Problem Set 3 (Due 1/29/21 11:59PM)

Your Name:

Names of Any Collaborators:

Please review the problem set guidelines in the syllabus before turning in your work.

P1. Let U be a domain. Prove that if f is a real-valued function analytic on U, then f is constant on U.

P2. Let $f(z) = u + iv$ be a complex-valued function defined on an open set $U \subseteq \mathbb{C}$. Suppose that the first-order partial derivatives of $\operatorname{Re} f = u$ and $\operatorname{Im} f = v$ exist and are continuous on U. Define the differential operators[1]

$$\frac{\partial f}{\partial z} \stackrel{\text{def}}{=} \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \quad \text{and} \quad \frac{\partial f}{\partial \overline{z}} \stackrel{\text{def}}{=} \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

Note: we also define $\frac{\partial f}{\partial x} \stackrel{\text{def}}{=} \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$, and similarly for $\frac{\partial}{\partial y}$.

(a) Show that f is analytic on U if and only if and only if $\frac{\partial f}{\partial \overline{z}} = 0$.

(b) If f is analytic on U, show that $f' = \frac{\partial f}{\partial z}$.

P3. **(Optional.)**[2] Let U be an open set and let f be a function that is continuous on U with the property

$$e^{f(z)} = z, \ \ z \in U.$$

(a) Show that f is analytic on U.[3]

(b) Consider the function $f(z) = \log z \stackrel{\text{def}}{=} \ln|z| + i \arg z,\ |z| > 0,\ -\pi < \arg z \leq \pi$. What, if anything, does part (a) tell you about f? Justify your conclusions.

P4. Find all solutions to the equation $e^{2z} - 2ie^z = 1$.

P5. (a) Find real valued functions $u, v : \mathbb{R}^2 \to \mathbb{R}$ such that $e^z = u(x, y) + iv(x, y)$.

(b) Find real valued functions $U, V : \mathbb{R}^2 \to \mathbb{R}$ such that $\cos z = U(x, y) + iV(x, y)$.

(c) Show that $e^{\overline{z}} = \overline{e^z}$ and $\cos \overline{z} = \overline{\cos z}$.

P6. Determine the points at which the following functions are analytic:

(a) $e^{\overline{z}}$;

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[1] These operators are inspired by the following formal (meaningless) calculation. Consider the functions $x(z, \overline{z}) = \frac{z+\overline{z}}{2}$ and $y(z, \overline{z}) = \frac{z-\overline{z}}{2i}$ (note $z = x + iy$) and formally apply the chain rule

$$\begin{aligned} \frac{\partial}{\partial z} &= \frac{\partial}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial}{\partial y}\frac{\partial y}{\partial z} = \frac{\partial}{\partial x}\frac{1}{2} + \frac{\partial}{\partial y}\frac{1}{2i}, \\ \frac{\partial}{\partial \overline{z}} &= \frac{\partial}{\partial x}\frac{\partial x}{\partial \overline{z}} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \overline{z}} = \frac{\partial}{\partial x}\frac{1}{2} - \frac{\partial}{\partial y}\frac{1}{2i}. \end{aligned}$$

[2] This is worth 3 points extra credit, added to your score on this assignment.

[3] This shows that a continuously defined logarithm on an open set is automatically analytic.

(b) $\cos \overline{z}$.

P7. Let $z \in \mathbb{C}$.

(a) Prove that ~~1^z~~ $|1^z|$ is single-valued if and only if $\operatorname{Im} z = 0$.

(b) Find a necessary and sufficient condition for $|i^{iz}|$ to be single-valued.

(c) **(Optional.)**[4] Show by counterexample that the statement is false: 1^z is single-valued if and only if $\operatorname{Im} z = 0$.

[4] Worth 1 point extra credit, added to your score on this assignment.