MATH 117: Daily Assignment 6

WRITE YOUR NAME HERE

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See the daily assignment webpage for due dates, templates, and assignment description. Try to explain your reasoning and justify your computations for every problem. You should not appeal to any theorems that we have not proved yet.

1. Let $F = \mathbb{Z}_2$, $V = F_3[x]$ and $W = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in F^{2\times 2} : a + d = 0 \right\}$. Define a linear map $L : V \to W$ via

$$L\left(a + bx + cx^2 + dx^3\right) = \left(\begin{smallmatrix} b+c & c+d \\ c+d & b+c \end{smallmatrix} \right).$$

Complete the following steps (the point is that computations involving linear maps are best done using matrices).

(a) Compute a basis B for V and a basis C for W. We have already seen bases for both, but you should make sure you can come up with them on your own.

(b) Compute $[L]_B^C$.

(c) Compute a basis for the column space of $[L]_B^C$ using methods from Section 2.9.

(d) What is the rank of $[L]_B^C$? What is the nullity of $[L]_B^C$? You should be able to compute the nullity without computing a basis for the nullspace.

(e) Recall that C defines a coordinate isomorphism $\varphi_C : W \to F^3,\ w \mapsto [w]_C$. This isomorphism sends any basis for $\operatorname{im}(L)$ to a basis for the column space of $[L]_B^C$, and vice-versa. Compute a basis for $\operatorname{im}(L)$ by evaluating φ^{-1} at the basis you found in part (c).

(f) Compute a basis for the null space of $[L]_B^C$. It's easy to find a spanning set using a standard trick. Explain using part (d) why it's actually a basis.

(g) The coordinate isomorphism $\varphi_B : V \to F^4,\ v \mapsto [v]_B$ sends any basis for $\ker(L)$ to a basis for the null space of $[L]_B^C$, and vice-versa. Compute a basis for $\ker(L)$ by evaluating φ_B^{-1} at the basis you found in part (f).

2. Let $V = \mathbb{R}[x]$ and let $W = \{p(x)(1 + x^2) : p(x) \in V\}$. Convince yourself that W is a subspace. Then use the First Isomorphism Theorem to construct an isomorphism $V/W \to \mathbb{C}$ of $\mathbb{R}$-vector spaces.