MATH 117: Daily Assignment 8 Solutions

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See the daily assignment webpage for due dates, templates, and assignment description. Try to explain your reasoning and justify your computations for every problem. You should not appeal to any theorems that we have not proved yet.

1. Let $V = \mathbb{R}^2$ and let $E = (e_1, e_2)$ denote the standard basis for V. Then

$$E \otimes E := (e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2)$$

is a basis for $V \otimes V$. Compute $[(5,3) \otimes (-1,2)]_{E\otimes E}$.[1]

Solution. Using bilinearity of $- \otimes -$, we have

$$\begin{aligned}(5,3) \otimes (-1,2) &= (5e_1 + 3e_2) \otimes (2e_2 - e_1) \\ &= 5e_1 \otimes 2e_2 + 5e_1 \otimes (-1)e_1 + 3e_2 \otimes 2e_2 + 3e_2 \otimes (-1)e_1 \\ &= 10(e_1 \otimes e_2) - 5(e_1 \otimes e_1) + 6(e_2 \otimes e_2) - 3(e_2 \otimes e_1).\end{aligned}$$

Thus,

$$[(5,3) \otimes (-1,2)]_{E\otimes E} = (-5, 10, -3, 6).$$

□

2. (a) Show that the cross product (from calculus) defines a bilinear map $- \times - : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$.

(b) By the Universal Property of the Tensor Product, there is a unique linear map $L : \mathbb{R}^3 \otimes \mathbb{R}^3 \to \mathbb{R}^3$ satisfying $L(u \otimes v) = u \times v$. Compute $[L]^E_{E\otimes E}$ where is the standard basis $E = (e_1, e_2, e_3)$ and $E \otimes E = (e_i \otimes e_j)$ is ordered lexicographically.

Solution. (a) I am not going to check this.

(b) Compute $L(e_i \otimes e_j)$ for each i, j using the right-hand rule. Each output will be in E, so we obtain

$$[L \otimes L]^E_{E\otimes E} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The zero columns reflect the fact that L is alternating.

□

3. (Optional, in case you want more practice)

(a) Show that matrix multiplication $- \cdot - : \mathbb{R}^{2\times 1} \times \mathbb{R}^{1\times 2} \to \mathbb{R}^{2\times 2}$ defines an $\mathbb{R}$-bilinear map.[2]

[1] Hint: Take advantage of the relations in Proposition 3.3.3.

[2] More generally, matrix multiplication defines an F-bilinear map $- \cdot - : F^{m\times n} \times F^{n\times k} \to F^{m\times k}$ where F is any field.

(b) By the Universal Property of the Tensor Product, there is a unique linear map $L : \mathbb{R}^{2\times 1} \otimes \mathbb{R}^{1\times 2} \to \mathbb{R}^{2\times 2}$ satisfying $L\left(\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} z & w \end{smallmatrix}\right)\right) = \left(\begin{smallmatrix} xz & xw \\ yz & yw \end{smallmatrix}\right)$. Compute $[L]_B^E$ where

$$B = \{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 0 & 1 \end{smallmatrix}\right)\}$$

and E is the standard basis for $\mathbb{R}^{2\times 2}$.

Solution. (a) Straightforward to verify.

(b) The images of B under L are precisely the elements of E, in the usual order. Thus, $[L]_B^E = I_4$, the 4×4 identity matrix. I didn't realize how boring this problem would be!

□