Homeworks. Riemann surface

1.1. Prove that any holomorphic function on a compact Riemann surface is constant.

1.2. Prove that a meromorphic function on a Riemann surface defines a holomorphic map from that Riemann surface to $\mathbb{CP}^1$. Conversely, show that all such maps arise from meromorphic functions.

1.3. Show that the space of meromorphic functions on $\mathbb{CP}^1$ with a a simple pole at infinity and no other poles can be identified with the space of linear functions $az + b, a \neq 0$.

1.4. Repeat problem 3 for higher degree by showing that the vector space of meromorphic functions on $\mathbb{CP}^1$ all of whose poles are at ∞ and of order at most d can be identified with the polynomials of degree at most d.

1.5. We have natural holomorphic inclusions $\mathbb{D} \to \mathbb{C} \to \hat{\mathbb{C}}$. Show that any 'inverses' to these inclusions are constant. That is, prove (a) any holomorphic map $\hat{\mathbb{C}} \to \mathbb{C}$ is constant. And (b) any holomorphic map $\mathbb{C} \to \mathbb{D}$ is constant.

1.6. Let $\mathbb{T} = \mathbb{C}/\Lambda$ be the standard torus. Construct a smooth map $f : \mathbb{CP}^1 \to \mathbb{T}$ which is onto. But prove that if we require f to be holomorphic then there is no such map.

1.7. Use the Schwartz lemma to prove the following generalization of the Schwartz lemma. If $f : \mathbb{D} \to \mathbb{D}$ is holomorphic and has a fixed point p then either f is a hyperbolic isometry, or f strictly decreases the hyperbolic distance of points from p: that is $d(p, f(q)) \leq d(p, q)$.

Homework 2. In the 1st 2 problems X_g denotes a compact Riemann surface of genus g and $g > 0$.

2.1. Construct a smooth map of $\mathbb{CP}^1$ onto X_g.

2.2. Prove that there is no holomorphic map of $\mathbb{CP}^1$ onto X_g.

2.3. If $P(0, 0) \neq 0$ show that dx/y and dy/x are holomorphic differential forms on the algebraic variety $P(x, y) = 0$, assumed non-singular.

2.4. If $P(x, y) = y^2 - p(x)$ defines a hyperelliptic surface, find a basis for the space of holomorphic differentials on the surface one of whose elements is of the form dx/y.

7. The problems from ch. 7 of Kirwan.

8. Sketch $x^3 + y^3 = 1$ as a curve in $\mathbb{R}^2$.

9. Consider the surface as $\mathbb{C}/\Lambda$. Let $z \in \mathbb{C}$ be the standard complex coordinate. a) show that dz is a holomorphic differential on X.

b) Show that any holomorphic differential is a complex multiple of dz.

c) Suppose, instead we represent the torus as a cubic in $\mathbb{CP}^1$, say $y^2 z = x^3 - g_2 x^2 z - g_3 z^3$ where as usual the polynomial $x^3 - g_2 x - g_3$ has no multiple roots. Then, viewed in the affine chart $z = 1$, the form dx/y is a holomorphic differential, globally. It seems that $dx/y, xdx/y, x^2 dx/y$ are three linearly independent holomorphic one-forms. But they cannot be: the space of such one-forms is one-dimensional. What is the resolution to this seeming paradox?

HW 3.

3.1. Verify that the genus of the Fermat curve $x^d + y^d + z^d$ is $\binom{d-1}{2}$ by using the Riemann-Hurwitz formula in conjunction with the map $[x, y, z] \to [x, y]$ (projection from $[0, 0, 1]$) restricted to the curve.

3.2. Show that the affine hyperelliptic curve $y^2 = p_4(x)$, when projectivized, has singularities on the line at infinity $z = 0$. Classify and count these singularities. (Are they nodes, cusps?) Here we are using homogeneous coordinates $[x, y, z]$ so the affine curve lies in the chart $z \neq 0$.

3.3. Show that $xy = 0$ is equivalent to $x^2 + y^2 = 0$ over $\mathbb{C}$.

3.4. Suppose that the holomorphic function f defined in a neighborhood of the origin of $\mathbb{C}^2$ satisfies $f(0,0) = 0, \frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0$ but $\frac{\partial^2 f}{\partial x^2} \neq 0$. Show that a) IF in addition $\frac{\partial^2 f}{\partial y^2} \neq 0$ THEN there is a local holomorphic change of coordinates $(x, y) \mapsto (u, v)$ taking f to $u^2 + v^2$ b) IF in addition $\frac{\partial^2 f}{\partial y^2} = 0$ but $\frac{\partial^3 f}{\partial y^3} \neq 0$ show that there is a local holomorphic change of coordinates taking f to $u^2 - v^3 = 0$

3.5. Discriminant of a cubic. A general cubic in one variable is given by $Ax^3 + Bx^2 + Cx + D = 0$, where we assume $A \neq 0$.

a. Show that by a scaling and translation: $x \mapsto ax + b$ we can put the cubic into the form $x^3 + qx + p$.

b. The discriminant Δ of a monic polynomial is the product $\Pi_{i<j}(r_i - r_j)^2$ of the squared of the differences of its roots. Verify that the discriminant of the cubic is $\Delta = -4p^3 - 27q^2$.

3.6. Let V be a finite-dimensional complex vector space and $\xi \to \mathbb{P}(V)$ be its "hyperplane bundle" which is the dual of the canonical line bundle. We saw in class that $V^* \subset H^0(\mathbb{P}(V), \xi)$. Show that $V^* = H^0(\mathbb{P}(V), \xi)$.