Weekly Assignment 3 Solutions

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MATH 117: Advanced Linear Algebra

August 22, 2023

Some hints for this assignment are written in the footnotes. See the weekly assignment webpage for due dates, templates, and assignment description. Make sure to justify any claims you make. You may not appeal to any results that we have not discussed in class.

1. Let V be finite-dimensional vector space and W a subspace. Suppose that $\{b_1, \ldots, b_k\}$ is a basis for W and extend this to a basis $\{b_1, \ldots, b_k, b_{k+1}, \ldots, b_n\}$ for V using Proposition 1.4.11. Prove that the set of vectors $\{b_{k+1} + W, \ldots, b_n + W\}$ is a basis for the quotient space V/W.

Proof. It suffices to show that $\{b_{k+1} + W, \ldots, b_n + W\}$ is an independent set. Indeed, if it is an independent set, then the vectors are all distinct which implies that

$$|\{b_{k+1} + W, \ldots, b_n + W\}| = n - k.$$

But we know that $V \cong W \oplus V/W$ so that

$$\dim V/W = \dim V - \dim W = n - k = |\{b_{k+1} + W, \ldots, b_n + W\}|.$$

Thus, if $\{b_{k+1} + W, \ldots, b_n + W\}$ is an independent set, then it is automatically a basis.

In order to show that the set is independent, suppose that

$$\sum_{i=k+1}^{n} \alpha_i (b_i + W) = W.$$

Then

$$\sum_{i=k+1}^{n} \alpha_i b_i \in W.$$

Hence, there exist $\alpha_1, \ldots, \alpha_k \in F$ such that

$$\sum_{i=k+1}^{n} \alpha_i b_i = \sum_{i=1}^{k} \alpha_i b_i$$

But $\{b_1, \ldots, b_n\}$ is a basis for V, hence, $\alpha_i = 0$ for all $i = 1, \ldots, n$. ☐

Definition 1. *Let V be a vector space and let S be a subset of V. The* annihilator S^0 *of S is the set of linear functionals whose kernel contains S, that is,*

$$S^0 := \{f \in V^* : f(v) = 0 \text{ for all } v \in S\} \subset V^*.$$

2. Let V be a vector space.

(a) Let S be a subset of V. Prove that S^0 is a subspace of V^*.

Proof.
The zero function annihilates S, so S^0 is nonempty. If $f, g \in S^0$, $\alpha \in F$, and $v \in S$, then

$$(\alpha f + g)(v) = \alpha f(v) + g(v) = 0 + 0 = 0.$$

Hence, $\alpha f + g \in S^0$. This proves S^0 is a subspace. □

(b) Let W be a subspace of V. Prove that W^0 is isomorphic to $(V/W)^*$.[1]

Proof. Define a function $\Phi : W^0 \to (V/W)^*$ as follows. Every $f \in W^0$ annihilates W, so the Unviersal Property of the Quotient can be invoked. Given $f \in W^0$, define $\Phi(f) : V/W \to F$ to be the unique linear functional satisfying $\Phi(f) \circ \pi = f$, where $\pi : V \to V/W$ is the quotient map. This function is actually a linear map. Indeed, let $f, g \in W^0$ and $\alpha \in F$. Then for any $v \in V$,

$$\begin{aligned}(\Phi(\alpha f + g))(v + W) &= (\Phi(\alpha f + g) \circ \pi)(v) \\ &= (\alpha f + g)(v) \\ &= \alpha f(v) + g(v) \\ &= \alpha(\Phi(f) \circ \pi)(v) + (\Phi(g) \circ \pi)(v) \\ &= \alpha(\Phi(f))(v + W) + (\Phi(g))(v)\end{aligned}$$

which proves that $\Phi(\alpha f + g) = \alpha\Phi(f) + \Phi(g)$. Define another function $\Psi : (V/W)^* \to W^0$ via $\Phi(\bar{f}) = \bar{f} \circ \pi$. Clearly, $\Phi(\bar{f}) \in W^0$ since π annihilates W. The Universal Property of the Quotient guarantees that Φ and Ψ are mutually inverse bijections. □

(c) Suppose that V is finite-dimensional and let W be a subspace of V. By part (b), $\dim(W^0) = \dim V - \dim W$. Provide another proof of this equation using dual bases.[2]

Proof. Start with a basis $\{b_1, \ldots, b_k\}$ for W and extend to a basis $B = \{b_1, \ldots, b_n\}$ for V. Let $B^* = \{\varphi_1, \ldots, \varphi_n\}$ be the dual basis. I claim that $\{\varphi_{k+1}, \ldots, \varphi_n\}$ is a basis for W^0. Let $w \in W$ and $k + 1 \le j \le n$. Then $w = \sum_{i=1}^k \alpha_i b_i$ for some $\alpha_1, \ldots, \alpha_k \in F$ and

$$\varphi_j(w) = \sum_{i=1}^k \alpha_i \varphi_j(b_i) = \sum_{i=1}^k \alpha_i \delta_{ji} = 0$$

since $j > k$. Thus, $\{\varphi_{k+1}, \ldots, \varphi_n\} \subset W^0$. By definition of dual basis, they are already independent. Thus, it suffices to show that they span W^0. Let $f \in W^0$. Since $f \in V$, we can write $f = \sum_{i=1}^n \alpha_i \varphi_i$ for some $\alpha_1, \ldots, \alpha_n \in F$. Evaluating the equation at b_j for $1 \le j \le k$ yields

$$0 = f(b_j) = \sum_{i=1}^n \alpha_i \varphi_i(b_j) = \sum_{i=1}^n \alpha_i \delta_{ij} = \alpha_j$$

since f annihilates W. Thus,

$$f = \sum_{i=k+1}^n \alpha_i \varphi_i$$

which proves the claim. Now the dimension formula follows immediately because $\dim(W^0) = n - k = \dim(V) - \dim(W)$. □

[1]Hint: Universal Property of the Quotient.

[2]Hint: Start with a basis for W and extend to a basis for V. Can you use the corresponding dual basis to construct a basis for W^0?

Definition 2. *Let V be a vector space. A bilinear form $B : V \times V \to F$ is called* reflexive *if $B(v, v') = 0$ implies $B(v', v) = 0$ for all $v, v' \in V$. The* radical *of a reflexive bilinear form is the set*

$$\mathrm{rad}(V) := \{v \in V : B(v, v') = 0 \textit{ for all } v' \in V\}.$$

A reflexive bilinear form is called nondegenerate *if* $\mathrm{rad}(V) = \{0\}$.

3. Let V be a vector space. Let $B : V \times V \to F$ be a bilinear form on V.

(a) For any $v \in V$, define a function $\Phi_B(v) : V \to F$ by the rule $(\Phi_B(v))(w) = B(v, w)$. Show that $\Phi_B(v)$ is a linear functional and show that the assignment $v \mapsto \Phi_B(v)$ defines a linear map $\Phi_B : V \to V^*$.

Proof. The function $\Phi_B(v)$ is linear because B is linear the second component - easy to check. Thus, Φ_B defines a function from V to V^*. The function Φ_B is linear because B is linear in the first component, but its slightly less obvious because Φ_B takes values in V^*. Indeed, let $u, v, w \in V$ and $\alpha \in F$. Then

$$\begin{aligned}(\Phi_B(\alpha u + v))(w) &= B(\alpha u + v, w) \\ &= \alpha B(u, w) + B(v, w) \\ &= \alpha(\Phi_B(u))(w) + (\Phi_B(v))(w)\end{aligned}$$

which shows that $\Phi_B(\alpha u + v) = \alpha\Phi_B(u) + \Phi_B(v)$. This proves the claim. □

(b) Suppose that V is finite-dimensional and that B is reflexive and nondegenerate.

(i) Prove that Φ_B is an isomorphism.

Proof. Since V and V^* are isomorphic, it suffices to prove that Φ_B is injective. Suppose that $v \in \ker \Phi_B$. Then $\Phi_B(v)$ is the zero map. Then for any $w \in V$, we have

$$0 = (\Phi_B(v))(w) = B(v, w).$$

This implies that $v \in \mathrm{rad}(W) = \{0\}$. Thus, $v = 0$ and Φ_B is injective.

Note: the hypothesis that B is reflexive was not used. The only reason to include this hypothesis was to avoid defining left and right radicals. Evidently, this statement is true for an arbitrary bilinear form whose right (left?) radical is the zero space. □

(ii) Let W be a subspace of V. Describe the preimage $W^\perp := \Phi_B^{-1}(W^0)$ of W^0 under Φ_B. In particular, $W^0 \cong W^\perp$.

Proof. We have

$$\begin{aligned}W^\perp = \Phi_B^{-1}(W^0) &= \{v \in V : \varphi_B(v) \in W^0\} \\ &= \{v \in V : W \subseteq \ker \Phi_B(v)\} \\ &= \{v \in V : B(v, w) = 0 \text{ for all } w \in W\}.\end{aligned}$$

This should convince you that the notation $W^\perp$ is appropriate. For example, if B is the dot product on $\mathbb{R}^n$ (an example of a reflexive, nondegenerate bilinear form), then $W^\perp$ is just the orthogonal complement of W! □

(iii) Suppose that B is nondegenerate when restricted to W, i.e., $\mathrm{rad}(W) = \{0\}$. Prove that $V = W \oplus W^{\perp}$.

Proof. First, observe that $W^{\perp} \cong W^0 \cong (V/W)^* \cong V/W$ by the preceding results. Also, $V \cong W \oplus V/W$. Thus,

$$\begin{aligned}\dim(W + W^{\perp}) &= \dim(W) + \dim(W^{\perp}) - \dim(W \cap W^{\perp}) \\ &= \dim(W) + \dim(V/W) - \dim(W \cap W^{\perp}) \\ &= \dim(W \oplus V/W) - \dim(W \cap W^{\perp}) \\ &= \dim(V) - \dim(W \cap W^{\perp}).\end{aligned}$$

Thus, in order to prove $V = W \oplus W^{\perp}$, it suffices to show that $W \cap W^{\perp} = \{0\}$. But this follows directly from the hypothesis because

$$\mathrm{rad}(W) = \{w \in W : B(w, w') = 0 \text{ for all } w' \in W\} = W \cap W^{\perp}.$$

This completes the proof. □

4. (i) Suppose that $L_1 : V_1 \to W_1$ and $L_2 : V_2 \to W_2$ are linear maps. Prove that there is a unique linear map

$$L_1 \otimes L_2 : V_1 \otimes V_2 \to W_1 \otimes W_2$$

with the property that $(L_1 \otimes L_2)(v_1 \otimes v_2) = L(v_1) \otimes L(v_2)$ for all $v_1 \in V_1$ and $v_2 \in V_2$.[3]

Proof. Define a map $B : V_1 \times V_2 \to W_1 \otimes W_2$ via $B(v_1, v_2) = L(v_1) \otimes L(v_2)$. Then B is bilinear because L_1, L_2 are linear and $- \otimes -$ is bilinear. Details left to the motivated student. Thus, according to the Universal Property of the Tensor Product, there is a unique linear map $L_1 \otimes L_2 : V_1 \otimes V_2 \to W_1 \otimes W_2$ with the property that

$$(L_1 \otimes L_2)(v_1 \otimes v_2) = B(v_1, v_2) = L(v_1) \otimes L(v_2).$$

This proves the claim. □

(ii) Let $F = \mathbb{Z}_5$ and let $V = F^2$. Let $L : V \to V$ be the linear map defined by left multiplication with the matrix $A = \left(\begin{smallmatrix} 0 & 1 \\ 4 & 2 \end{smallmatrix}\right)$. Let $E = (e_1, e_2)$ denote the standard basis for V. Compute the matrix

$$[L \otimes L]_B$$

for linear map $L \otimes L : V \otimes V \to V \otimes V$, where B is the basis $(e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2)$ for $V \otimes V$.

Solution. Note that $B = E \otimes E$ as we saw in the lecture. One can easily show that $[(a, b) \otimes (c, d)]_B = (ac, ad, bc, bd)$ for any $a, b, c, d \in F$, so coordinate vectors are actually easy to compute. We have

$$\begin{aligned}[(L \otimes L)(e_1 \otimes e_1)]_B &= [L(e_1) \otimes L(e_1)]_B = [(0, 4) \otimes (0, 4)]_B = (0, 0, 0, 1), \\ [(L \otimes L)(e_1 \otimes e_2)]_B &= [L(e_1) \otimes L(e_2)]_B = [(0, 4) \otimes (1, 2)]_B = (0, 0, 4, 3), \\ [(L \otimes L)(e_2 \otimes e_1)]_B &= [L(e_2) \otimes L(e_1)]_B = [(1, 2) \otimes (0, 4)]_B = (0, 4, 0, 3),\end{aligned}$$

and

$$[(L \otimes L)(e_2 \otimes e_2)]_B = [L(e_2) \otimes L(e_2)]_B = [(1, 2) \otimes (1, 2)]_B = (1, 2, 2, 4).$$

Thus,

$$[L \otimes L]_B = \left(\begin{smallmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 4 & 2 \\ 0 & 4 & 0 & 2 \\ 1 & 3 & 3 & 4 \end{smallmatrix}\right).$$

□

[3]Hint: Universal Property of the Tensor Product.