Problem Set 5 (Due 2/12/21 11:59PM)

Your Name:

Names of Any Collaborators:

Please review the problem set guidelines in the syllabus before turning in your work. Make use of theorems like the Triangle Inequality for Contour Integrals, the Fundamental Theorem of Contour Integrals, and the Cauchy-Goursat theorem, where applicable.

P1. Let

$$f(z) = \frac{z^2 + 2}{(z^2 + 3)(z^2 + 2z + 1)}$$

and let C_R denote the semicircle of radius R parameterized by $z(t) = Re^{it}$ with $t \in [0, \pi]$. Show that

$$\lim_{R\to\infty} \int_{C_R} f(z)\, dz = 0.$$

P2. Let C be a positively oriented simply closed contour and let R be the region consisting of C and its interior.

(a) Show that the area A of the region R is given by the formula[1]

$$A = \frac{1}{2i} \int_C \overline{z}\, dz.$$

(b) Compute the area A of the region enclosed by the *cardioid* C with parameterization $z(t) = \frac{1}{2} + e^{it} + \frac{1}{2}e^{2it}$ where $t \in [0, 2\pi]$.

P3. Let C be a closed contour and let $z_0 \in \mathbb{C}$ be a point not lying on C. The *winding number* of C about z_0 is defined by the integral

$$n(C, z_0) = \frac{1}{2\pi i} \int_C \frac{1}{z - z_0}\, dz.$$

(a) Compute $n(C_1, z_0)$ where C_1 is parameterized by $z(t) = z_0 + Re^{it},\ t \in [0, 2k\pi], k \in \mathbb{Z},\ R > 0$.

(b) Compute $n(C_2, z_0)$, where C_2 is any circle and z_0 is any point not lying on or interior to C_2.

(c) Let C_3 be any closed contour and z_0 any point not lying on C_3, parameterized by $z : [a, b] \to \mathbb{C}$. For any such contour, we can always find real-valued (piece-wise) differentiable functions $r, \theta : [a, b] \to \mathbb{R}$ with $r(t) > 0$ such that $z(t) = z_0 + r(t)e^{i\theta(t)}$. Compute $n(C_3, z_0)$.

(d) Give a geometric interpretation of the winding number. You must justify your interpretation.

[1] Hint: Green's theorem

P4. Let $f(z) = \frac{1}{z^2-1}$. Determine all possible values of the integral

$$\int_{C^{(k)}} f(z)\, dz$$

where $C^{(k)}$ is any circle, traversed k-times, and not passing through 1 and -1. You must justify your claim.

P5. Let $a, b \in \mathbb{C}$ and let C_R be the circle of radius R centered at the origin, traversed once in the positive orientation. If $|a| < R < |b|$, show that

$$\int_{C_R} \frac{1}{(z-a)(z-b)}\, dz = \frac{2\pi i}{a-b}.$$

P6. Let $f(z) = \frac{1}{z^2+1}$. Determine whether f has an antiderivative on the given domain D. You must prove your claims.

(a) $D = \mathbb{C} \setminus \{i, -i\}$;

(b) $D = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$