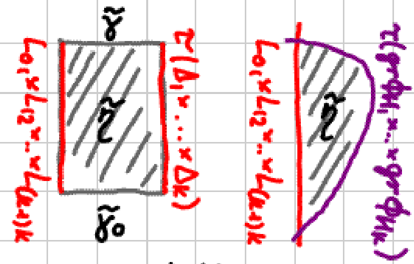


(II) <u>$\mathbb{Z}_{N_{\underline{\mathcal{L}}}}$-grading on $CF(\underline{\mathcal{L}}, \underline{H})$</u>

$|\langle \underline{p} \rangle| := I_{Maslov}(\underline{\eta})$ for $\underline{\eta} : [0,1] \to \mathcal{P}$ $\quad \underline{\eta}(0) = \underline{\gamma}_0$, $\underline{\eta}(1) = \underline{\gamma} \,\hat{=}\, \underline{p} \in \pitchfork_{\underline{H}} \underline{\mathcal{L}}$

NOT IN LECTURE

Fold $\underline{\eta}$ to $\tilde{\eta} : [0,1] \times [0, \frac{1}{2}] \longrightarrow N_0 \times N_1 \times N_1 \times \ldots \times N_k$



then $\eta(s,t) = (\phi^t_{H_k} \times Id_1 \times \phi^t_{H_1} \times Id_2 \times \ldots \times \phi^t_{H_{k-1}} \times Id_k)(\tilde{\eta}(s,t))$

induces 2 paths of Lagrangian subspaces that are transverse at the ends

$\eta|^*_{t=0} T(L_{01} \times \ldots \times L_{(k-1)k}),\ \eta|^*_{t=1} T\tau(\mathrm{gr}\,\phi_{H_1} \ldots \mathrm{gr}\,\phi_{H_k}) : [0,1] \longrightarrow \mathcal{L}ag(\eta^* T(N_0 \times N_1 \times \ldots \times N_k))$
$\cong$ trivialize
$\mathbb{C}^N$

These have a Maslov index [Robbin-Salamon].

(III) Energy and index identities

$\forall \underline{u} \in \mathcal{M}(\underline{p}^-, \underline{p}^+) \qquad \mathcal{E}(\underline{u}) = \mathcal{A}_{\underline{H}}(\underline{p}^-) - \mathcal{A}_{\underline{H}}(\underline{p}^+) \qquad \text{mod } \tau N_{\underline{L}}$

$\text{index } D_{\underline{u}} = |\underline{p}^+| - |\underline{p}^-| \qquad \text{mod } N_{\underline{L}}$

$\Rightarrow \mathcal{E}(\underline{u}) = \tau \cdot \text{index } D_{\underline{u}} + C_{\underline{p}^-, \underline{p}^+}$

$\left(\text{"} \mathcal{A}_{\underline{H}}(\underline{p}^-) + \tau|\underline{p}^-| - \mathcal{A}_{\underline{H}}(\underline{p}^+) - \tau|\underline{p}^+| \right.$
using same paths to $\underline{p}^-, \underline{p}^+$ in $\mathcal{P}$ $\left. \right)$

$\boxed{\mathbb{R}\text{-shift}}$ $\quad \underline{u} \mapsto \sigma * \underline{u} = (u_j(\sigma + \cdot, \cdot))$ maps $\mathcal{M}$ to $\mathcal{M}$ $\quad \forall \sigma \in \mathbb{R}$

$\Rightarrow (\partial_s u_j) \in \ker D_{\underline{u}} = T_{\underline{u}} \mathcal{M} \Rightarrow \text{index } D_{\underline{u}} \geq 1$ unless $\underline{u} = \underline{p}^- = \underline{p}^+$ $\mathbb{R}$-independent
$(\leadsto \mathcal{E}(\underline{u}) = 0, \text{ index } D_{\underline{u}} = 0)$

⊛ $\mathbb{R}$ acts properly discontinuously on $\{\underline{u} \in \mathcal{M} \mid \text{index } D_{\underline{u}} \geq 1\}$

($\underline{u}$ cannot be periodic in $s \in \mathbb{R}$ since $0 < \mathcal{E}(\underline{u}) = \int_{\mathbb{R}} \sum \int |\partial_s u_j|^2 < \infty$)

$\Rightarrow$ For $k \geq 0$ $\quad \mathcal{M}^k(\underline{p}^-, \underline{p}^+) := \{\underline{u} \in \mathcal{M}(\underline{L}, \underline{p}^-, \underline{p}^+, \underline{H}, \underline{\delta}, \underline{J}) \mid \text{index } D_{\underline{u}} = k+1\} / \mathbb{R}$

is a smooth manifold of dimension k.

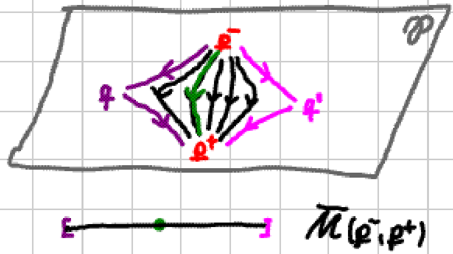
Thm (Gromov compactness, Gluing) (& orientations from "relative spin structure")

Assuming monotonicity and $N_{\underline{L}} \geq 2$ (e.g. $L_{(j-1)j}$ oriented)
and for (ii) minimal Maslov $N_{L_{(j-1)j}} \geq 3$ for disks $(D, \partial D) \to (N_{j-1} \times N_j, L_{(j-1)j})$

(i) $\mathcal{M}^0(\underline{p}^-, \underline{p}^+)$ is compact $\quad$ (& oriented $\mathcal{M}^0(\underline{p}^-, \underline{p}^+) \to \{\pm 1\}$)

(ii) $\mathcal{M}^1(\underline{p}^-,\underline{p}^+)$ can be compactified to a $\overset{\text{(oriented)}}{}$ 1-manifold $\overline{\mathcal{M}}^1(\underline{p}^-,\underline{p}^+)$ with boundary $\partial\overline{\mathcal{M}}^1(\underline{p}^-,\underline{p}^+) = \bigcup_{\underline{q}\in\cap_{\underline{L}}} \mathcal{M}^0(\underline{p}^-,\underline{q})\times\mathcal{M}^0(\underline{q},\underline{p}^+)$ "broken trajectories"

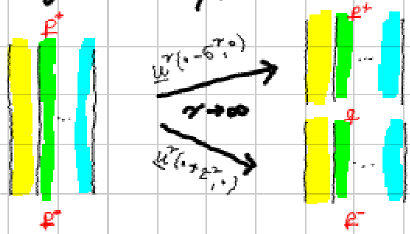
""Proof"": as in Morse theory



""Proof of compactness"" Any sequence $\underline{u}^\nu\in\mathcal{M}^k(\underline{p}^-,\underline{p}^+)$ of bounded energy $E(\underline{u}^\nu)=\tau(k+1)+c_{\underline{p}^-,\underline{p}^+}$ has a convergent subsequence unless

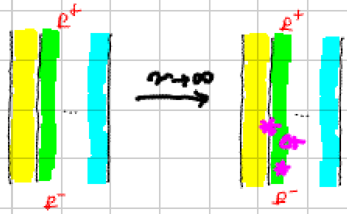
(a) energy escapes to $\pm\infty$

"breaking of trajectory"



(b) energy concentrates at a point *

"bubbling off"



In the image, a sphere $S^2\to N_j$ or disc $(D^2,\partial D^2)\to(N_{j-1}^-\times N_j, L_{(j-1)j})$ forms.

On the domain, $\underline{u}^\nu\to\underline{u}'$ converges on the complement of the point(s) *, and the singularity can be removed to obtain a new solution $\underline{u}'\in\mathcal{M}(\underline{p}^-,\underline{p}^+)$ with less energy $\underset{\text{monotonicity}}{\Longrightarrow}$ less index

(i) new index $\leq 1-N_{\underline{L}}<0 \Rightarrow \nexists\,\underline{u}' \Rightarrow$ no bubbling

(ii) new index $\leq 2-N_{L_{(j-1)j}}<0 \Rightarrow \nexists\,\underline{u}' \Rightarrow$ no bubbling "■"

Define $\partial : CF \to CF$ by $\quad \partial \langle \underline{p}^- \rangle := \sum_{\underline{p}^+ \in \cap_{\underline{H}} \underline{L}} \underbrace{\left(\sum_{\underline{u} \in \mathcal{M}^0(\underline{p}^-, \underline{p}^+)} \pm 1 \right)}_{\in \mathbb{Z}_2} \langle \underline{p}^+ \rangle$

$\in \mathbb{Z}_2$ or $\mathbb{Z}$ with orientations

then $\quad \partial^2 \langle \underline{p}^- \rangle = \sum_{\underline{p}^+} \underbrace{\sum_{\underline{q}} \# \mathcal{M}^0(\underline{p}^-, \underline{q}) \cdot \# \mathcal{M}^0(\underline{q}, \underline{p}^+)}_{= \# \partial \overline{\mathcal{M}}^1(\underline{p}^-, \underline{p}^+) = 0} \langle \underline{p}^+ \rangle = 0 .$

<u>Thm</u>: Floer (co)homology groups $HF(\underline{L}) := \ker \partial / \operatorname{im} \partial$ are independent of the choice of $\underline{H}, \underline{\delta}, \underline{J}$; up to isomorphism.

L10 - quilted Floer homology - invariance

Note Title 2/25/2008

<u>Thm (Gromov compactness)</u> $\underline{\mathcal{L}}$ monotone cyclic correspondence

(i) $\mathcal{M}^0(\underline{p}^-,\underline{p}^+)$ is compact   if $N_{\underline{\mathcal{L}}} \geq 2$ $(\Rightarrow N_{L_{(j-1)j}} \geq 2)$

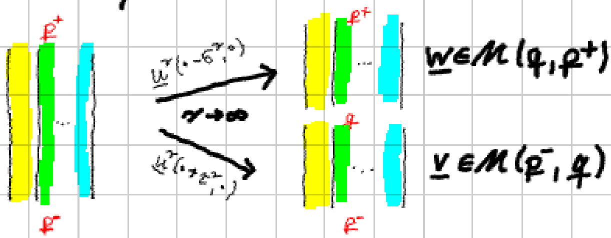
(ii) $\mathcal{M}^1(\underline{p}^-,\underline{p}^+)$ is compact "up to breaking of trajectories" if $N_{L_{(j-1)j}} \geq 3$

<u>"Proof"</u>: Any sequence $\underline{u}^\nu \in \mathcal{M}^k(\underline{p}^-,\underline{p}^+)$ has bounded energy $E(\underline{u}^\nu) = \tau \underbrace{(k+1)}_{\text{index } D_{\underline{u}^\nu}} + c_{\underline{p}^-,\underline{p}^+}$

"and hence" (!analysis!) has a convergent subsequence unless

(a) energy escapes to $\pm\infty$

"breaking of trajectory"

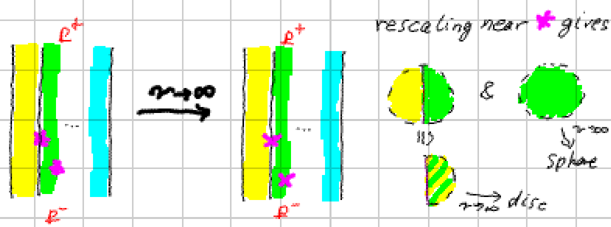


(i): $\underline{v}$ or $\underline{w}$ has index 0 ↯ constant has no energy

$$\text{index } D_{\underline{u}^\nu} = \text{index } D_{\underline{v}} + \text{index } D_{\underline{w}}$$
$$\text{(ii)}: \quad 2 = 1 + 1$$

(b) energy concentrates at point(s) $*$

"bubbling off"



In the image, a sphere $S^2 \to N_j$ or disc $(D^2, \partial D^2) \to (N_{j-1}^- \times N_j, L_{(j-1)j})$ forms.

On the domain, $\underline{u}^\nu \to \underline{u}'$ converges on the complement of the point(s) $\times$, and the singularity can be removed to obtain a new solution $\underline{u}' \in \mathcal{M}(\underline{p}^-,\underline{p}^+)$

with <u>less energy</u> $\underset{\text{monotonicity}}{\Longrightarrow}$ <u>less index</u>

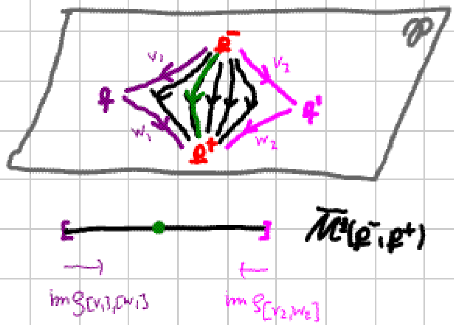
$\mathcal{E} = \tau \cdot \text{Ind} + \text{const}, \quad \tau > 0$

(i) new index $\le 1 - N_{\underline{L}} < 0 \;\Rightarrow\; \nexists u' \;\Rightarrow$ no bubbling

(ii) new index $\le 2 - N_{L_{(j-1)j}} < 0 \;\Rightarrow\; \nexists u' \;\Rightarrow$ no bubbling "$\blacksquare$"

To establish $\partial \overline{\mathcal{M}}^1(\underline{p}^-,\underline{p}^+) = \bigcup_{\underline{q} \in \cap_{\underline{L}}\underline{L}} \mathcal{M}^0(\underline{p}^-,\underline{q}) \times \mathcal{M}^0(\underline{q},\underline{p}^+)$

it remains to prove a



<u>Gluing theorem:</u>

There exist embeddings

$$g_{[\underline{v}],[\underline{w}]} : (R_0,\infty) \hookrightarrow \mathcal{M}^1(\underline{p}^-,\underline{p}^+)$$

for each $([\underline{v}],[\underline{w}]) \in \bigcup_{\underline{q} \in \cap_{\underline{L}}\underline{L}} \mathcal{M}^0(\underline{p}^-,\underline{q}) \times \mathcal{M}^0(\underline{q},\underline{p}^+)$

- with disjoint images
- such that $\mathcal{M}^1(\underline{p}^-,\underline{p}^+) \setminus \bigcup \operatorname{im} g_{[\underline{v}],[\underline{w}]}$ is compact

"<u>Proof</u>": some more in [Salamon, Lectures..., §3.3]

- pregluing : define $\underline{v} \#_R \underline{w} \in \mathcal{B}(\underline{L},\underline{p}^-,\underline{p}^+)$ by interpolating $\underline{v}(\cdot + R,\cdot)$ and $\underline{w}(\cdot - R,\cdot)$, then $\bar\partial\, \underline{v}\#_R\underline{w}$ = small
- implicit function theorem gives a nearby zero $\bar\partial\, g_{\underline{v},\underline{w}}(R) = 0$ (based on estimates for $D_{\underline{v}\#_R\underline{w}}$)

⋮

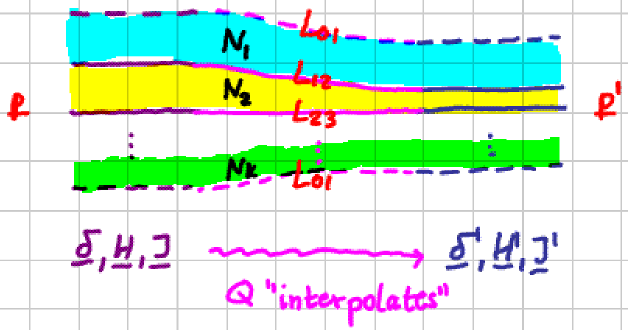
"$\blacksquare$"

Thm: Floer (co)homology groups $HF(\underline{L}) := {}^{\ker\partial}/_{\mathrm{im}\,\partial}$ are independent of the choice of $\underline{H}, \underline{\delta}, \underline{J}$; up to isomorphism.

"Proof": Construction of isomorphism $HF(\underline{L},\underline{H},\underline{\delta},\underline{J}) \xrightarrow{\sim} HF(\underline{L},\underline{H}',\underline{\delta}',\underline{J}')$

(i) Define $\Phi_Q : CF(\underline{L},\underline{H}) \longrightarrow CF(\underline{L},\underline{H}')$, $\langle \underline{p}\rangle \longmapsto \sum_{\underline{p}'\in \cap_{\underline{H}'}\underline{L}} \#\tilde{\mathcal{M}}^0(\underline{p},\underline{p}')\,\langle \underline{p}'\rangle$

from 0-dim. moduli space $\tilde{\mathcal{M}}^0(\underline{L},Q,\underline{p},\underline{p}')$ of holomorphic quilts



- index 0
- no $\mathbb{R}$-symmetry

∘ pick "regular" Q

(ii) Φ_Q is a chain homomorphism $\Phi_Q\partial = \partial'\Phi_Q$ "because"

we can exclude bubbling and compactify $\tilde{\mathcal{M}}^1(\underline{L},Q,\underline{p},\underline{p}')$ with boundary

$$\bigcup_{\underline{q}\in\cap_{\underline{H}}\underline{L}} \mathcal{M}^0(\underline{L},\underline{H},\underline{J},\underline{\delta},\underline{p},\underline{q})\times\tilde{\mathcal{M}}^0(\underline{L},Q,\underline{q},\underline{p}') \;\cup\; \bigcup_{\underline{q}'\in\cap_{\underline{H}'}\underline{L}} \tilde{\mathcal{M}}^0(\underline{L},Q,\underline{p},\underline{q}')\times\mathcal{M}^0(\underline{L},\underline{H}',\underline{J}',\underline{\delta}',\underline{q}',\underline{p}')$$

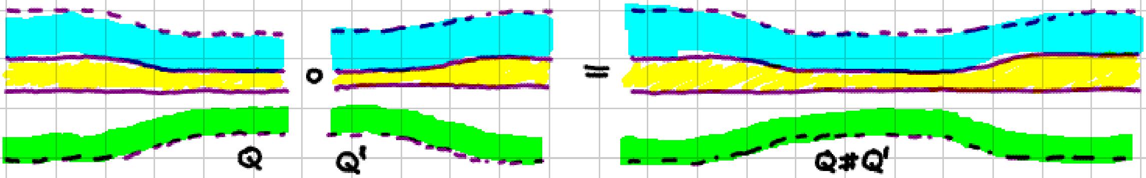
index $\quad 1 + 0 \quad$ or $\quad 0 + 1 \quad = 1$

Hence Φ_Q descends to a map $H\Phi_Q : HF(\underline{L},\underline{H},\underline{\delta},\underline{J}) \to HF(\underline{L},\underline{H}',\underline{\delta}',\underline{J}')$.

(iii) Similarly construct $H\Phi_{Q'}: HF(\underline{\mathcal{L}}, \underline{H}', \underline{\delta}', \underline{J}') \to HF(\underline{\mathcal{L}}, \underline{H}, \underline{\delta}, \underline{J})$, then

- $H\Phi_{Q'} \circ H\Phi_Q = H\Phi_{Q\#Q'} \circlearrowright HF(\underline{\mathcal{L}}, \underline{H}, \underline{\delta}, \underline{J})$

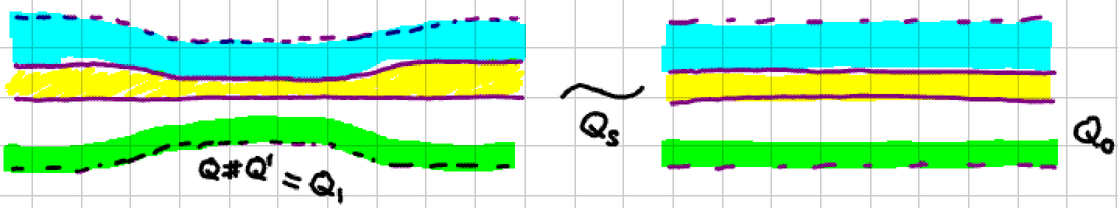
"because composition is given by gluing"



$$\left(\tilde{\mathcal{M}}^0(Q, p, p') \times \tilde{\mathcal{M}}^0(Q', p', q) \xrightarrow{\simeq} \tilde{\mathcal{M}}^0(Q \#_R Q', p, q) \right)$$

$(\underline{v}, \underline{w})$ $\xrightarrow{\text{pregluing}}$ $\underline{v} \#_R \underline{w}$ ↗ implicit function theorem for fixed large R

- $H\Phi_{Q\#Q'} = H\Phi_{Q_0}$ "because" (see (iv)) $Q\#Q'$ is homotopic to trivial quilt $Q_0 = (\underline{\delta}, \underline{H}, \underline{J})$



- $H\Phi_{Q_0} = Id_{CF(\underline{\mathcal{L}}, \underline{H})}$ since $\tilde{\mathcal{M}}^0(\underline{\mathcal{L}}, Q_0, p, q)$ has an $\mathbb{R}$-action

$\Rightarrow$ solutions have index ≥ 1 except for constant strips

$\Rightarrow \# \tilde{\mathcal{M}}^0(p, q) = \delta_{p,q} \Rightarrow \Phi = Id$

(iv) A "homotopy $(Q_s)_{s\in[0,1]}$ of quilts" with fixed ends $(\underline{H}_i, \underline{\delta}_i, \underline{J}_i)$, $i=1,2$

defines a chain homotopy equivalence $T: CF(\underline{\mathcal{L}},\underline{H}_1) \to CF(\underline{\mathcal{L}},\underline{H}_2)$

$\Phi_{Q_1} - \Phi_{Q_2} = \partial_2 \circ T + T \circ \partial_1$, thus $HF_{Q_1} = HF_{Q_2} : HF(\underline{\mathcal{L}},\underline{H}_1,\dots) \to HF(\underline{\mathcal{L}},\underline{H}_2,\dots)$.

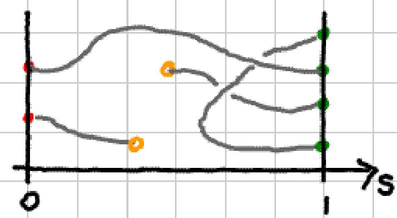
We construct $T\langle p_1\rangle := \sum_{p_2} \#\hat{\mathcal{M}}^{-1}(\underline{\mathcal{L}},\{Q_s\},p_1,p_2)\,\langle p_2\rangle$

from the index $k=-1$ moduli spaces

$$\hat{\mathcal{M}}^k(\underline{\mathcal{L}},\{Q_s\},p_1,p_2) := \{(s,\underline{u}) \mid s\in[0,1],\ \underline{u} \in \tilde{\mathcal{M}}^k(\underline{\mathcal{L}},Q_s,p_1,p_2)\}$$

(index $D_{\underline{u}} = k$)

The identity follows from $\hat{\mathcal{M}}^0$ having



- true boundary $\tilde{\mathcal{M}}^0(Q_0) \cup \tilde{\mathcal{M}}^0(Q_1)$
- ends (→ compactified boundary)

$$\mathcal{M}^0(\underline{H}_1,\dots p_1,q_1) \times \hat{\mathcal{M}}^{-1}(\{Q_s\},q_1,p_2) \cup \hat{\mathcal{M}}^{-1}(\{Q_s\},p_1,q_2) \times \mathcal{M}^0(\underline{H}_2,\dots,q_2,p_2)$$

index $\quad 1 + -1 \quad$ or $\quad -1 + 1 = 0$

"■"

for (a few) more details see [Salamon, Lectures on F.H., §3.4.]

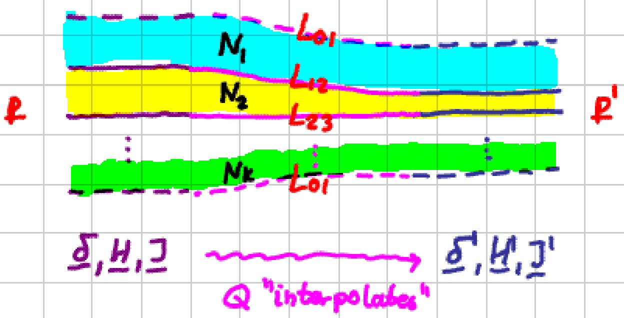
L11 – quilted Floer homology & geometric compositio

Note Title 2/25/2008

<u>Thm</u>: $HF(\underline{L})$ is independent of $\underline{H}, \underline{\delta}, \underline{J}$.

<u>"Proof"</u>: [Salamon, Lectures on F.H., §3.4.] [Schwarz, Morse Homology]

(i)
(ii) Counting holomorphic quilts $\tilde{\mathcal{M}}^0(\underline{L}, Q, \underline{p}, \underline{p}')$ for regular "quilt data" Q



interpolating $(\underline{\delta}, \underline{H}, \underline{J})$ to $(\underline{\delta}', \underline{H}', \underline{J}')$

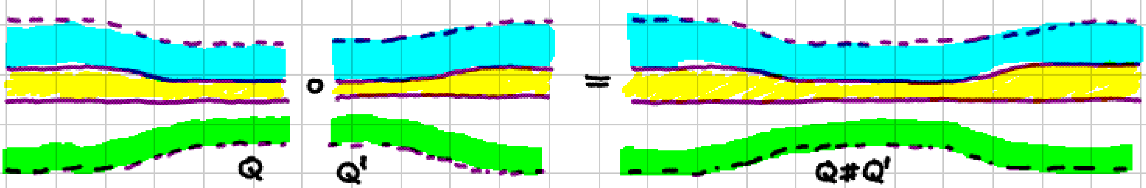
defines a map

$$H\Phi_Q : HF(\underline{L}, \underline{H}, \underline{\delta}, \underline{J}) \to HF(\underline{L}, \underline{H}', \underline{\delta}', \underline{J}')$$

Similarly construct $H\Phi_{Q'} : HF(\underline{L}, \underline{H}', \underline{\delta}', \underline{J}') \to HF(\underline{L}, \underline{H}, \underline{\delta}, \underline{J})$, then

(iii) $H\Phi_{Q'} \circ H\Phi_Q = H\Phi_{Q \# Q'} \circlearrowright HF(\underline{L}, \underline{H}, \underline{\delta}, \underline{J})$

"because composition is given by gluing"

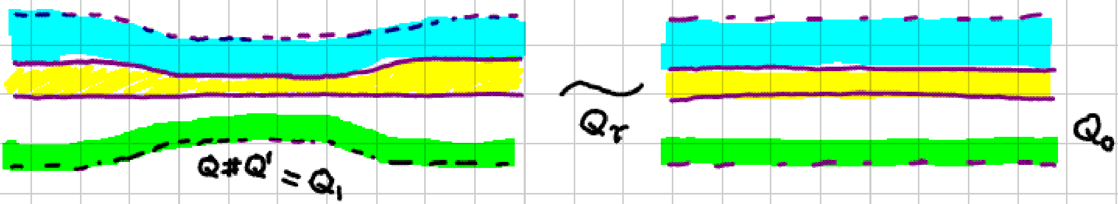


$$\left(\tilde{\mathcal{M}}^0(Q, \underline{p}, \underline{p}') \times \tilde{\mathcal{M}}^0(Q', \underline{p}', \underline{q}) \xrightarrow{\simeq} \tilde{\mathcal{M}}^0(Q \#_R Q', \underline{p}, \underline{q}) \right)$$

$(\underline{v}, \underline{w})$ $\xrightarrow{\text{pregluing}}$ $\underline{v} \#_R \underline{w}$ — implicit function theorem for fixed large R

(iv) $H\Phi_{Q\#Q'} = H\Phi_{Q_0}$ "because" $Q\#Q'$ is homotopic to trivial quilt $Q_0=(\underline{\delta},\underline{H},\underline{J})$

below



(v) $H\Phi_{Q_0} = Id_{CF(\underline{L},\underline{H})}$ since $\tilde{\mathcal{M}}^0(\underline{L},Q_0,p,q)$ has an $\mathbb{R}$-action

$\Rightarrow$ solutions are index ≥ 1 or constant strip $\Rightarrow \#\tilde{\mathcal{M}}^0(p,q) = \delta_{p,q} \Rightarrow \Phi = Id$

(iv) A "homotopy $(Q_\tau)_{\tau\in[0,1]}$ of quilts" with fixed ends $(\underline{H}_i,\underline{\delta}_i,\underline{J}_i)$, $i=1,2$

defines a chain homotopy equivalence $T: CF(\underline{L},\underline{H}_1)\to CF(\underline{L},\underline{H}_2)$

$\boxed{\Phi_{Q_1}-\Phi_{Q_2} = \partial_2\circ T + T\circ\partial_1}$, thus $HF_{Q_1} = HF_{Q_2}: HF(\underline{L},\underline{H}_1,\dots)\to HF(\underline{L},\underline{H}_2,\dots)$.

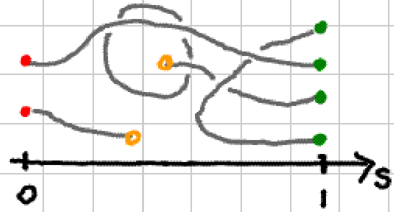
We construct $T\langle p_1\rangle := \sum_{p_2} \#\hat{\mathcal{M}}^{-1}(\underline{L},\{Q_\tau\},p_1,p_2)\,\langle p_2\rangle$

from the index $k=-1$ moduli spaces

$$\hat{\mathcal{M}}^k(\underline{L},\{Q_\tau\},p_1,p_2) := \Big\{(\tau,\underline{u}) \;\Big|\; \tau\in[0,1],\ \underline{u}\in \underset{(\text{index } D_{\underline{u}}=k)}{\tilde{\mathcal{M}}^k(\underline{L},Q_\tau,p_1,p_2)}\Big\}$$

The $\boxed{\text{identity}}$ follows from $\hat{\mathcal{M}}^0$ having

- true boundary $\tilde{\mathcal{M}}^0(Q_0) \cup \tilde{\mathcal{M}}^0(Q_1)$
- ends ($\to$ compactified boundary)



$$\mathcal{M}^0(\underline{H}_1\dots p_1,q_1)\times\hat{\mathcal{M}}^{-1}(\{Q_\tau\},q_1,p_2) \cup \hat{\mathcal{M}}^{-1}(\{Q_\tau\},p_1,q_2)\times\mathcal{M}^0(\underline{H}_2\dots,q_2,p_2)$$

index $1 + -1$ or $-1 + 1 = 0$

Thm [W-Woodward]: $\underline{\mathcal{L}} \sim \underline{\mathcal{L}}'$ equivalent cyclic correspondences

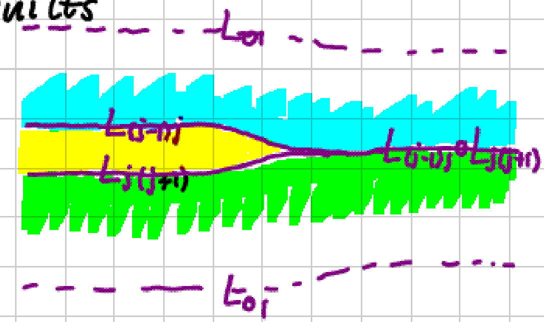
$\Rightarrow HF(\underline{\mathcal{L}}) \cong HF(\underline{\mathcal{L}}')$ isomorphism (induced by sequence of good moves from $\underline{\mathcal{L}}$ to $\underline{\mathcal{L}}'$).

Dream Proof for a good move $\underline{\mathcal{L}} = (\ldots\ L_{(j-1)j}\,,\, L_{j(j+1)}\,, \ldots)$

define $HF(\underline{\mathcal{L}}) \xrightarrow{\sim} HF(\underline{\mathcal{L}}')$ $\qquad \underline{\mathcal{L}}' = (\ldots\ L_{(j-1)j} \circ L_{j(j+1)}\,, \ldots)$

transverse & embedded

by counting "quilts"



...but cannot make sense of  ...

INSTEAD: Define $\Phi : CF(\underline{\mathcal{L}}, \underline{H}) \to CF(\underline{\mathcal{L}}', \underline{H}')$ from a canonical

isomorphism $\cap_{\underline{H}} \underline{\mathcal{L}} \xrightarrow{\sim} \cap_{\underline{H}'} \underline{\mathcal{L}}'$, $(p_1, \ldots p_{j-1}, p_j, p_{j+1}, \ldots p_k) \longmapsto (p_1 \ldots p_{j-1}, p_{j+1} \ldots p_k)$

- choose $\underline{H}'$ regular for $\underline{\mathcal{L}}'$, then $\underline{H} := (\ldots H'_{j-1}, H_j = 0, H'_{j+1}, \ldots)$ is regular for $\underline{\mathcal{L}}$

$$L_{01} \times \ldots L_{(j-1)j} \circ L_{j(j+1)} \ldots L_{(k-1)k} \pitchfork \tau(\ldots \Delta_{j-1} \times \Delta_{j+1} \ldots)$$

$$L_{01} \times \ldots \underbrace{L_{(j-1)j} \times L_{j(j+1)}} \ldots L_{(k-1)k} \pitchfork \tau(\ldots \Delta_{j-1} \times \Delta_j \times \Delta_{j+1} \ldots)$$

$T(..) \cong T(L_{(j-1)j} \circ L_{j(j+1)}) \oplus D_j$; $D_j \pitchfork \Delta_j \subset N_j^- \times N_j$ since $L_{(j-1)j} \circ L_{j(j+1)}$ is transverse

- $\cap_{\underline{H}} \underline{\mathcal{L}} \xrightarrow{\sim} \cap_{\underline{H}'} \underline{\mathcal{L}}'$ is bijective since $(\varphi(p_{j-1}), p_{j+1}) \in L_{(j-1)j} \circ L_{j(j+1)}$ embedded

has a unique $p_j \in N_j$ with $(\varphi(p_{j-1}), p_j) \in L_{(j-1)j}$, $(p_j, p_{j+1}) \in L_{j(j+1)}$

Clearly, $\Phi \circ \Phi^{-1} = \mathrm{Id} = \Phi^{-1} \circ \Phi$, but we need to show that Φ descends to homology $H\Phi : HF(\underline{\mathcal{L}}) \to HF(\underline{\mathcal{L}}')$, i.e. $\partial = \partial'$ on $CF(\underline{\mathcal{L}}) \cong CF(\underline{\mathcal{L}}')$.

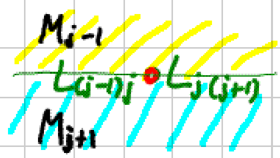
- Fix regular $\underline{H}', \underline{\delta}', \underline{J}'$ for $\underline{\mathcal{L}}'$, then $\underline{H} := (H_1' \dots H_j = 0 \dots)$,

  $\underline{\delta} := (\delta_1' \dots \delta_j = \varepsilon \dots)$, $\underline{J} := (J_1' \dots J_j(t) = \text{some fixed } J_j \in \mathrm{End}(TN_j) \dots)$

  is regular for $\underline{\mathcal{L}}$ for all $\varepsilon > 0$ sufficiently small.

- For $\varepsilon > 0$ suff. small there is a (oriented) bijection

$$\mathcal{M}^0(\underline{\mathcal{L}}, \underline{H}, \underline{\delta}, \underline{J}, \underline{p}^-, \underline{p}^+) \xrightarrow{\sim} \mathcal{M}^0(\underline{\mathcal{L}}', \underline{H}', \underline{\delta}', \underline{J}', \Phi(\underline{p}^-), \Phi(\underline{p}^+))$$



"$\blacksquare$"

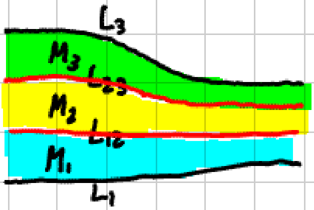
<u>Corollary</u>: We can define groups of 2-morphisms in the symplectic category *Symp*: For $M_0, M_1 \in \mathrm{Ob}$ (sympl. mfds) and $[\underline{L}], [\underline{L}'] \in \mathrm{Mor}(M_0, M_1)/\sim$ (generalized correspondences $M_0 \to M_1$)

$${}^2\mathrm{Mor}([\underline{L}], [\underline{L}']) := HF(\underline{L} \# (\underline{L}')^t) \qquad \text{is well defined.}$$

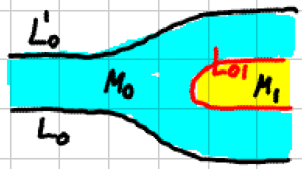
$$\begin{tikzcd} & \underline{L} \arrow[dr] & \\ M_0 \arrow[ur] & & M_1 \arrow[dl] \\ & \underline{L}' \arrow[ul] & \end{tikzcd}$$

L12 - Pseudoholomorphic quilts

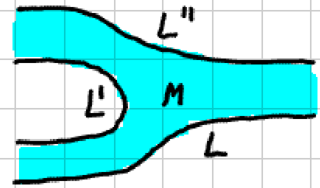
Note Title 3/31/2008



$HF(\underline{L}, .. \underline{\delta} ..) \to HF(\underline{L}, .. \underline{\delta}' ..)$



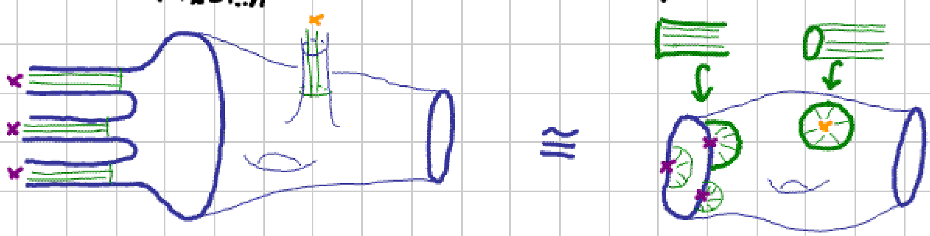
$HF(L_0, L_0') \to HF(L_0, L_{01}, L_{01}^T, L_0')$



$HF(L, L') \otimes HF(L', L'') \to HF(L, L'')$

<u>Def^n</u>: A <u>quilted surface</u> $\underline{S} = \big((\bar{S}_k), (z_{k,e}), (j_k), (\varepsilon_{k,e}), (\delta_{k,e}), \mathcal{S}, (\varphi_\sigma)\big)$ consists of

① patches: $(S_k)_{k=1..n}$ surfaces with strip-like ends



a) $\bar{S}_k$ <u>compact Riemann surface with boundary</u>

$\mathcal{E}_k^b = \mathcal{E}_k^{b+} \cup \mathcal{E}_k^{b-}$, $\mathcal{E}_k^i = \mathcal{E}_k^{i+} \cup \mathcal{E}_k^{i-}$ finite sets

$z_{k,e} \in \partial \bar{S}_k \ \forall e \in \mathcal{E}_k^b$, $z_{k,e} \in \bar{S}_k \setminus \partial \bar{S}_k \ \forall e \in \mathcal{E}_k^i$ <u>distinct marked points</u>

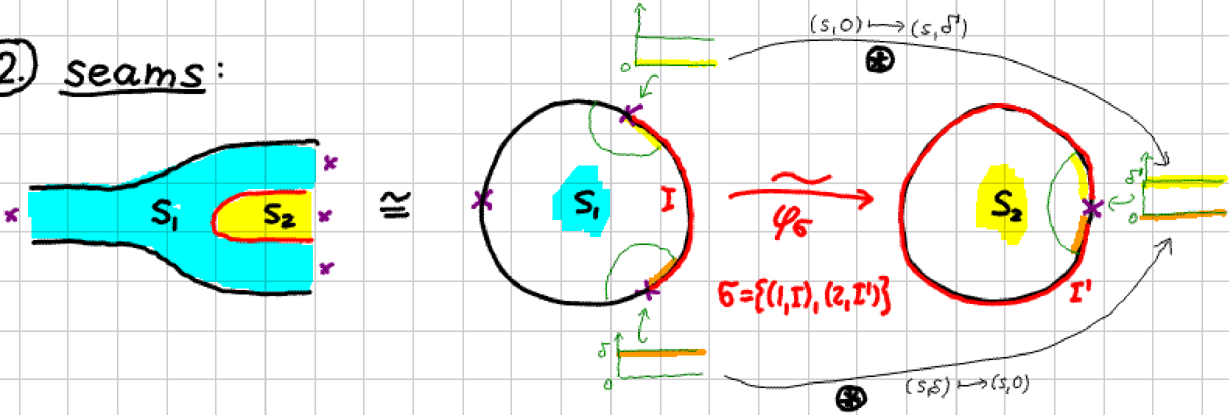
b) j_k <u>complex structure</u> on $S_k = \bar{S}_k \setminus \{ z_{k,e} \mid e \in \mathcal{E}_k^b \cup \mathcal{E}_k^i \}$

c) <u>strip-like ends</u> $\varepsilon_{k,e} : \mathbb{R}^\pm \times [0, \delta_{k,e}] \hookrightarrow S_k$ $\quad e \in \mathcal{E}_k^{b\pm}$, width $\delta_{k,e} > 0$

$\qquad\qquad\qquad\qquad \mathbb{R}^\pm \times S^1 \hookrightarrow S_k$ $\quad e \in \mathcal{E}_k^{i\pm}$

- disjoint images
- $\varepsilon_{k,e}(\pm\infty, \cdot) = z_{k,e}$, $\varepsilon_{k,e}(\cdot, \{0,1\}) \subset \partial \bar{S}_k$
- $\varepsilon_{k,e}^* j_k = i$ standard complex structure on $\mathbb{R} \times \mathbb{R} = \mathbb{C}$ $\quad (S^1 = \mathbb{R}/\mathbb{Z})$

② seams:



a) $\mathcal{S}$ collection of pairwise disjoint 2-element subsets

$$\sigma = \{(k_\sigma, I_\sigma), (k'_\sigma, I'_\sigma)\} \subset \bigcup_{k=1}^{n} \{k\} \times \pi_0\left(\partial \bar{S}_k \setminus \{z_{ke} \mid e \in \mathcal{E}_k^b\}\right)$$

$\cup$

I_σ, I'_σ connected components $\cong \mathbb{R}$ or S^1

b) for each $\sigma \in \mathcal{S}$ a diffeomorphism $\varphi_\sigma : I_\sigma \xrightarrow{\sim} I'_\sigma$

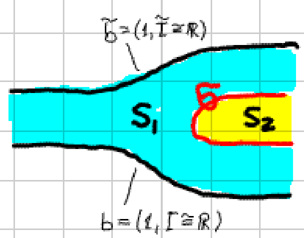
⊛ compatible with strip-like ends $\left(\text{i.e. } \varepsilon_{k'e'}^{-1} \circ \varphi_\sigma \circ \varepsilon_{ke} : (s,0) \mapsto (s, \delta_{k,e'}) \text{ or } (s, \delta_{ke}) \mapsto (s,0) \right)$

③ various orderings (for orientations)

Def^n: The boundary of $\underline{S}$ is

$$\mathcal{B} := \bigcup_{k=1}^{n} \{k\} \times \pi_0\left(\partial \bar{S}_k \setminus \{z_{ke} \mid e \in \mathcal{E}_k^b\}\right) \setminus \bigcup_{\sigma \in \mathcal{S}} \sigma .$$

It indexes the true boundary components $b = (k_b, I_b)$ with $I_b \cong \mathbb{R}$ or S^1.

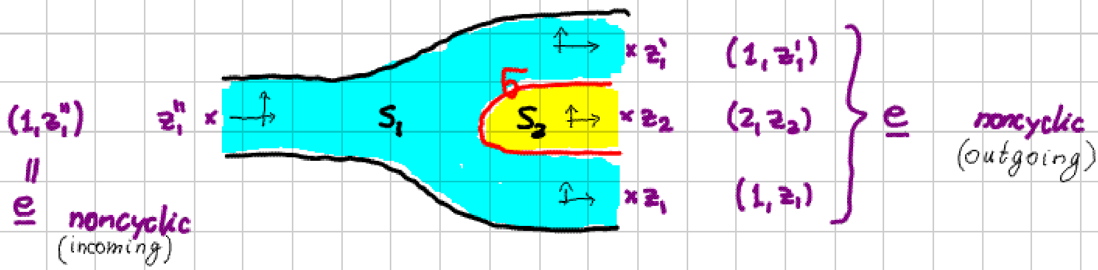


Def^{n}: The ends of $\underline{S}$ are $\underline{e} \in \mathcal{E}(\underline{S}) = \mathcal{E}^+(\underline{S}) \sqcup \mathcal{E}^-(\underline{S})$

given by maximal sequences $\underline{e} = (k_i, z_{k_i,e_i})_{i=1..N}$ of

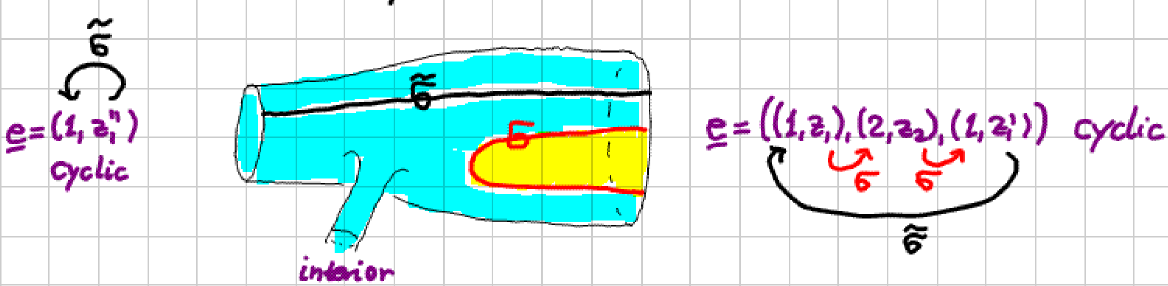
marked points $z_{k_i,e_i} \in \overline{S}_{k_i}$ identified by seams $\sigma_i = \{(k_i, I_i), (k_{i+1}, I'_{i+1})\}$ with $I_i \xrightarrow{\varphi_{\sigma_i}} I'_{i+1}$

(i.e. $\lim_{z \to z_{k_i,e_i}} \varphi_{\sigma_i}(z) = z_{k_{i+1},e_{i+1}}$).



Note • By compatibility of seams with strip-like ends, all marked points in an end $\underline{e}$ are either incoming ($\underline{e} \in \mathcal{E}^-(\underline{S})$) or outgoing ($\underline{e} \in \mathcal{E}^+(\underline{S})$).

• Ends can be
- interior $\underline{e} = (k, z_{k,e} \in \overline{S}_k \setminus \partial \overline{S}_k)$ without seams
- cyclic : with seam $\sigma_N = \{(k_N, z_{k_N,e_N}), (k_1, z_{k_1,e_1})\}$
- noncyclic : without ——— " ———



Symplectic targets for $\underline{S}$ is a tuple $\underline{M} = (M_k)_{k=1..n}$

of symplectic manifolds M_k for each patch S_k.

Lagrangian boundary and seam conditions for $(\underline{S}, \underline{M})$

is a collection $\underline{LBS} = (L_b)_{b\in\mathcal{B}} \cup (L_\sigma)_{\sigma\in\mathcal{S}}$ of

- Lagrangian submanifolds $L_b \subset M_{k_b}$ for each boundary component $b \in \mathcal{B}$,
- Lagrangian correspondences $L_\sigma \subset M_{k_\sigma}^- \times M_{k'_\sigma}$ for each seam $\sigma \in \mathcal{S}$.

A quilt map from $\underline{S}$ to $(\underline{M}, \underline{LBS})$ is a tuple $\underline{u} = (u_k)_{k=1..n}$

of maps $u_k : S_k \to M_k$ satisfying

- Lagrangian boundary conditions $u_{k_b}(I_b) \subset L_b \quad \forall b = (k_b, I_b) \in \mathcal{B}$
- Lagrangian seam conditions $\left(u_{k_\sigma} \times (u_{k'_\sigma} \circ \varphi_\sigma)\right)(I_\sigma) \subset L_\sigma$

$$\forall \; \sigma = \{(k_\sigma, I_\sigma), (k'_\sigma, I'_\sigma)\} \in \mathcal{S}.$$

φ_σ

Fix almost complex structures $\underline{J} = (J_k \in \mathrm{End}(TM_k))$, then

$\underline{u}$ is $\underline{J}$-holomorphic iff $\overline{\partial}_{j_k, J_k} u_k = 0 \qquad \forall k = 1..n$.

$$\overline{\partial}_{j_k, J_k} u_k = \tfrac{1}{2}\left(du_k \circ j_k - J_k \circ du_k\right) \in \Omega^1(S_k, u_k^* TM_k)$$

Note: $\underline{LBS}$ associates a cyclic generalized Lagr. correspondance to each end $\underline{e} = (k_i, z_{k_i, e_i})_{i=1..N}$

- $\underline{e}$ interior : $\underline{\mathcal{L}}_{\underline{e}} = \emptyset$ (⇝ Hamiltonian Floer homology with $\cap_H \underline{L}_{\underline{e}} = \mathrm{Fix}\,\varphi_H$)
- $\underline{e}$ cyclic with seams $\sigma_1, \dots, \sigma_{N-1}, \sigma_N$: $\underline{\mathcal{L}}_{\underline{e}} = (L_{\sigma_N}, L_{\sigma_1}, \dots, L_{\sigma_{k-1}})$
- $\underline{e}$ noncyclic with seams $\sigma_1, \dots, \sigma_{N-1}$ and boundary components b_0, b_N (adjacent to z_{k_1, e_1}, z_{k_N, e_N})

  $\underline{\mathcal{L}}_{\underline{e}} = (L_{b_0}, L_{\sigma_1}, \dots, L_{\sigma_{N-1}}, L_{b_N})$

Thm: For any "monotone" $(\underline{M}, \underline{LBS})$ $\underline{S}$ induces a relative invariant

$$\Phi_{\underline{S}} : \bigotimes_{\underline{e}^- \in \mathcal{E}^-(\underline{S})} HF(\underline{\mathcal{L}}_{\underline{e}^-}) \longrightarrow \bigotimes_{\underline{e}^+ \in \mathcal{E}^+(\underline{S})} HF(\underline{\mathcal{L}}_{\underline{e}^+})$$

L13 - quilt invariants

Note Title 3/31/2008

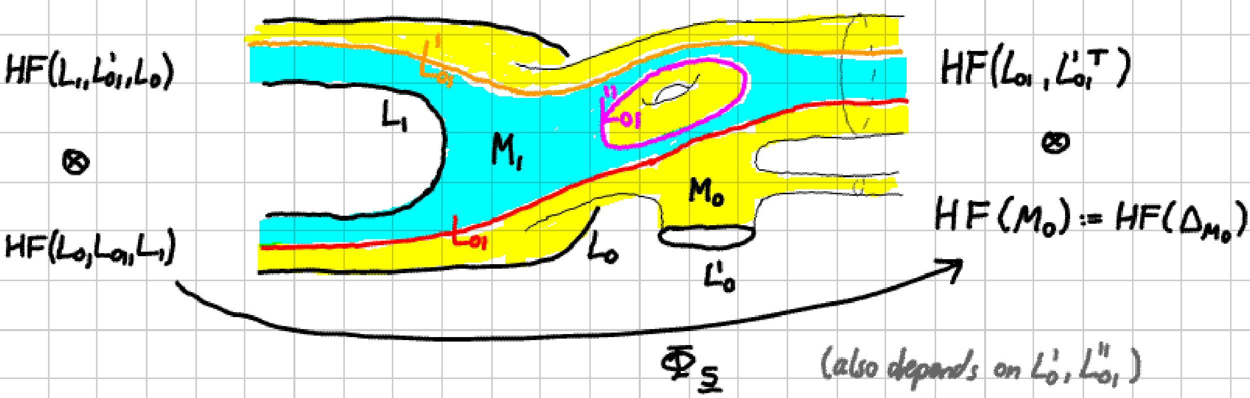
Thm: $\underline{S}$ quilted surface. Suppose $(\underline{M}, \underline{LBS})$ is monotone

(i.e. energy-index relation for all quilt maps $\underline{u}: \underline{S} \to \underline{M}$ satisfying $\underline{LBS}$ conditions)

Then there is a well defined "relative invariant"

$$\Phi_{\underline{S}} : \bigotimes_{\underline{e}^- \in \mathcal{E}^-(\underline{S})} HF(\underline{\mathcal{L}}_{\underline{e}^-}) \longrightarrow \bigotimes_{\underline{e}^+ \in \mathcal{E}^+(\underline{S})} HF(\underline{\mathcal{L}}_{\underline{e}^+})$$

$$\bigotimes_{\underline{e}^-} \underset{\cap_{\underline{H}_{\underline{e}^-}} \underline{\mathcal{L}}_{\underline{e}^-}}{\underline{x}_{\underline{e}^-}} \longmapsto \sum_{\underline{y}_{\underline{e}^+} \in \cap_{\underline{H}_{\underline{e}^+}} \underline{\mathcal{L}}_{\underline{e}^+}} \# \left\{ \begin{array}{l} \text{perturbed } \underline{J}\text{-holomorphic} \\ \text{quilt maps } \underline{u}: \underline{S} \to \underline{M} \\ \text{with limits } \underline{x}_{\underline{e}^-}, \underline{y}_{\underline{e}^+} \end{array} \right\} \bigotimes \underline{y}_{\underline{e}^+}$$



Construction: (i) For each end $\underline{e} \in \mathcal{E}(\underline{S})$

$$\left. \begin{array}{l} \bullet \text{ pick regular } \underline{H}_{\underline{e}} = (H_{k_i, e_i}) \\ \bullet \text{ widths } \underline{\delta}_{\underline{e}} \text{ are given by strip-like ends of } \underline{S} \\ \bullet \text{ pick regular } \underline{J}_{\underline{e}} = (J_{k_i, e_i}) \end{array} \right\} \text{defines } HF(\underline{\mathcal{L}}_{\underline{e}})$$

This determines for each end $\varepsilon_{k,e} : \mathbb{R}^{\pm} \times \left\{ {[0,\delta_{k,e}] \atop S^1} \right\} \hookrightarrow S_k$ a unique

- Hamiltonian $H_{k,e} \in \mathcal{C}^\infty\left(\left\{ {[0,\delta_{k,e}] \atop S^1} \right\} \times M_k\right)$
- ω_{M_k}-compatible almost complex structure $J_{k,e} \in \mathcal{C}^\infty\left(\left\{ {[0,\delta_{k,e}] \atop S^1} \right\}, \mathcal{E}nd(TM_k)\right)$

(ii) Pick "interpolating Hamiltonians" $\underline{K} = \left(K_k \in \Omega^1(S_k, \mathcal{C}^\infty(M_k))\right)_{k=1..n}$

- $\varepsilon_{k,e}^* K_k = H_{k,e}\, dt$ on $\underset{s}{\mathbb{R}^{\pm}} \times \underset{t}{\left\{ {[0,\delta_{k,e}] \atop S^1} \right\}}$
- $K_k|_{\partial S_k} \equiv 0$

and define the vector field valued 1-forms $\underline{Y} = \left(Y_k \in \Omega^1(S_k), \Gamma(TM_k)\right)$

by $\omega_{M_k}(Y_k, \cdot) = d^{M_k} K_k \in \Omega^1(S_k \times M_k)$. (Then $\varepsilon_{k,e}^* Y_k = X_{H_{k,e}} dt$.)

$T(S_k \times M_k) = TS_k \times TM_k$

(iii) Pick <u>regular</u> "interpolating almost complex structures"

$$\underline{J} = \Big(J_k \in \mathcal{C}^\infty(S_k, \underbrace{\mathcal{J}(M_k, \omega_k)}_{\text{space of } \omega_k\text{-compatible almost complex structures}})\Big)$$

- $\varepsilon_{k,e}^* J_k = J_{k,e}$
- The Fredholm section $\bar\partial_{\underline{H},\underline{J}}$ is <u>transverse</u> to 0.

$$\bar\partial : \{\text{quilt maps}\} \longrightarrow \bigoplus_{k=1}^{n} \Omega^{0,1}(S_k, TM_k)$$
$$\underline{u} \longmapsto \left(J_k(u_k)(du_k - Y_k(u_k)) - (du_k - Y_k(u_k))\circ j_k\right)$$

$\boxed{i\partial_s = \partial_t}$

$$\boxed{\text{on ends: } = J_{k,e}(u_k)\begin{pmatrix} \partial_s u_k\, ds \\ + (\partial_t u_k - X_{H_{k,e}}(u_k))\, dt \end{pmatrix} - \begin{pmatrix} -\partial_s u_k\, dt \\ + (\partial_t u_k - X_{H_{k,e}}(u_k))\, ds \end{pmatrix}}$$

Consider the moduli spaces of <u>holomorphic quilts</u>

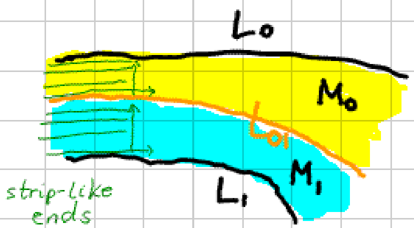
$\underline{u} : \underline{S} \to \underline{M}$ satisfying $\begin{cases} \overline{\partial}_{\underline{H},\underline{J}} \, \underline{u} = 0 \\ \underline{LBS}\text{ - conditions} \end{cases}$

- finite energy $E(\underline{u}) = \sum_{k=1}^{n} \int_{S_k} u_k^* \omega_k - d(H_k \circ u_k) = \frac{1}{2} \sum_{k=1}^{n} \int_{S_k} |du_k - Y_k|^2$

  $\Updownarrow$

  near each end $\underset{(k_i, z_{k_i,e_i})_{i=1..N}}{\underline{e}}$ $\underline{u}$ converges to $\cap_{\underline{H}_{\underline{e}}} \underline{\mathcal{L}}_{\underline{e}}$

[$\underline{u}$ near (z_{k_i,e_i}) is "half a Floer trajectory" $(u_{k_i} : \mathbb{R}^{\pm} \times [0, \delta_{k_i,e_i}] \to M_{k_i})$



so $u_{k_i}(s, \cdot) \xrightarrow[s \to \pm\infty]{} \gamma_{k_i} : [0, \delta_{k_i,e_i}] \to M_{k_i}$

$(\gamma_{k_i})_{i=1..N} \in \cap_{\underline{H}_{\underline{e}}} \underline{\mathcal{L}}_{\underline{e}}$]

- Gromov compactness & monotonicity:

  0-dim. moduli spaces are compact $\longrightarrow$ count defines $\Phi_{\underline{S}}$ on chains

  1-dim. moduli spaces are compact up to "energy escaping off an end"

- gluing $\Rightarrow \left(\bigoplus_{\underline{e}^+ \in \mathcal{E}^-} \partial_{\underline{\mathcal{L}}_{\underline{e}^+}} \right) \circ \Phi_{\underline{S}} + \Phi_{\underline{S}} \circ \left(\bigoplus_{\underline{e}^- \in \mathcal{E}^-} \partial_{\underline{\mathcal{L}}_{\underline{e}^-}} \right) = 0$

  $\Rightarrow \Phi_{\underline{S}}$ descends to homology

$\blacksquare$

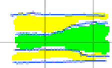
<u>Thm</u>: $\Phi_{\underline{S}}$ is independent of perturbations and only depends on $\underline{S}$ "up to homotopy", i.e. $\Phi_{\underline{S}}$ is determined by

- the surfaces $S_k = \overline{S}_k \setminus \{z_{k,e}\}$ up to diffeomorphism
- incoming (−) / outgoing (+) labels on ends
- combinatorial seams $\mathcal{S}$ and orientation of seam diffeomorphisms φ_σ.

<u>Proof</u>: • continuation maps intertwine between quilt invariants $\Phi_{\underline{S}}, \Phi_{\underline{S}'}$ with different end data $(\underline{\delta}_{\underline{e}}, \underline{H}_{\underline{e}}, \underline{J}_{\underline{e}})_{\underline{e}\in\mathcal{E}(\underline{S})}$, $(\underline{\delta}'_{\underline{e}}, \underline{H}'_{\underline{e}}, \underline{J}'_{\underline{e}})_{\underline{e}\in\mathcal{E}(\underline{S}')}$

$$\begin{tikzcd}
\bigotimes_{\underline{e}^-} HF(\ldots\underline{\delta}_{\underline{e}^-}\ldots) \arrow[r, "\Phi_{\underline{S}}"] \arrow[d, "\bigotimes_{\underline{e}^-}\Phi_{\underline{\delta}_{\underline{e}^-}\rightsquigarrow\underline{\delta}'_{\underline{e}^-}}", "\wr"'] & \bigotimes_{\underline{e}^+} HF(\ldots\underline{\delta}_{\underline{e}^+}\ldots) \arrow[d, "\bigotimes_{\underline{e}^+}\Phi_{\underline{\delta}_{\underline{e}^+}\rightsquigarrow\underline{\delta}'_{\underline{e}^+}}", "\wr"'] \\
\bigotimes_{\underline{e}^-} HF(\ldots\underline{\delta}'_{\underline{e}^-}\ldots) \arrow[r, "\Phi_{\underline{S}'}"] & \bigotimes_{\underline{e}^+} HF(\ldots\underline{\delta}'_{\underline{e}^+}\ldots)
\end{tikzcd}$$






- composition is gluing: $\Phi_{\underline{S}} \circ \Phi^{+}_{\delta\rightsquigarrow\delta'} \underset{(Q_\tau)}{\sim} \Phi^{-}_{\delta\rightsquigarrow\delta'} \circ \Phi_{\underline{S}'}$

- homotopies $(Q_\tau)_{\tau\in[0,1]} = (\underline{S}_\tau, \underline{H}_\tau, \underline{J}_\tau)_{\tau\in[0,1]}$ between quilted surfaces $\underline{S}_0, \underline{S}_1$ and perturbation data $(\underline{H}_0, \underline{J}_0)$, $(\underline{H}_1, \underline{J}_1)$ with "fixed ends" (widths $(\delta_{k,e})$ and Floer data $(\underline{H}_{\underline{e}}), (\underline{J}_{\underline{e}})$ fixed) provide chain homotopy equivalences $\Phi_{\underline{S}_0, \underline{H}_0, \underline{J}_0} \sim \Phi_{\underline{S}_1, \underline{H}_1, \underline{J}_1}$ ∎

UPSHOT: We can define maps $\Phi_{\underline{S}} : \bigotimes_{\underline{e}^-} HF(\underline{\mathcal{L}}_{\underline{e}^-}) \longrightarrow \bigotimes_{\underline{e}^+} HF(\underline{\mathcal{L}}_{\underline{e}^+})$

by drawing a quilted surface as one surface (with boundary & ends)

with seams indicated by embedded non-intersecting 1-manifolds ($\cong \mathbb{R}$ or S^1)

and labeling the patches / seams / boundary components by

symplectic manifolds / Lagrangian correspondances / Lagrangian submanifolds.

We have calculation rules

- composition is gluing $\Phi_{\underline{S}_0} \circ \Phi_{\underline{S}_1} = \Phi_{\underline{S}_0 \# \underline{S}_1}$

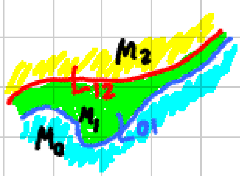
  in $HF(\underline{\mathcal{L}}_{\underline{e}_0^+} = \underline{\mathcal{L}}_{\underline{e}_1^-})$ at $\underline{e}_0^+ \sim \underline{e}_1^-$

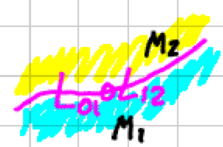
- for $\underline{S}$ = quilt of strips (with $\mathbb{R}$-symmetry after deformation)

  $\Phi_{\underline{S}} = Id$

- $\Phi_{\underline{S}}$ is invariant under

  - deformation of quilt

  - replacing a strip 

by a seam 

  with transverse & embedded composition

L14 – The symplectic 2-category

Note Title 4/2/2008

holomorphic quilt invariants – summary

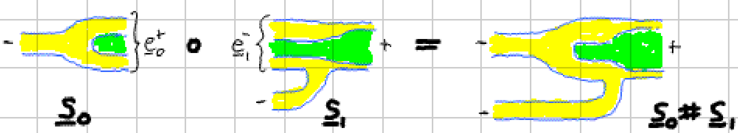
$\Phi_{\underline{S}} : \bigotimes_{\underline{e}^- \in \underline{\mathcal{E}}^-} HF(\underline{\mathcal{L}}_{\underline{e}^-}) \longrightarrow \bigotimes_{\underline{e}^+ \in \underline{\mathcal{E}}^+} HF(\underline{\mathcal{L}}_{\underline{e}^+})$ is defined by

- a quilted surface – given as one surface (with boundary & +/- ends)
  – seams indicated by embedded non-intersecting 1-manifolds
- labeling patches, seams, boundary components by symplectic manifolds, Lagrangian correspondences, Lagrangian submanifolds

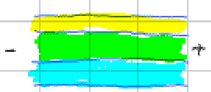
calculation rules

(i) composition is gluing $\qquad \Phi_{\underline{S}_0} \circ \Phi_{\underline{S}_1} = \Phi_{\underline{S}_0 \# \underline{S}_1}$

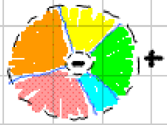
in $HF(\underline{\mathcal{L}}_{\underline{e}_0^+} = \underline{\mathcal{L}}_{\underline{e}_1^-})$ $\qquad$ at $\underline{e}_0^+ \sim \underline{e}_1^-$



(ii) for $\underline{S}$ = quilt of strips (with $\mathbb{R}$-symmetry) $\qquad \Phi_{\underline{S}} = Id$

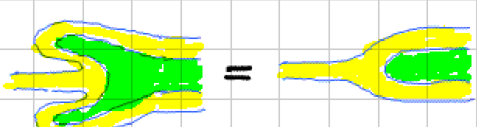


= Id



= Id

(iii) $\Phi_{\underline{S}}$ is invariant under deformation of quilt



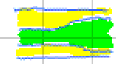
(deformation $Q_\tau = (\underline{S}_\tau, \underline{H}_\tau, \underline{J}_\tau)_{\tau \in [0,1]}$ with fixed ends induces chain homotopy equivalence $T_{\{Q_\tau\}}$, i.e. $\Phi_{\underline{S}_1} - \Phi_{\underline{S}_0} = (\oplus \partial_{\underline{e}^+}) \circ T + T \circ (\oplus \partial_{\underline{e}^-})$)

(iv) continuation maps intertwine between quilt invariants $\Phi_{\underline{S}}, \Phi_{\underline{S}'}$

for homotopic $\underline{S}, \underline{S}'$ with different end data $(\underline{\delta}_{\underline{e}}, \underline{H}_{\underline{e}}, \underline{J}_{\underline{e}})_{\underline{e}\in\mathcal{E}(\underline{S})}, (\underline{\delta}'_{\underline{e}'},\dots)_{\underline{e}'\in\mathcal{E}(\underline{S}')}$.

Example:

$$\begin{tikzcd}
\bigotimes_{\underline{e}^-} HF(\underline{\mathcal{L}}_{\underline{e}^-},..\underline{\delta}_{\underline{e}^-}...) \arrow[r, "\Phi_{\underline{S}}"] \arrow[d, "\wr", "\bigotimes_{\underline{e}^-}\Phi_{\underline{\delta}_{\underline{e}^-}\rightsquigarrow\underline{\delta}'_{\underline{e}^-}}"'] \arrow[dr, bend left, "a"] \arrow[dr, bend right, "b"'] & \bigotimes_{\underline{e}^+} HF(\underline{\mathcal{L}}_{\underline{e}^+},..\underline{\delta}_{\underline{e}^+}..) \arrow[d, "\wr"', "\bigotimes_{\underline{e}^+}\Phi_{\underline{\delta}_{\underline{e}^+}\rightsquigarrow\underline{\delta}'_{\underline{e}^+}}"] \\
\bigotimes_{\underline{e}^-} HF(\underline{\mathcal{L}}_{\underline{e}^-},..\underline{\delta}'_{\underline{e}^-}...) \arrow[r, "\Phi_{\underline{S}'}"] & \bigotimes_{\underline{e}^+} HF(\underline{\mathcal{L}}_{\underline{e}^+},..\underline{\delta}'_{\underline{e}^+}..)
\end{tikzcd}$$

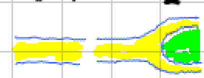


Proof:

$$a = \Phi_{\underline{S}} \circ \Phi^+_{\underline{\delta}\rightsquigarrow\underline{\delta}'} \overset{(i)}{=} \Phi_{Q_0} \overset{(iii)}{\underset{\text{homotopy } (Q_\tau)}{=}} \Phi_{Q_1} \overset{(i)}{=} \Phi^-_{\underline{\delta}\to\underline{\delta}'} \circ \Phi_{\underline{S}'} = b$$

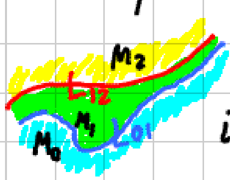
$\xrightarrow{\text{glue}}$

$\xleftarrow{\text{glue}}$



(v) isomorphisms $HF(\underline{\mathcal{L}}_{\underline{e}}) \cong HF(\underline{\mathcal{L}}'_{\underline{e}'})$ between equivalent generalized Lagrangian correspondences (related by a simple move $\underline{\mathcal{L}}_{\underline{e}} = (..\ L_{01}, L_{12}\ ..)$, $\underline{\mathcal{L}}'_{\underline{e}'} = (..\ L_{01}\circ L_{12}\ ..)$) intertwine between quilt invariants $\Phi_{\underline{S}}, \Phi_{\underline{S}'}$

where a strip



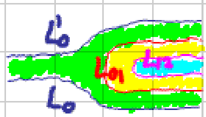
in $\underline{S}$ is replaced by a seam

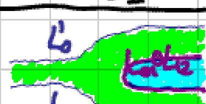
in $\underline{S}'$.

Example:

$$\begin{tikzcd}
HF(L_0, L_0') \arrow[r, "\Phi_{\underline{S}}"] \arrow[d, "\wr"] & HF(L_0, L_{01}, L_{12}, L_{12}^T, L_{01}^T, L_0') \arrow[d, "\wr", "(2\text{ simple moves})"'] \\
HF(L_0, L_0') \arrow[r, "\Phi_{\underline{S}'}"] & HF(L_0, L_{01}\circ L_{12}, (L_{01}\circ L_{12})^T, L_0')
\end{tikzcd}$$

$(L_{01}\circ L_{12})^T = L_{12}^T \circ L_{01}^T$





The (monotone) symplectic 2-category

objects: (M,ω) closed symplectic manifold, monotone: $[\omega]=c_1$ ($\tau=1$ by scaling of ω)

morphisms: $Mor(M_0,M_1)=$ generalized Lagrangian correspondences $M_0 \xrightarrow{\underline{L}} M_1$
modulo equivalence generated by good moves
(i.e. transverse, embedded geometric composition); monotone

composition: $Mor(M_0,M_1)\times Mor(M_1,M_2)\longrightarrow Mor(M_0,M_2)$

concatenation $\quad \left[M_0\xrightarrow{\underline{L}_{01}}M_1\right],\left[M_1\xrightarrow{\underline{L}_{12}}M_2\right]\longmapsto\left[M_0\xrightarrow{\underline{L}_{01}\#\underline{L}_{12}} M_1 \longrightarrow M_2\right]$

⊛ associativity

identity: $1_M:=[\Delta_M]\in Mor(M,M)$

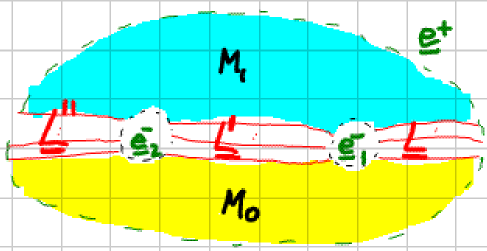
2-morphisms: ${}^2Mor\left(M_0\xrightarrow{\underline{L}}M_1,\ M_0\xrightarrow{\underline{L}'}M_1\right)=HF\left(\underline{L}\#(\underline{L}')^T\right)$

$$\begin{tikzcd} M_0 \arrow[r, bend left, "\underline{L}"] & M_1 \arrow[l, bend left, "(\underline{L}')^T"] \end{tikzcd}$$

2-composition: $\Phi_{\underline{S}}:\ {}^2Mor(\underbrace{\underline{L},\underline{L}'}_{\cong e_1^-})\otimes{}^2Mor(\underbrace{\underline{L}',\underline{L}''}_{\cong e_2^-})\longrightarrow{}^2Mor(\underbrace{\underline{L},\underline{L}''}_{\cong e^+})$

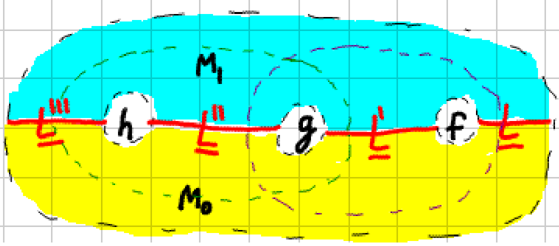
$(f,g)\longmapsto f\circ g$

$\underline{S}$:



⊛ associativity:

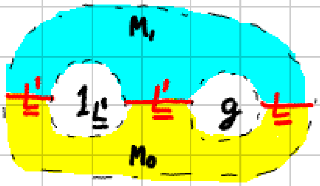
$(f\circ g)\circ h \underset{\text{gluing}}{=}$



$\underset{\text{gluing}}{=} f\circ(g\circ h)$

2-identity: $1_{\underline{L}} = \Phi_{\underline{S}} \in {}^2Mor\left(M_0 \xrightarrow{\underline{L}} M_1 ,\ M_0 \xrightarrow{\underline{L}} M_1\right)$

⊛ $\forall g \in HF(\underline{L},\underline{L}')$ $\quad g \circ 1_{\underline{L}'} = g$ and $1_{\underline{L}} \circ g = g$



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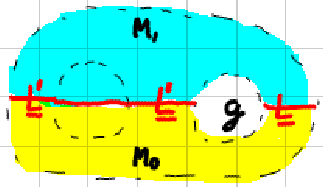

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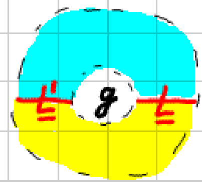
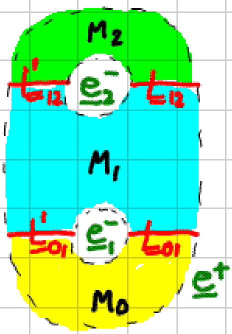


strips with $\mathbb{R}$-action

$= Id(g)$

composition functor: $\quad Mor(M_0,M_1) \times Mor(M_1,M_2) \longrightarrow Mor(M_0,M_2)$

$\underline{S}$:



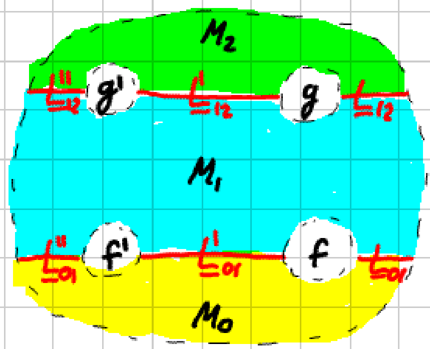
$\underline{L}_{01} \qquad \underline{L}_{12} \longmapsto \underline{L}_{01}\#\underline{L}_{12}$

$\underline{L}'_{01} \qquad \underline{L}'_{12} \longmapsto \underline{L}'_{01}\#\underline{L}'_{12}$

$$\Phi_{\underline{S}}: \underbrace{HF(\underline{L}_{01},\underline{L}'_{01})}_{\mathcal{I}_{e_1^-}} \otimes \underbrace{HF(\underline{L}_{12},\underline{L}'_{12})}_{\mathcal{I}_{e_2^-}} \to \underbrace{HF(\underline{L}_{01}\#\underline{L}_{12},\underline{L}'_{01}\#\underline{L}'_{12})}_{\mathcal{I}_{e^+}}$$

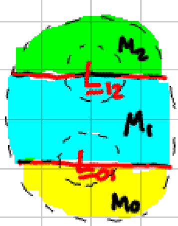
$$(f,g) \longmapsto f\#g$$

⊛ $(f\circ f')\#(g\circ g') =$



$= (f\#g)\circ(f'\#g')$

⊛ $1_{\underline{L}_{01}} \# 1_{\underline{L}_{12}} =$



=

$= 1_{\underline{L}_{01}\#\underline{L}_{12}}$

Corollary (Categorification): There exists a functor

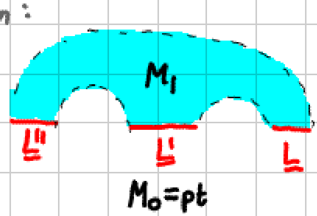
$$\mathrm{Symp}^{\#}_{\text{monotone}} \longrightarrow \mathcal{C}at$$

[Objects: categories
Morphisms: functors
– composition & identity functor]

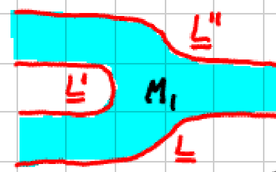
(i) M symplectic $\longmapsto$ $\mathrm{Mor}(pt, M) := \mathrm{Don}^{\#}(M)$ Donaldson-Fukaya category

Objects: generalized Lagrangians $pt \to \dots \to M$
Morphisms: HF-classes

composition:

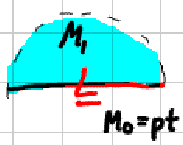


$\widehat{=}$ $HF(\underline{L}', \underline{L}'') \otimes HF(\underline{L}, \underline{L}')$



$\xrightarrow{\Phi_S}$ $HF(\underline{L}, \underline{L}'')$

identity:



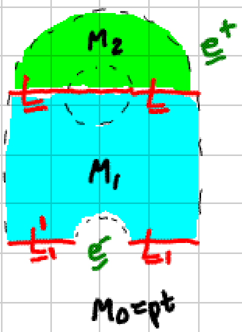
$\widehat{=}$

$\in HF(\underline{L}, \underline{L})$

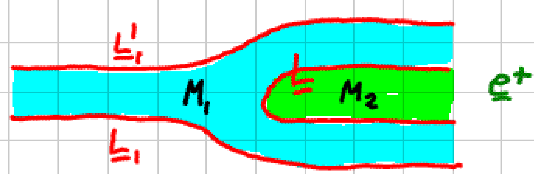
(ii) $M_1 \xrightarrow{\underline{L}} M_2$ generalized Lagrangian correspondence $\longmapsto$ $\Phi_{\underline{L}} : \mathrm{Don}^{\#}(M_1) \to \mathrm{Don}^{\#}(M_2)$

$\underline{L}_1 \longmapsto \underline{L}_1 \# \underline{L}$ $\quad pt \to \dots \to M_1 \to \dots M_2$

$HF(\underline{L}_1, \underline{L}_1') \ni f \longmapsto f \# 1_{\underline{L}} \in HF(\underline{L}_1 \# \underline{L}, \underline{L}_1' \# \underline{L})$



$=$ e^-



e^+

Proof: Given any 2-category $\mathcal{C}$ and distinguished object p_0

there is a functor of 1-categories $\mathcal{C} \to \mathcal{C}at$ given by

object $p \quad \longmapsto \quad \mathcal{M}or(p_0, p)$ is a category

morphism $p \xrightarrow{h} q \quad \longmapsto$

$$\begin{tikzcd} \mathcal{M}or(p_0,p) \arrow[r] \arrow[dr, "Id \times (h, 1_h)"'] & \mathcal{M}or(p_0,q) \\ & \mathcal{M}or(p_0,p) \times \mathcal{M}or(p,q) \arrow[u, "\text{composition functor of } \mathcal{C}"'] \end{tikzcd}$$

is a functor

We picked $p_0 = pt$ as distinguished object in $\mathcal{S}ymp^{\#}$.

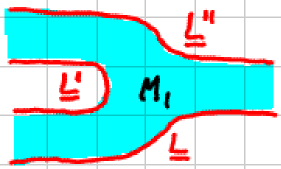
L15 - Floer Field Theory

Note Title 4/7/2008

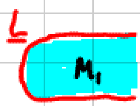
We defined a functor $Symp^{\#}_{monotone} \longrightarrow Cat$ by

(i) For each object, M symplectic, monotone, define the extended Donaldson-Fukaya category $Don^{\#}(M)$:

- Objects : generalized Lagrangians $pt \xrightarrow{\underline{L}} \cdots \to M$
- Morphisms : HF classes $Mor(\underline{L}, \underline{L}') = HF(\underline{L}, \underline{L}'^T)$ $\quad pt \xrightarrow{\underline{L}} M \xleftarrow{\underline{L}'} pt$, $\; M \xrightarrow{\underline{L}'^T} pt$
- composition :

$HF(\underline{L}', \underline{L}'') \to$
$HF(\underline{L}, \underline{L}') \to$  $\to HF(\underline{L}, \underline{L}'')$

- identity :

 $\to 1_{\underline{L}} \in HF(\underline{L}, \underline{L})$

(ii) For each generalized Lagrangian correspondence $M_1 \xrightarrow{\underline{L}} M_2$ define a functor $\Phi_{\underline{L}} : Don^{\#}(M_1) \to Don^{\#}(M_2)$:

- on objects $\underset{pt \to M_1}{\underline{L}_1} \longmapsto \underset{pt \to M_1 \to M_2}{\underline{L}_1 \# \underline{L}}$
- on morphisms $HF(\underline{L}_1, \underline{L}_1') \longrightarrow HF(\Phi_{\underline{L}}(\underline{L}_1), \Phi_{\underline{L}}(\underline{L}_1')) = HF(\underline{L}_1, \underline{L}, \underline{L}^T, \underline{L}_1'^T)$

is given by the quilt invariant $-$  $+$

<u>Remark</u>: To define the functor $Symp^\# \to Cat$ it suffices to fix

(i) M symplectic, monotone $\longmapsto$ $Don^\#(M)$ category

(ii') $L_{12} \subset M_1^- \times M_2$ Lagrangian correspondence "simple" $\longmapsto$ $\Phi_{L_{12}} : Don^\#(M_1) \to Don^\#(M_2)$ functor

and check

(a) Any morphism of $Symp^\#_{monotone}$ can be decomposed into simple morphisms

$$[\underline{L}] = [L_{01}] \circ [L_{12}] \circ \ldots \circ [L_{(k-1)k}] \qquad ; \; L_{(j-1)j} \subset N_{j-1}^- \times N_j \text{ "simple".}$$

(b) Any other decomposition $[\underline{L}] = [L'_{01}] \circ \ldots \circ [L'_{(k'-1)k'}]$

is obtained by a sequence of good moves.

Functoriality then determines for all $[\underline{L}]$

$$\Phi_{[\underline{L}]} = \Phi_{[L_{01}] \circ \ldots \circ [L_{(k-1)k}]} = \Phi_{L_{01}} \circ \ldots \circ \Phi_{L_{(k-1)k}}$$

which is independent of the decomposition since

<u>Thm</u>: good move in $(L_{(j-1)j})_{j=1..k}$ $\widehat{=}$ composition in $(\Phi_{L_{(j-1)j}})_{j=1..k}$

$L_{01} \circ L_{12}$ transverse, embedded $\Longrightarrow \Phi_{L_{01}} \circ \Phi_{L_{12}} = \Phi_{L_{01} \circ L_{12}}$

<u>Proof</u>:

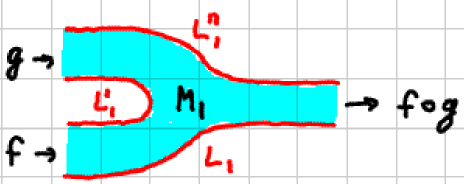
This reproduces the previous definition (ii)

$$\Phi_{\underline{L}} = \Phi_{L_{01}} \circ \Phi_{L_{12}} \circ \dots \circ \Phi_{L_{(k-1)k}}$$

Homework: Prove functoriality of $\Phi_{L_{12}} : \mathcal{D}on^{\#}(M_1) \to \mathcal{D}on^{\#}(M_2)$

$$\Phi_{L_{12}}(f \circ g) = \Phi_{L_{12}}(f) \circ \Phi_{L_{12}}(g) \qquad \forall f \in HF(L_1, L_1'),\ g \in HF(L_1', L_1'')$$

by pictures.



$\Phi_{L_{12}}$:

<u>Proposition</u>: $\mathcal{C}$ category with a subcollection of "simple morphisms"

To define a functor $\mathcal{C} \to \mathrm{Symp}^{\#}$ it suffices to fix

(i) X object $\longmapsto$ M_X symplectic

(ii) $X_1 \xrightarrow{Y} X_2$ simple morphism $\longmapsto$ $L_Y \subset M_{X_1}^{-} \times M_{X_2}$ Lagrangian corresp.,

and check

(a) Any morphism $\tilde{Y}$ of $\mathcal{C}$ can be decomposed into simple morphisms

$\tilde{Y} = Y_{01} \circ \ldots \circ Y_{(k-1)k}$, $Y_{(j-1)j}$ simple

(b) Any other decomposition $\tilde{Y} = Y'_{01} \circ \ldots \circ Y'_{(k'-1)k'}$, $Y'_{(j-1)j}$ simple

is obtained by a sequence of moves

$$\left\{ \begin{array}{l} \bullet\ Y_\alpha \circ Y_\beta = Y_\gamma \\ \bullet\ Y_\gamma = Y_\alpha \circ Y_\beta \\ \bullet\ Y_\alpha \circ Y_\beta = Y_\gamma \circ Y_\delta \end{array} \right\} \text{ that correspond to transverse embedded geometric composition } \left\{ \begin{array}{l} \bullet\ L_{Y_\alpha} \circ L_{Y_\beta} = L_{Y_\gamma} \\ \bullet\ L_{Y_\gamma} = L_{Y_\alpha} \circ L_{Y_\beta} \\ \bullet\ L_{Y_\alpha} \circ L_{Y_\beta} = L_{Y_\gamma} \circ L_{Y_\delta} \end{array} \right\}$$

<u>Corollary</u>: If all M_X and L_Y in (i), (ii) are monotone

then we obtain a functor $\mathcal{C} \longrightarrow \mathrm{Symp}^{\#}_{\text{monotone}}$

and hence a categorification functor $\mathcal{C} \longrightarrow \mathcal{C}at$

factoring through $\mathrm{Symp}^{\#}_{\text{monotone}}$.

Example: Floer field theory in 2+1 dimension (almost 2+1 TQFT)

from moduli spaces of "central curvature & fixed determinant" bundles

- $\mathcal{C} = \mathcal{C}ob_{2+1}$ objects : Σ Riemann surface (closed, oriented 2-mfds) !connected!

  morphisms : Y 3dim cobordism $\partial Y = \Sigma_0^- \sqcup \Sigma_1$

(i) [Narasimhan-Seshadri] smooth, monotone symplectic manifolds :

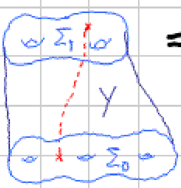
$$M_\Sigma := \left\{ \begin{array}{l} A \;\; U(r)\text{-connection on } \Sigma \\ (F_A)_{su(r)} = 0 \;\; , \; \det(A) = \delta \end{array} \right\} \Big/ \text{gauge} \;\cong\; \left\{ \begin{array}{l} \rho : \pi_1(\Sigma \setminus pt) \to SU(r) \\ \rho(\circlearrowleft_{pt}) = -\mathbb{1} \end{array} \right\} \Big/ SU(r)$$

(degree d part)

$\boxed{\begin{array}{l} \text{fix } r \in \mathbb{N},\; u(r) = su(r) \oplus u(1) \\ \delta \;\; U(1) \text{ bundle on } \Sigma, \text{ degree } d \text{ coprime to } r \end{array}}$

$$\cong \left\{ \begin{array}{l} \alpha_1 \dots \alpha_g, \beta_1 \dots \beta_g \in SU(r)^{2g} \\ \prod_{j=1}^{g} \alpha_j \beta_j \alpha_j^{-1} \beta_j^{-1} = -\mathbb{1} \end{array} \right\} \Big/ SU(r)$$

(ii) $L_Y := \left\{ (\tilde{A}|_{\Sigma_0}, \tilde{A}|_{\Sigma_1}) \;\middle|\; \tilde{A} \;\; U(r)\text{-connection on } \Sigma,\; (F_{\tilde{A}})_{su(r)} = 0,\; \det(\tilde{A}) = \tilde{\delta} \right\} \Big/ \text{gauge}$



$= \left\{ (\rho_0, \rho_1) \in M_{\Sigma_0} \times M_{\Sigma_1} \;\middle|\; \exists \text{ extension } \tilde{\rho} : \pi_1(Y \setminus \text{line}) \to SU(r) \right\}$

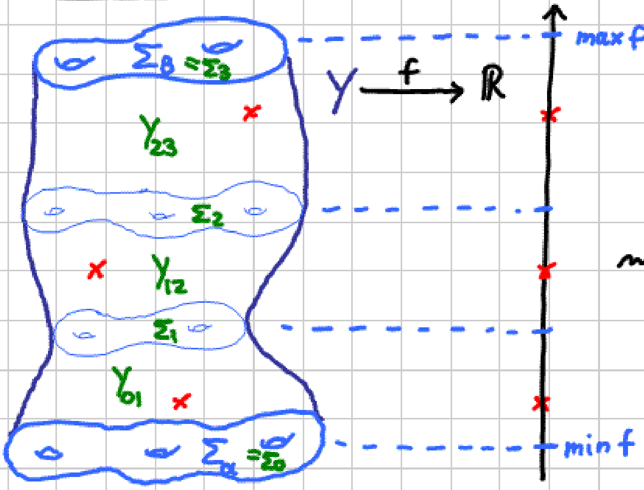
is <u>in general not smooth</u>

(ii') $L_Y \subset M_{\Sigma_0}^- \times M_{\Sigma_1}$ smooth, monotone Lagrangian correspondence for

- simple morphisms Y = cylinder $\Sigma \times [0,1]$ or handle attachment

  (i.e. $\exists f : Y \to \mathbb{R}$ Morse, 0 or 1 crit. pt., maximal on Σ_1, minimal on Σ_0)

ⓐ decomposition of general morphism $\partial Y = \bar{\Sigma}_\alpha \sqcup \Sigma_\beta$



$$\rightsquigarrow Y = \begin{cases} Y_{23} \\ \cup_{\Sigma_2} \\ Y_{12} \\ \cup_{\Sigma_1} \\ Y_{01} \end{cases} \quad \overset{(ii)}{\rightsquigarrow} \quad \underline{\underline{L}}_Y = \begin{array}{c} M_{\Sigma_3} \\ \uparrow L_{Y_{23}} \\ M_{\Sigma_2} \\ \uparrow L_{Y_{12}} \\ M_{\Sigma_1} \\ \uparrow L_{Y_{01}} \\ M_{\Sigma_0} \end{array}$$

cut between levels of critical points

ⓑ moves between decompositions : Cerf moves for Morse functions

- cancellation of critical points $\rightsquigarrow$

$$Y_{g\to g+1} \cup Y_{g+1\to g} \cong [0,1]\times\Sigma_g$$

$$\boxed{\begin{aligned} L_{Y_{g\to g+1}} &= \left\{ \begin{pmatrix}\alpha_i\\ \beta_i\end{pmatrix}_{i=1..g} \begin{pmatrix}\alpha_i'\\ \beta_i'\end{pmatrix}_{i=1..g+1} \;\middle|\; \alpha_i'=\alpha_i,\ \beta_i'=\beta_i,\ \beta_{g+1}'=1\!\!1 \right\} \\ L_{Y_{g+1\to g}} &= \left\{ \begin{pmatrix}\alpha_i'\\ \beta_i'\end{pmatrix}_{i=1..g+1} \begin{pmatrix}\alpha_i\\ \beta_i\end{pmatrix}_{i=1..g} \;\middle|\; \alpha_i'=\alpha_i,\ \beta_i'=\beta_i,\ \alpha_{g+1}'=1\!\!1 \right\} \\ L_{Y_{g\to g+1}} \circ L_{Y_{g+1\to g}} &= \Delta_{M_{\Sigma_g}} = L_{[0,1]\times\Sigma_g} \end{aligned}}$$

- change of order
- handle slide

$$Y_\alpha \cup_\Sigma Y_\beta = Y \cong Y_\gamma \cup_\Sigma Y_\delta \rightsquigarrow \boxed{L_{Y_\alpha}\circ L_{Y_\beta} = L_Y = L_{Y_\gamma}\circ L_{Y_\delta}}$$

- cancelation of trivial (no crit. pt.) cobordism $\quad Y_\alpha \cup [0,1]\times\Sigma \cong Y_\alpha \rightsquigarrow \boxed{L_{Y_\alpha}\circ\Delta_{M_\Sigma} = L_{Y_\alpha}}$

<u>Corollary</u>: Fix Riem. surfaces $\Sigma_\alpha, \Sigma_\beta$

and (gen.) Lagrangians $L_\alpha \subset M_{\Sigma_\alpha}, L_\beta \subset M_{\Sigma_\beta}$

then we have a topological invariant

Y cobordism from Σ_α to Σ_β

$\downarrow$

$\underline{L}_Y$ gen. Lagr. corresp. $M_{\Sigma_\alpha} \to M_{\Sigma_\beta}$

$\downarrow$

$HF(\underbrace{L_\alpha \# \underline{L}_Y \# L_\beta}_{pt \to M_{\Sigma_\alpha} \to M_{\Sigma_\beta} \to pt})$ quilted Floer homology

<u>Ex.</u>: $\Sigma_\alpha = \Sigma_\beta = T^2$

$pt \subset M_{T^2} = pt$

Y 3-manifold

$\downarrow$

$Y \# [0,1] \times T^2 =: \tilde{Y}$ cobordism

$\downarrow$

$HF(\underline{L}_{\tilde{Y}})$

<u>Conj</u>: $HF(\underline{L}_{Y \# [0,1] \times T^2})$ is closely related to

[Kronheimer - Mrowka] invariants of Y from singular instantons
Collin-Steer

L16 – Compactness – local estimates

Note Title 4/14/2008

Fix $\underline{S}$ quilted surface $\rightsquigarrow (S_k, j_k)$ surfaces (with strip-like ends)

$$\varphi_\sigma : \underset{\partial S_{k_\sigma}}{I_\sigma} \xrightarrow{\sim} \underset{\partial S_{k'_\sigma}}{I'_\sigma} \quad \text{seams}$$

$\underline{M}$ symplectic targets

$$\underline{LBS} = \begin{cases} (L_\sigma)_{\sigma \in \mathcal{S}} & \text{Lagrangian correspondences for seams } I_\sigma \simeq I'_\sigma \\ \cup & \\ (L_b)_{b \in \mathcal{B}} & \text{Lagrangian submanifolds for boundaries } I_b \subset \partial S_{k_b} \end{cases}$$

$\underline{K}$ Hamiltonian perturbation $\rightsquigarrow Y_k \in \Omega^1(S_k, \Gamma(TM_k))$

$\underline{J}$ almost complex structure $\rightsquigarrow J_k \in \mathcal{C}^\infty(S_k, \mathcal{J}(M_k, \omega_k))$

Consider holomorphic quilts $\left(\underline{u}^\nu = (u_k^\nu) : \underline{S} \to \underline{M}\right)_{\nu \in \mathbb{N}}$ of fixed energy

$\left(\text{e.g. } \underline{u}^\nu \in \mathcal{M}^\ell(\underline{S}, \ldots, (\underline{x}_e)_{e \in \mathcal{E}^-}, (\underline{y}_e)_{e \in \mathcal{E}^+}) \text{ – index } \ell \text{, end data } (\underline{x}_e), (\underline{y}_e) \text{ fixed}\right)$

i.e. $\forall \nu \in \mathbb{N} \quad \overline{\partial}_{\underline{J}, \underline{K}} \underline{u}^\nu = 0 \qquad \mathcal{E}(\underline{u}^\nu) = \frac{1}{2} \| d\underline{u}^\nu - \underline{Y}(\underline{u}) \|_{L^2}^2 = E = E(\ell, (\underline{x}_e), (\underline{y}_e))$

i.e. $\forall \nu \in \mathbb{N} \; \forall k$

$$\begin{cases} J_k(u_k^\nu)\left(du_k^\nu - Y_k(u_k^\nu)\right) = \left(du_k^\nu - Y_k(u_k^\nu)\right) \circ j_k \\ u_{k_b}(I_b) \subset L_b \;\; \forall b \in \mathcal{B}, \quad (u_{k_\sigma} \times (u_{k'_\sigma} \circ \varphi_\sigma))(I_\sigma) \subset L_\sigma \\ \sum_k \frac{1}{2} \int_{S_k} |du_k^\nu - Y_k(u_k^\nu)|^2 = E \end{cases}$$

$\boxed{\text{equations in local coordinates on } \underline{S}}$

<u>interior of S_k</u> : $\quad B_\delta = \{(s,t)\in\mathbb{R}^2 \mid s^2+t^2<\delta^2\} \qquad j_k\partial_s = \partial_t$

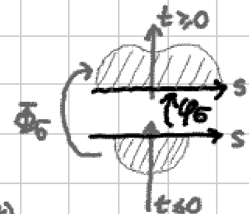
$$\partial_s u^r_k + J_k(u^r_k)\partial_t u^r_k = \underbrace{Y_k(u^r_k)(\partial_s) + J_k(u^r_k)Y_k(u^r_k)(\partial_t)}_{=:\,P_k(u^r_k)}$$

<u>near boundary</u> $I_b \subset S_{k=k_b}$: $\quad B_\delta\cap\mathbb{H} = \{s^2+t^2<\delta^2,\ t\geq 0\}$

$$\begin{cases} \partial_s u^r_k + J_k(u^r_k)\partial_t u^r_k = P_k(u^r_k) & \forall s,t \\ u^r_k(s,0)\in L_b \quad \forall s \end{cases}$$

<u>near seam</u> $I_\sigma \subset S_{k=k_\sigma}$, $I'_\sigma \subset S_{k'=k'_\sigma}$:

(pick (s,t)-coordinates $t\leq 0$ on S_k, $t\geq 0$ on $S_{k'}$)



$\varphi_\sigma : [-\delta,\delta]\to\mathbb{R}$
$\varphi_\sigma(0) = 0$
$\varphi_\sigma' > 0$ due to compatibility with ends

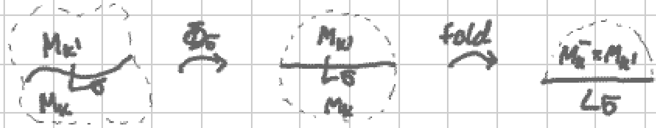
$$w^r(s,t) := \Big(u^r_k(s,-t),\ u^r_{k'}(\overbrace{\varphi_\sigma(s),\ \varphi'_\sigma(s)\cdot t}^{\Phi_\sigma(s,t)})\Big) : B_\delta\cap\mathbb{H} \longrightarrow M_k\times M_{k'}$$

satisfies
$$\begin{cases} w^r(s,0)\in L_\sigma \quad \forall s \\ \underset{\begin{pmatrix}\partial_s u^r_k \\ \varphi'_\sigma\cdot\partial_s u^r_{k'}\end{pmatrix}}{\underset{\parallel}{\partial_s w^r}} + \tilde{J}(w^r)\ \underset{\begin{pmatrix}-\partial_t u^r_k \\ \varphi'_\sigma\cdot\partial_t u^r_{k'}\end{pmatrix}}{\underset{\parallel}{\partial_t w^r}} = \tilde{P}(w^r) \qquad \forall s,\ t\geq 0 \end{cases}$$

$$\tilde{J}(s,t,\underbrace{v,v'}_{M_k\times M_{k'}}) = -J_k(s,t,v) \oplus J_{k'}(\varphi_\sigma(s),\varphi'_\sigma(s)\cdot t, v') \qquad \in \mathrm{End}(T_vM_k\times T_{v'}M_{k'})$$

$$\tilde{P}(s,t,\overbrace{v,v'}) = P_k(s,-t,v) + \varphi'_\sigma\cdot P_{k'}(\varphi_\sigma(s),\varphi'_\sigma(s)\cdot t, v') \qquad \in T_v(M_k\times M_{k'})$$

<u>Rmk</u>: this uniformizes and folds the seam



energy: $E \geq \frac{1}{2}\int_{B_\delta} |d\tilde{u}_k - Y_k(\tilde{u}_k)|^2$; $\sup_{z\in S_k, u\in M_k} |Y_k(z,u)| = C_Y < \infty$

$$\Rightarrow \|d\tilde{u}_k\|^2_{L^2(B_\delta)} \leq 2E + C_Y^2 \pi \delta^2$$

i.e. $W^{1,2}$-bound on $\underline{\tilde{u}}$

higher bounds ($W^{k,p} \subset W^{1,2}$) from Laplace equation

$$\partial_s u + J(u)\partial_t u = P(u)$$

$$\Rightarrow (\partial_s^2 + \partial_t^2) u = \underbrace{(\partial_s - J\partial_t)(\partial_s + J\partial_t) u}_{\nabla_{\partial_s u} P(u) - J(u)\nabla_{\partial_t u} P(u)} - \nabla_{\partial_s u} J(u)\cdot\partial_t u + J(u)\nabla_{\partial_t u} J(u)\,\partial_t u$$

Calderon-Zygmund : $\|\nabla^2 v\|_{L^p} \leq C_{p,n} \|\Delta v\|_{L^p}$ $\quad \forall v \in W^{2,p}(\mathbb{R}^n)$, $1<p<\infty$

$$\Rightarrow \|u\|_{W^{2,p}} \leq C_p \|du\|_{L^p} + C_J \|\,|du|^2\|_{L^p} + C \|u\|_{W^{1,p}}$$

[$v = h\cdot u : \mathbb{R}^2 \to M \hookrightarrow \mathbb{R}^N$

$\|u\|_{W^{2,p}(B_{\delta/2})} \leq \|\nabla^2(h\cdot u)\|_{L^p} + \|u\|_{W^{1,p}} \leq C_{p,2}\|h\cdot\Delta u\|_{L^p} + C_h \|u\|_{W^{1,p}}$]

To get $W^{2,p}$-bounds on $\tilde{u}_k$ we need

- $W^{1,2p}$-bounds on $\tilde{u}_k$; $1<p<\infty$ $\qquad \|\,|du|^2\|_{L^p} \leq \|du\|^2_{L^{2p}}$

or

- $\|d\tilde{u}_k\|_{L^2}$ small ; $1<p<2$ $\qquad \|\,|du|^2\|_{L^p} \leq \|du\|_{L^2}\cdot\|du\|_{L^{\frac{2p}{2-p}}} \leq \underbrace{\|du\|_{L^2}}_{\text{small}} C \|u\|_{W^{2,p}}$

(Hölder $L^2\cdot L^q \hookrightarrow L^p$ and Sobolev embedding $W^{1,p}\hookrightarrow L^q$ with $\frac{1}{p} = \frac{1}{2} + \frac{1}{q}$) $\quad \to$ absorb into LHS

<u>bootstrapping</u>:

small energy $\|du_k^\nu\|_{L^2} \le \varepsilon$ $\forall \nu$ on $B_\delta \subset S_k$

$\to$ $W^{2,p}$-bounds $(p<2)$ on $(u_k^\nu)_{\nu\in\mathbb{N}}$

$\to$ $W^{1,\frac{2p}{2-p}}$-bounds $\quad \frac{2p}{2-p} =: q > 2$

$\to$ $W^{2,q}$-bounds

$\boxed{W^{k,q}\text{-bounds}, q>2 \to W^{k+1,q}\text{-bounds}}$ from Calderon-Zygmund $(v = \partial_s^{k-1} u)$

$\to$ $W^{k,q}$-bounds $\forall k$

$\to$ $W^{k-1,q}$-convergent subsequence $\forall k$

$\xrightarrow[\text{subsequence}]{\text{diagonal}}$ $\mathcal{C}^\infty$-convergent subsequence

($\mathcal{C}^\ell$-convergence on all compact subsets $\forall \ell$)

So compactness holds on interior balls of small energy.

However, we still need to understand

- compactness near the boundary
- dependence of energy quantum ε on ball radius δ

L17 – Compactness – boundary and mean value inequalities

$\underline{u}^\nu : \underline{S} \to \underline{M}$ holomorphic quilts with fixed energy $\mathcal{E}(\underline{u}^\nu) = E$

Last time: $B_\delta \hookrightarrow S_k$ interior ball

small "energy" $\|du_k^\nu\|_{L^2(B_\delta)} \le \varepsilon_\delta \; \forall \nu \;\Rightarrow\; \exists\, \mathcal{C}^\infty$-convergent subsequence

higher bounds with boundary conditions

Assume $\mathcal{C}^0$-convergence (e.g. from $W^{1,p}$-bounds and $W^{1,p} \underset{\text{compact}}{\hookrightarrow} \mathcal{C}^0$ for $p>2$)

so we can use local coordinates on $\begin{matrix} M_{k_b} & \supset & \mathbb{C}^n \\ \cup & & \cup \\ L_{k_b} & \supset & \mathbb{R}^n \end{matrix}$ resp. $\begin{matrix} M_{k_\sigma} \times M_{k_\sigma'} & \supset & \mathbb{C}^n \\ \cup & & \cup \\ L_\sigma & \supset & \mathbb{R}^n \end{matrix}$

with $J|_L = i$

$$\partial_s u + J(u)\partial_t u = P(u), \quad u(s,t=0) \in \mathbb{R}^n \quad \forall s$$

$u = v + iw \;;\; v, w : B_\delta \cap \mathbb{H} \to \mathbb{R}^n$

$\Rightarrow \quad \Delta v + i \Delta w = \text{lower order }(v,w)$

$w|_{t=0} = 0 \qquad \rightsquigarrow$ Dirichlet problem for w

$\partial_t v|_{t=0} = \underbrace{-\partial_s w|_{t=0}}_{=0} + \underbrace{\mathrm{Im}\, P(v+iw)}_{\text{lower order}} \qquad \rightsquigarrow$ Neumann problem for v

$\rightarrow$ same bootstrapping as in interior, starting from $W^{1,p>2}$-bounds

$\boxed{\text{mean value inequalities}}$

local energy : $\frac{1}{2}\int_{\substack{B_\delta \\ \text{or } B_\delta\cap H}} |du - Y(u)|^2 = \int \underbrace{|\partial_s u - Y(u)(\partial_s)|^2}_{= |\partial_t u - Y(u)(\partial_t)|^2} = \int e(u)$

energy density $e = e(u) : B_\delta \longrightarrow [0,\infty)$
$(s,t) \longmapsto |\partial_s u - Y(u)\partial_t|^2$

- $\Delta e = 2\underbrace{|\nabla(\partial_s u - Y(u)\partial_s)|^2}_{①} + 2\underbrace{\langle \partial_s u - Y(u)\partial_s , \Delta(\partial_s u - Y(u)\partial_s)\rangle}_{②}$

  $\geq -ae^2 - A_0$; $a, A_0 > 0$ constants (estimate linear term $2e \leq 1 + e^2$)

[[MS, Lemma 4.3.1] using $\partial_s u + J\partial_t u$ = lower order

② $= \langle \partial_s u, \nabla_s((\nabla_{\partial_t u}J)\partial_s u - (\nabla_{\partial_s u}J)\partial_t u) - R(\partial_s u, \partial_t u)\partial_t u\rangle$ + l.o.

$\geq -C(\nabla J, \nabla^2 J)\cdot|du|^4 - C(\nabla J)\cdot\underbrace{|du|^2|\nabla du|}_{\leq \frac{1}{2}\varepsilon^2|\nabla du|^2 + \frac{1}{2}\varepsilon^{-2}|du|^4}$ + l.o. $\boxed{2ab \leq a^2+b^2}$

$\geq -ae^2 - A_1 e - A_0$ absorb in ①

using $|\partial_s u|, |\partial_t u| \leq \sqrt{e} + \|Y\|_\infty$]

- $\partial_t e|_{t=0} = \partial_t\, \omega(\partial_s u - Y(u)\partial_s, J(u)(\partial_s u - Y(u)\partial_s))$

  $= \omega(\underbrace{\nabla_t\partial_s u}_{= \nabla_s\partial_t u = -\nabla_s J\partial_s u}, J(u)\partial_s u) + \omega(\partial_s u, J(u)\nabla_t\partial_s u)$ + lower order

  e.g. $\nabla_{\partial_t u}\omega(\partial_s u, J\partial_s u)$

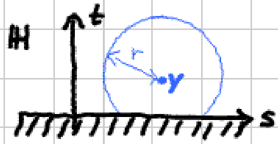
  $= \underbrace{2\omega(\underbrace{\partial_s u}_{TL}, \nabla_s \underbrace{\partial_s u}_{TL})}_{=0 \text{ if } L \text{ linear}} - \langle(\nabla_{\partial_s u}J)\partial_s u, \partial_s u\rangle + (\nabla_{\partial_t u}\omega)(\partial_s u, \partial_t u) + \ldots$

  (e.g. in coordinates, but then $\nabla J \neq 0$ or $\nabla\omega \neq 0$)

  $\leq C|du|^3$ + lower order $\leq b e^{3/2} + B_0$

Thm: $\exists C \,,\, \forall a,b \geq 0 \; \exists \mu(a,b)>0 \;:\; \forall y \in \mathbb{H}^2, r>0$

If $e \in \mathcal{C}^2(B_r(y)\cap\mathbb{H}, [0,\infty))$ satisfies



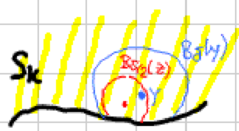
$$\begin{cases} \Delta e \geq -a e^2 - A_1 e - A_0 \\ -\frac{\partial}{\partial t} e\big|_{t=0} \geq -b e^{3/2} - B_1 e - B_0 \end{cases} \quad \text{and} \quad \int_{B_r(y)\cap\mathbb{H}} e \leq \mu(a,b)\,,$$

then $$e(y) \leq C\Big(A_0 r^2 + B_0 r + (A_1 + B_1^2 + r^{-2}) \int_{B_r(y)\cap\mathbb{H}} e \Big)$$

Corollary: $W^{1,\infty}$-bounds on balls of small energy

- $\displaystyle\int_{B_\delta(y)\subset S_k} |du_k^\nu - Y_k(u_k^\nu)|^2 \leq \mu \quad \forall \nu \;\Rightarrow\; \sup_{B_{\delta/2}(y)} |du_k^\nu| \leq C \quad \forall \nu$

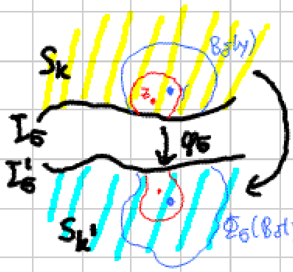
$\displaystyle \| \quad \int_{B_\delta(y)\cap\mathbb{H}} e(u_k^\nu)$



$z \in B_{\delta/2}(y)$

- $\displaystyle\int_{B_\delta(y)\subset S_k} |du_k^\nu - Y_k(u_k^\nu)|^2 + \int_{\Phi_\delta(B_\delta(y))} |du_{k'}^\nu - Y_{k'}(u_{k'}^\nu)|^2 \leq \mu \;\Rightarrow\; \sup_{B_{\delta/2}(y)} |du_k^\nu| + \sup_{\Phi_\delta(B_{\delta/2}(y))} |du_{k'}^\nu| \leq C \quad \forall \nu$

$\displaystyle \| \quad \int_{B_\delta(y)\cap\mathbb{H}} e(w_\delta^\nu)$



$\Phi_\delta(s,t) = (\varphi_\delta(s), \varphi'_\delta(s)\cdot t)$

Proof: Apply mean value inequalities to $e(u_k^\nu) = |du_k^\nu - Y_k(u_k^\nu)|^2$

resp. $e(w_\delta^\nu)(s,t) = |du_k^\nu - Y_k(u_k^\nu)|^2(s,-t) + |du_{k'}^\nu - Y_{k'}(u_{k'}^\nu)|^2(\Phi_\delta(s,t))$

on balls of radius $\delta/2$ centered at $z \in B_{\delta/2}(y)$.

L18 – Compactness – up to energy concentration

Note Title 4/14/2008

LAST TIME: $u : B_\delta \to M$, $B_\delta \subset \mathbb{R}^2$

energy density $e = e(u) : B_\delta \to [0,\infty)$

$(s,t) \mapsto |\partial_s u - Y(u)\partial_s|^2$

Lemma (i): $\partial_s u + J(u)\partial_t u = Y(u)(\partial_s) + J(u)Y(u)(\partial_t)$

$$\Rightarrow \quad \Delta e \geq -a e^2 - A_0 \qquad ; \; a, A_0 > 0 \text{ constants}$$

NEXT: $u : B_\delta \cap \mathbb{H} \to M$

Lemma (i) still holds

Lemma (ii): $u|_{t=0} \in L$ Lagrangian and $\partial_s u + J(u)\partial_t u = Y(u)(\partial_s) + J(u)Y(u)(\partial_t)$

$$\Rightarrow \quad -\partial_t e|_{t=0} \geq -b e^{3/2} - B_0 \qquad ; \; b, B_0 > 0 \text{ constants}$$

Proof: ⚠ Note that $Y(u)(\partial_s)|_{t=0} = 0$ since $*K_k|\partial S_k = 0$

$$\partial_t e|_{t=0} = \partial_t\, \omega\big(\partial_s u - Y(u)\partial_s, J(u)(\partial_s u - Y(u)\partial_s)\big)$$

pick any connection ∇

$$= \omega(\nabla_t \partial_s u, J(u)\partial_s u) + \omega(\partial_s u, J(u)\nabla_t\partial_s u) + \text{lower order}$$

$\nabla_t \partial_s u = \nabla_s \partial_t u = -\nabla_s J \partial_s u$ e.g. $\nabla_{\partial_t u}\omega(\partial_s u, J\partial_s u)$

$$= \underbrace{2\omega(\underset{TL}{\partial_s u}, \nabla_s \underset{TL}{\partial_s u})}_{=0 \text{ if } L \text{ linear}} - \langle(\nabla_{\partial_s u} J)\partial_s u, \partial_s u\rangle + (\nabla_{\partial_t u}\omega)(\partial_s u, \partial_s u) + \ldots$$

(pick ∇ s.t. L geodesic, but then $\nabla J \neq 0$ or $\nabla\omega \neq 0$)

$$\leq C|du|^3 + \text{lower order} \leq b e^{3/2} + B_0 \qquad \blacksquare$$

Corollary: $W^{1,\infty}$-bounds on balls of small energy

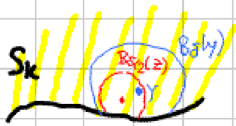
$\exists\, \mu = \mu(a,b) = \mu(\underline{S}, \underline{M}, \underline{L}, \underline{J}, \underline{Y}) > 0$ s.t.t.f.h.

$(\underline{u}^\nu : \underline{S} \to \underline{M})_{\nu\in\mathbb{N}}$ holomorphic quilts

- $y \in S_k$, $B_\delta(y) \subset S_k$ intersects no seams (but possibly a boundary component)

$$\int_{B_\delta(y)\subset S_k} |du_k^\nu - Y_k(u_k^\nu)|^2 \le \mu \quad \forall \nu \quad \Rightarrow \quad \sup_{B_{\delta/2}(y)} |du_k^\nu| \le C \quad \forall \nu$$

$\| \; \int_{B_\delta(y)\cap\mathbb{H}} e(u_k^\nu)$



$z \in B_{\delta/2}(y)$

- $y \in S_k$, $B_\delta(y) \subset S_k$ intersects one seam $I_\sigma \subset \partial S_k$

$$\int_{B_\delta(y)\subset S_k} |du_k^\nu - Y_k(u_k^\nu)|^2 + \int_{\Phi_\sigma(B_\delta(y))} |du_{k'}^\nu - Y_{k'}(u_{k'}^\nu)|^2 \le \mu \quad \forall \nu \Rightarrow \sup_{B_{\delta/2}(y)} |du_k^\nu| + \sup_{\Phi_\sigma(B_{\delta/2}(y))} |du_{k'}^\nu| \le C \quad \forall \nu$$

$\| \; \int_{B_\delta(y)\cap\mathbb{H}} e(w_\sigma^\nu)$

$\Phi_\sigma(s,t) = (\varphi_\sigma(s), \varphi_\sigma'(s)\cdot t)$

Proof: Apply mean value inequalities to $e(u_k^\nu) = |du_k^\nu - Y_k(u_k^\nu)|^2$

resp. $e(w_\sigma^\nu)(s,t) = |du_k^\nu - Y_k(u_k^\nu)|^2(s,-t) + |du_{k'}^\nu - Y_{k'}(u_{k'}^\nu)|^2(\Phi_\sigma(s,t))$

on balls of radius $\delta/2$ centered at $z \in B_{\delta/2}(y)$.

$(\underline{u}^\nu : \underline{S} \to \underline{M})_{\nu\in\mathbb{N}}$ holomorphic quilts of energy E

pick a j_k-compatible metric on each S_k and consider

$e(u_k^\nu)$ as function on S_k, $e(w_\sigma^\nu)$ as function on $S_{k\sigma}$

<u>Claim</u>: There exists a finite "bubbling set" $(z_i)_{i=1..N}$, $N < E/\mu$

and a subsequence $(\underline{u}^{\nu_i})_{i\in\mathbb{N}}$ such that

- for $z \in S_k \setminus \bigcup_\sigma I_\sigma \setminus \{z_1..z_N\}$ $\exists \delta_z > 0 : \int_{B_{\delta_z}(z)} e(u_k^{\nu_i}) \leq \mu \quad \forall i$
- for $z \in I_\sigma \setminus \{z_1..z_N\}$ (with $I_\sigma \subset \partial S_k$) $\exists \delta_z > 0 : \int_{B_{\delta_z}(z)} e(w_\sigma^{\nu_i}) \leq \mu \quad \forall i$

<u>Proof</u>: Either the claim holds with $N=0$ and the original sequence or

there is $z_1 \in S_k \setminus \bigcup I_\sigma$ resp. $z_1 \in I_\sigma$ and a subsequence $(\underline{u}^{\nu_i})_{i\in\mathbb{N}}$ s.t.

$\int_{B_{2^{-i}}(z_1)} e(u_k^{\nu_i})$ resp. $e(w_\sigma^{\nu_i}) > \mu \quad \forall i$ (and hence $E > \mu$)

<u>Iteration</u>:

Either the claim holds with $\{z_1,..,z_N\}$ and this subsequence or

there is another z_{N+1} and a further subsequence s.t.

$\int_{B_{2^{-i}}(z_j)} e(u_{k_j}^{\nu_i})$ resp. $e(w_{\sigma_j}^{\nu_i}) > \mu \quad \forall i \ \forall j = 1..N+1$

(and hence $E > (N+1)\mu$ since $B_{2^{-i}}(z_{1..N+1})$ disjoint for $i \gg 1$)

Iteration stops since $E < \infty$. ∎

<u>Corollary</u>: There exists a subsequence that converges in

$$\mathcal{C}^\infty\Big(\bigsqcup_k S_k \setminus \bigcup_{i=1}^N z_i\Big) \qquad \big(\text{i.e. in } \mathcal{C}^\ell(K) \text{ for all } \ell\in\mathbb{N},\ K \text{ compact}\big)$$

<u>Proof</u>: For $j\in\mathbb{N}$ we can cover

$$\bigsqcup_k S_k \setminus \bigcup_{i=1}^N B_{2^{-j}}(z_i) \setminus \bigcup_{k,e\,\text{end}} \varepsilon_{ke}\big(\{|s|>j\}\big)$$

with $\varepsilon_{ke}(\{|s|>j\}) \subset \mathbb{R}^\pm\times(S^1 \text{ or } [0,\delta_e])$ and $\varepsilon_{ke}(\dots)\subset S_k$,

by finitely many balls $B_{\delta_i/4}(z_i)$ $\big(\text{resp. } B_{\delta_i/4}(z_i)\sqcup\Phi_\sigma(B_{\delta_i/4}(z_i)) \text{ for } z_i\in I_\sigma\big)$

with small energy ($\leq\mu$) on B_{δ_i}

$\Rightarrow$ $W^{1,\infty}$-bounds on $B_{\delta_i/2}$

$\Rightarrow$ $W^{k,p}$-bounds $\forall k,p$ on $B_{\delta_i/4}$

$\Rightarrow$ $\mathcal{C}^\infty$-convergent subsequence on $\bigcup_i B_{\delta_i/4}$ (fixed j)

Finally, take a diagonal subsequence over $j\in\mathbb{N}$ ■

<u>Note</u>: The above subsequence concentrates energy at every z_i

in the bubbling set: $\forall\delta>0\quad\exists N_\delta:\ \forall\nu\geq N_\delta$

$$\int_{B_\delta(z_i)} e(u^\nu_{k_i}) \text{ resp. } e(w^\nu_{\sigma_i}) \quad >\mu$$

Hence the limit $\underline{\hat{u}}\in\mathcal{C}^\infty\big(\bigsqcup_k S_k\setminus\bigcup_{i=1}^N z_i\big)$ has energy

$$E(\underline{\hat{u}}) := \int_{\underline{S}\setminus\text{bubbling set}} \tfrac{1}{2}\,|d\hat{u}_k - Y_k(\hat{u}_k)|^2 \leq E - \underbrace{N}_{\#\text{bubbling points}}\cdot\mu$$

Thm (Removable Singularities) $\quad \underline{\hat{u}} \in \mathcal{C}^\infty(\bigsqcup_k S_k \setminus \bigcup_{i=1}^N z_i)$

"singular" holomorphic quilt with finite energy $\quad E(\underline{\hat{u}}) < \infty$

$\Rightarrow \lim_{z \to z_i} u_{k_i}(z_i)$ resp. $w_{\sigma_i}(z_i)$ exists $\forall i = 1..N$ and defines a

smooth extension to a holomorphic quilt $\underline{\tilde{u}} : \underline{S} \to \underline{M} \quad \in \mathcal{C}^\infty(\sqcup S_k)$

of energy $E(\underline{\tilde{u}}) = E(\underline{\hat{u}})$.

Cor: "compactness" for monotone, minimal index moduli spaces

Assume $\mathcal{M}^k := \mathcal{M}^k(\underline{S}, \underline{K}, \underline{J}, (\underline{x}_e)_{e \in \mathcal{E}^-}, (\underline{y}_e)_{e \in \mathcal{E}^+})$ satisfies

- monotonicity: $\mathcal{E}|_{\mathcal{M}^k} = \text{const} = E_k \; ; \; \underline{\tilde{u}} \in \mathcal{M}^\ell, E(\underline{\tilde{u}}) < E_k \Rightarrow \ell < k$
- minimal index: $\mathcal{M}^\ell = \emptyset \quad \forall \ell < k_{min}$

Then $\mathcal{M}^{k_{min}}$ is compact w.r.t. $W^{m,p}(\sqcup S_k)$-norm (any m, p)

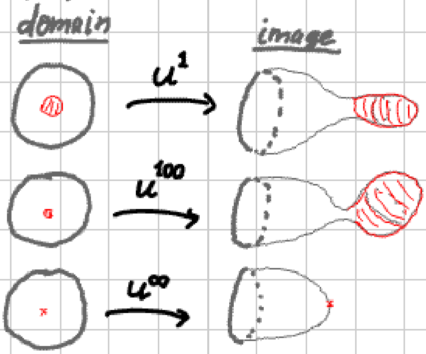
"up to breaking of trajectories on the cyl./strip-like ends."

L19 - Compactness - bubbling

Note Title 4/24/2008

Compactness for monotone, minimal index moduli spaces

- $\mathcal{C}^\infty_{loc}$ - convergence on complement of bubbling points
- energy loss at bubbling points
- removal of bubbling singularities



skipping:
- index identities
- exponential decay on quilted ends

corrected

model case: $L_0 \pitchfork L_1 \subset M$ compact Lagrangian, J ω-comp. a.c.s.

$$\exists\, C, \delta, \hbar > 0 \;:\; \forall T > 2,\; u : [-T,T]\times[0,1] \longrightarrow M$$

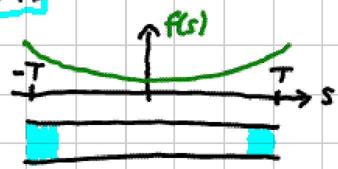
$$\begin{cases} \partial_s u + J\partial_t u = 0 \\ u|_{t=0} \in L_0,\; u|_{t=1} \in L_1 \end{cases} \qquad \int_{[-T,T]\times[0,1]} |\partial_s u|^2 < \hbar$$

$$\Rightarrow \forall S \le T-1 \quad \sup_{[-S,S]\times[0,1]} d(u(s,t), L_0 \pitchfork L_1) + \|\nabla u\|_{L^\infty([-S,S]\times[0,1])}$$

$$\le C e^{-\delta(T-S)} \Big(\int_{\boxed{([-T,-T+1] \cup [T-1,T])\times[0,1]}} |\partial_s u|^2 \Big)^{1/2}$$

Proof: $f(s) := \int_0^1 |\partial_s u(s,t)|^2 dt$

satisfies $f'' \ge \delta^2 f$



Cor.: $\mathcal{M}^0$ compact in $W^{k,p}(\sqcup S_k)$-topology

$\mathcal{M}^1$ compact up to "one Floer trajectory breaking off at one end"

$\boxed{\text{geometric description of bubbling}}$

(I) interior $\quad u^\nu : B_1 = \{s^2+t^2<1\} \longrightarrow M \quad , \quad \partial_s u^\nu + J\partial_t u^\nu = 0$

$(s^\nu, t^\nu) \longrightarrow 0 \quad , \quad |du^\nu(s^\nu, t^\nu)| = R^\nu \xrightarrow[\nu\to\infty]{} \infty \quad , \quad \sup_\nu \int_{B_1} |\partial_s u^\nu|^2 < \infty$

Hofer trick: can assume $\|du^\nu\|_{L^\infty(B_{\varepsilon^\nu}(s^\nu,t^\nu))} \leq 2R^\nu \quad ; \quad \varepsilon^\nu R^\nu \to \infty$, $\varepsilon^\nu \to 0$

"Little Lemma" [Hofer–Zehnder Ch.6 Lemma 5]

X complete metric space, $f: X \to [0,\infty)$ continuous

$\forall x_0 \in X, \varepsilon_0 > 0 \; \exists x \in B_{2\varepsilon_0}(x_0), \; \varepsilon \in (0, \varepsilon_0] : \quad \sup_{y \in B_\varepsilon(x)} f(y) \leq 2f(x)$

$\varepsilon f(x) \geq \varepsilon_0 f(x_0)$

(Apply this to $f = |du^\nu|$, $x_0 = (s_\nu, t_\nu)$, $\varepsilon_0 = R_\nu^{-1/2}$ to find $x = (s'_\nu, t'_\nu)$, $0 < \varepsilon_\nu < R_\nu^{-1/2}$, $R'_\nu = f(x) = |du^\nu(s'_\nu, t'_\nu)|$ with $\begin{cases} (s'_\nu, t'_\nu) \to 0 \\ \varepsilon_\nu \to 0 \\ \varepsilon_\nu R'_\nu \geq R_\nu^{-1/2} R_\nu \to \infty \end{cases}$)

Rescaling: $v^\nu : B_{\varepsilon^\nu R^\nu} \longrightarrow M \quad , \quad (\sigma, \tau) \longmapsto u^\nu(s^\nu + \frac{\sigma}{R^\nu}, t^\nu + \frac{\tau}{R^\nu})$

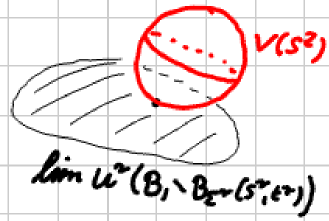
satisfies $\partial_\sigma v^\nu + J(v^\nu)\partial_\tau v^\nu = 0 \quad , \quad \sup_\nu \int_{B_{\varepsilon^\nu R^\nu}} |\partial_s v^\nu|^2 = \sup_\nu \int_{B_{\varepsilon^\nu}} |\partial_s u^\nu|^2 < \infty$

$\|\partial_s v^\nu\|_{L^\infty} \leq 2 \quad , \quad |\partial_s v^\nu(0)| = 1$

Compactness: $\exists$ subsequence $v^{\nu_i} \xrightarrow{C^\infty_{loc}} v^\infty \in C^\infty(\overset{\cong S^2 \setminus pt}{\mathbb{R}^2}, M)$

$$\partial_s v^\infty + J(v^\infty)\partial_t v^\infty = 0 \;, \quad \int |\partial_s v^\infty|^2 < \infty \;, \quad |\partial_s v^\infty(0)| = 1$$

Removal of singularity $\Rightarrow$ "the bubble" is a J-hol. sphere



$v: S^2 \to M$, $\bar\partial_J v = 0$, nonconstant

$$\Rightarrow \frac{1}{2}\int_{S^2} |dv|^2 = \int_{S^2} v^*\omega \geq \hbar > 0$$

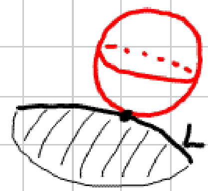
$\hbar > 0$ positive generator of $\langle [\omega], \pi_2(M)\rangle \subset \mathbb{R}$

or $\hbar > 0$ by Gromov compactness

(II) boundary $\quad u^\nu: B_1 \cap \{t \geq 0\} \longrightarrow M \;, \quad u^\nu|_{t=0} \in L$

$\rightsquigarrow v^\nu : B_{\varepsilon^\nu R^\nu} \cap \{\tau \geq -t^\nu R^\nu\} \longrightarrow M, \quad v^\nu|_{\tau = -t^\nu R^\nu} \in L$

- subsequence with $t^\nu R^\nu \to \infty \rightsquigarrow$ limit $v^\infty : \mathbb{R}^2 \to M$

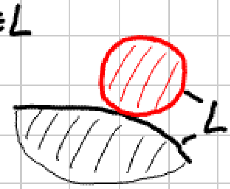


$\rightarrow$ bubble is a J-hol. sphere $v: S^2 \to M$

- subsequence with $t^\nu R^\nu \to T < \infty \rightsquigarrow$ limit $v^\infty : \mathbb{R} \times [-T, \infty) \to M$, $v^\infty|_{\tau=-T} \in L$

$\rightarrow$ bubble is a J-hol. disc $\quad v: \underset{\{|z|\leq 1\} \subset \mathbb{C}}{\overset{\shortparallel}{\mathbb{D}}} \to M, \quad \begin{cases} \bar\partial_J v = 0 \\ v|_{\partial \mathbb{D}} \in L \end{cases}$

$$\Rightarrow \frac{1}{2}\int |dv|^2 = \underset{\langle [\omega], \pi_2(M,L)\rangle}{\overset{\cap}{\int v^*\omega}} \geq \hbar > 0$$



L20 - bubbling at seams

Note Title 5/5/2008

(III) bubbling point on a seam

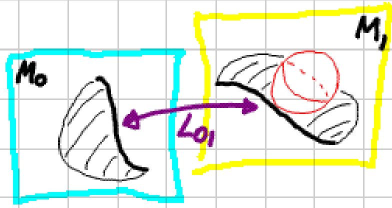
$$|\partial_s u_0^\nu(s_1^\nu, -|t^\nu|)|^2 + |\partial_t u_1^\nu(s_1^\nu, |t^\nu|)|^2 = (R^\nu)^2 \to \infty$$

fold

$$|\partial_s w^\nu(s_1^\nu, |t^\nu|)| = R^\nu \to \infty$$

- subsequence with $t^\nu R^\nu \to \infty$ (and $|\partial_s u_1^\nu| \to \infty$)

  $\to$ bubble is a J_1-hol. sphere $v : S^2 \to M_1$



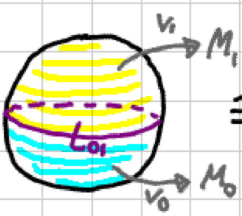
- subsequence with $t^\nu R^\nu \to -\infty$ (and $|\partial_s u_0^\nu| \to \infty$)

  $\to$ bubble is a J_0-hol. sphere $v : S^2 \to M_0$

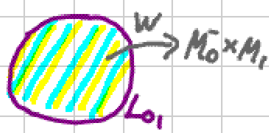
- subsequence with $t^\nu R^\nu \to T$

  $\to$ bubble is a $(-J_0, J_1)$-hol. disc $\underset{(v_0, v_1)}{W} : \mathbb{D} \to M_0^- \times M_1$, $\quad \bar\partial_{(-J_0, J_1)} w = 0$, $\quad w|_{\partial D} \in L_{01}$

or, equivalently, quilted sphere



$\hat{=}$



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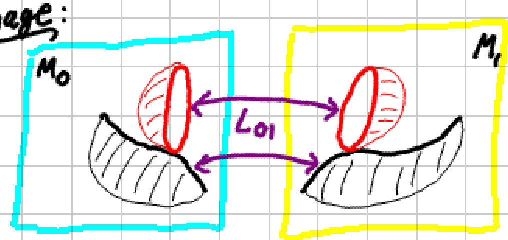
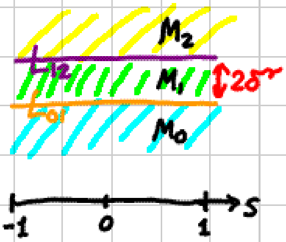


bubbling in shrinking strip



$$u_2^\nu : [-1,1]\times[0,1) \to M_2 \qquad \bar\partial_{J_2} u_2^\nu = 0 \qquad (u_1^\nu(s,\delta^\nu), u_2^\nu(s,0)) \in L_{12}$$
$$u_1^\nu : [-1,1]\times[-\delta^\nu,\delta^\nu] \to M_1 \qquad \bar\partial_{J_1} u_1^\nu = 0 \qquad (u_0^\nu(s,1), u_1^\nu(s,-\delta^\nu)) \in L_{01}$$
$$u_0^\nu : [-1,1]\times(-1,0] \to M_0 \qquad \bar\partial_{J_0} u_0^\nu = 0$$

$$\sup_\nu \sum_{i=0}^{2} \int |\partial_s u_i^\nu|^2 < \infty$$

For simplicity consider the case of a bubbling sequence $(s^\nu, t^\nu) = (0,0)$ for u_1^ν

$$|du_1^\nu(0,0)| = R^\nu \to \infty \qquad , \ \|du_i^\nu\|_{L^\infty} \le 2R^\nu \quad \text{for } i = 0,1,2$$

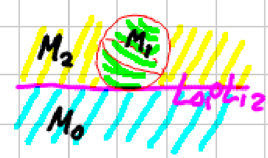
<u>Rescaling</u>: $v_1^\nu : [-R^\nu,R^\nu]\times[-\delta^\nu R^\nu, \delta^\nu R^\nu] \longrightarrow M_1$, $(\sigma,\tau) \longmapsto u_1^\nu(R^\nu\sigma, R^\nu\tau)$
$v_0^\nu : [-R^\nu,R^\nu]\times(-R^\nu,0] \longrightarrow M_0$
$v_2^\nu : [-R^\nu,R^\nu]\times[0,R^\nu) \longrightarrow M_2$

$$\|dv_i^\nu\|_{L^\infty} \le 2 \ , \quad |dv_1^\nu(0)| = 1$$

- $\delta^\nu R^\nu \longrightarrow \infty$: limit $\quad v_1^\infty : \mathbb{R}^2 \longrightarrow M_1$

  bubble $\quad v : S^2 \longrightarrow M_1$

<u>bubble in domain</u>

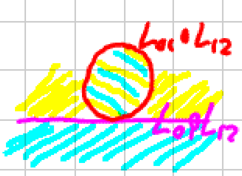


- $\delta^\nu R^\nu \longrightarrow 0$: limit $\quad v_0^\infty : \mathbb{R}\times(-\infty,0] \to M_0$
  $v_1^\infty : \mathbb{R} \to M_1$
  $v_2^\infty : \mathbb{R}\times[0,\infty) \to M_2$

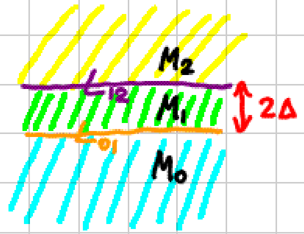
$$(v_0^\infty(s,0), v_1^\infty(s)) \in L_{01} \ , \ (v_1^\infty(s), v_2^\infty(s,0)) \in L_{12} \iff (v_0^\infty(s,0), v_2^\infty(s,0)) \in L_{01}\circ L_{12}$$

$\xrightarrow{\text{fold}}$ $\quad w^\infty = (v_0^\infty(-\cdot,\cdot), v_2^\infty) : \mathbb{R}\times[0,\infty) \to M_0^- \times M_2$

$$\begin{cases} \bar\partial_{(-J_0,J_2)} w = 0 \\ w|_{s=0} \in L_{01}\circ L_{12} \end{cases} \quad \rightsquigarrow \text{bubble } w : \mathbb{D}^2 \to M_0^- \times M_2$$



• $\delta^2 R^2 \longrightarrow \Delta > 0$: limit



$$\begin{aligned} v_0^\infty &: \mathbb{R}\times(-\infty,0] \longrightarrow M_0 & \quad (v_0^\infty(s,0), v_1^\infty(s,-\Delta)) \in L_{01} \\ v_1^\infty &: \mathbb{R}\times[-\Delta,\Delta] \longrightarrow M_1 & \\ v_2^\infty &: \mathbb{R}\times[0,\infty) \longrightarrow M_2 & \quad (v_1^\infty(s,\Delta), v_2^\infty(s,0)) \in L_{12} \end{aligned}$$

finite energy: $\sum_{i=0}^{2} \int |dv_i^\infty|^2 < \infty$

nonconstant: $|dv_1^\infty(0,0)| = 1$ $\quad \left(\text{in general } |dv_i^\infty(0,\tau)| = 1 \text{ for some } i,\tau \right)$

$$\boxed{\underline{\text{Conjecture}}: \quad \lim_{s^2+t^2\to\infty} v_i^\infty(s,t) = p_i \in M_i \quad \text{for } i = 0,1,2}$$

<u>Remark</u>: This is true in the trivial cases

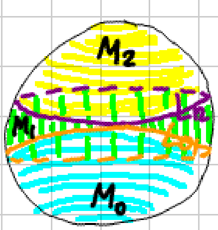
$M_1 = pt$ by removal of singularity for discs v_0^∞ and v_2^∞
since $L_{01} \subset M_0$, $L_{12} \subset M_2$ are simple Lagrangian boundary conditions

$M_0 = M_2 = pt$: by finite energy $v_1^\infty(s,t) \xrightarrow[s\to\pm\infty]{} p_1^\pm \in M_1$ uniformly in $t \in [-\Delta,\Delta]$

embeddedness of $L_{01} \circ L_{12} \subset pt \times pt$ means $L_{01} \cap L_{12} \subset M_1$ is a single point
so, since $p_1^+, p_1^- \in L_{01} \cap L_{12}$, they are automatically equal.

If Conj. is true then $(p_0,p_1) \in L_{01}$, $(p_1,p_2) \in L_{12}$ $(\Rightarrow (p_0,p_2) \in L_{01}\circ L_{12})$

and $(v_0^\infty, v_1^\infty, v_2^\infty)$ can be compactified to a <u>"figure 8 bubble"</u>



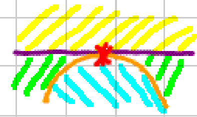
$$\begin{aligned} v_0 &: D_0 \longrightarrow M_0 & (v_0,v_1)|_{\partial D_0} \in L_{01} \\ v_1 &: S^2 \setminus (\mathring{D}_0 \cup \mathring{D}_2) \longrightarrow M_1 & \\ v_2 &: D_2 \longrightarrow M_2 & (v_1,v_2)|_{\partial D_2} \in L_{12} \end{aligned}$$

and hence $(v_0,v_2)|_{D_0\cap D_2} \in L_{01}\circ L_{12}$

$D_0, D_2 \subset S^2$ closed discs with $D_0 \cap D_2 = pt$

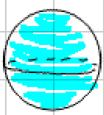
We call this "singularly quilted sphere" figure eight because, with smaller discs, the seams touch like a figure eight 8.

The conjecture is a removable singularity statement for a quilt with tangentially intersecting seams at the singularity *.



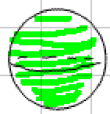
Complete list of possible bubbles:

$S^2 \to M_0$



$D^2 \to M_0^- \times M_1$

$S^2 \to M_1$



$D^2 \to M_1^- \times M_2$

$S^2 \to M_2$

figure 8

$D^2 \to M_0^- \times M_2$

Note: Can view all others as degeneration of figure 8.

Energy Quantization in case of figure 8 bubbling:

$$\liminf \sum_{i=1}^{3} \int_{B_{\epsilon^\nu R^\nu}} |du_i^\nu|^2 \geq \liminf \sum_{i=0}^{2} \int |dv_i^\infty|^2 \geq \hbar_\Delta > 0$$

follows from mean value inequality (for bubbling point $(0,0)$ in u_1^ν)

If $\int |dv_1^\infty|^2 < \hbar$ then $1 = |dv_1^\infty(0,0)|^2 \leq C + C\Delta^{-2} \int_{B_\Delta(0,0)} |dv_1^\infty|^2$

$$\Rightarrow \int |dv_1^\infty|^2 \geq \Delta^2(C^{-1}-1),$$

otherwise $\int |dv_1^\infty|^2 \geq \hbar$; so in any case $\int |dv_1^\infty|^2 \geq \hbar_\Delta := \min(\hbar, \Delta^2(C^{-1}-1))$.

Conj.: This holds with $\hbar>0$ independent of $\Delta>0$.

possible reason: $\sum_{i=0}^{2}\int|dv_i^\infty|^2 = \sum_{i=0}^{2}\int v_i^*\omega_i \in \langle[\omega_0,\omega_1,\omega_2]$, figure 8 homotopy class $\rangle$

We do obtain compactness for strip shrinking under the assumption of monotonicity and minimal index from

Lemma: $\exists\hbar>0$ s.t.t.f.h.

$(u_0^\nu,u_1^\nu,u_2^\nu)_{\nu\in\mathbb{N}}$ any sequence as above with $\delta^\nu\to0$

If $\liminf_{\nu\to\infty}\sum_i\|du_i^\nu\|_{L^\infty(B_\varepsilon(0))}=\infty$ $\forall\varepsilon>0$

then $\exists$ subsequence $(\nu_j)_{j\in\mathbb{N}}$ and $\varepsilon_j\to0$ s.t. $\liminf_{j\to\infty}\sum_i\int_{B_{\varepsilon_j}(0)}|du_i^{\nu_j}|^2\geq\hbar$.

(these domains need to be somewhat enlarged, depending on δ^ν, as a result of folding)

Sketch of proof by contradiction

- Find a (diagonal) sequence $(u_0^\nu,u_1^\nu,u_2^\nu)$ with $\delta^\nu\to0$, $\sum\|du_i^\nu\|_{L^\infty}=R^\nu\to\infty$

  but $\sum\int|du_i^\nu|^2\to0$.

- Deduce $\delta^\nu R^\nu\to0$ from width-dependent energy quantization.

- Show that the limit is $w_{02}=(u_0^\infty,u_2^\infty):\mathbb{R}\times[0,\infty)\to M_0^-,M_2$

$$\begin{cases}\bar\partial_{(-J_0,J_2)}w_{02}=0\\ w_{02}|_{t=0}\in L_{01}\circ L_{12}\end{cases}\qquad \int|dw_{02}|^2=0 \quad \text{but} \quad \underbrace{|dw_{02}(0)|>0}_{\text{need to prove } \mathcal{C}^1\text{-convergence}} \quad \lightning$$

QED

More open questions

Is $\left\{ \begin{array}{lll} v_0^\infty : \mathbb{R}\times(-\infty,0] \to M_0 & (v_0^\infty(s,0), v_1^\infty(s,-\Delta)) \in L_{01} & \\ v_1^\infty : \mathbb{R}\times[-\Delta,\Delta] \to M_1 & & \bar\partial_{J_i} v_i^\infty = 0 \\ v_2^\infty : \mathbb{R}\times[0,\infty) \to M_2 & (v_1^\infty(s,\Delta), v_2^\infty(s,0)) \in L_{12} & \end{array} \right\}$

a Fredholm problem?

If so, what does the moduli space of figure 8 bubbles look like?

(dimension, transversality, ...)

Is there a gluing map?

$$\left\{ \begin{array}{l} \text{moduli space of} \\ \text{figure 8 bubbles} \end{array} \right\} \times \left\{ \begin{array}{l} \text{holomorphic quilts with marked} \\ \text{point on } L_{01}\circ L_{12} \text{ seam} \end{array} \right\} \to \left\{ \begin{array}{l} \text{holomorphic quilts} \\ \text{with } L_{01}\text{- and } L_{12}\text{-seam} \\ \text{at distance } \Delta \end{array} \right\}$$

What algebraic structure (à la FOOO obstructions / A_∞-algebra)

results from that? E.g. the canonical map

$$I : \underset{\circlearrowleft \delta_\Delta}{CF(\ldots L_{01}, L_{12} \ldots)} \longrightarrow \underset{\circlearrowleft \delta_0}{CF(\ldots L_{01}\circ L_{12} \ldots)}$$

should intertwine the differentials δ_Δ and δ_0

up to a count of figure 8 bubbles.

... now build a general symplectic A_∞-2-category, allowing

non-monotone symplectic and Lagrangian manifolds!