Riemann surface HW, WINTER 2022

Homework Assignment I: One-forms and integration

1. Background. Before starting this problem recall the usual way of identifying the edges of a square to construct a torus. You may also want to look into how to construct the projective plane, Mobius strip and Klein bottle by identifying edges of a square. Then you may want to look at the standard construction of a genus g compact oriented surface out of a $4g$-gon.

This problem, particularly the final open-ended part of it, was inspired by a reading of the first five or so pages of the article by Karcher and Weber titled 'The Geometry of Klein's Riemann Surface" which makes up the second chapter of the book 'The eightfold way: the beauty of Klein's quartic curve'. There you will find an example of a perfect pairing amongst the edges of a decagon that leads to a genus 2 surface, and for which the image of the decagon's vertices form two points on the surface. You will also find an octagon with a perfect pairing that is different from the standard "$4g$" pairing of the previous paragraph but nevertheless leads to a genus $g = 2$ surface.

Set-up. Replace the square by the regular N-gon in the plane, denoted $P = P_N$, orienting its boundary, hence each of its N edges. All edges have the same length so there is an isometry between any two edges. In fact there are exactly two isometries between any two chosen edges, one is orientation preserving and the other is orientation reversing. By pairing off all the edges, either with an orientation preserving or reversing isometry we can build a surface $S = P/\sim$ out of P where $\sim$ is the equivalence relation generated by this pairing, so that two points which are not the same on P are considered equivalent if and only if they lie on the boundary on paired edges and are identified by the chosen isometry.

1. Take $N = 6$. Draw the hexagon.

a) Indicate on the figure the pairing that leads to a topological sphere. Make sure to indicate whether or not the isometries chosen in pairing are orientation preserving or reversing.

b) Repeat to build a torus.

c) Show that the only oriented compact surfaces you can build by identifying edges of the hexagon are the sphere and the torus .

2. Show that if *any* of the edge-pairing isometries are orientation preserving then the resulting surface is non-orientable.

3. Suppose instead of identifying edges in pairs, we identify some edges in triples or quadruples, by choosing three or four edges and isometries between each. Explain why the resulting quotient space $P/\sim$ is not a topological surface.

4. Take $N = 5$. Then not all edges can be paired. So pair 4 of the 5. Show that the resulting surface has the left-out edge as a boundary component.

a) Show how to construct a torus with a disc cut out.

5. When $N = 2k$ describe a pairing that leads to a sphere. In your pairing, the image of the N vertices lead to some number of vertices on the

[Open ended problem]

Notation. Label the edges of our N-gon by $[N] = \{1, 2, 3, 4, \ldots, N\}$ in order. A perfect pairing is a partition of $[N] = \{1, 2, 3, 4, \ldots, N\}$ into couples, so a collection of pairs whose union is $[N]$. Of course N must be even for a perfect pairing to exist. For each pair in a perfect pairing choose the orientation reversing isometry. Then the resulting quotient S is guaranteed to be a compact oriented surface without boundary.

a) [Combinatorics] How many perfect pairings are there among $N = 2k$ objects?

We want to know what is the probability that a "randomly chosen" surface made from our N-gon has a particular genus?

You can take your own route here. Here are some ideas

i) Among all these perfect pairings, how many yield spheres?

ii) What is the maximum genus surface you can make with a perfect pairing of the $N = 2k$ edges?

iii) P has N vertices. Their image in S consists of less than N points since some are identified. What are the maximum and minimum number of points you can get in S as images of vertices?

iv) For each genus g less than this maximum possible genus, what is the number of pairings which leads to a surface of genus g?