MATH 117: Daily Assignment 5 Solutions

Jadyn V. Breland

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Some hints may be written in the footnotes. See the daily assignment webpage for due dates, templates, and assignment description. Try to explain your reasoning and justify your computations for every problem. You should not appeal to any theorems that we have not proved yet.

1. For each part, you are given a vector space V over a field F with ordered bases B and C and a linear operator $L : V \to V$. Compute $[L]_B^C$.[1]

(a) $F = \mathbb{Z}_3$, $V = F_2[x]$, $B = (1+x, 1+x^2, x+x^2)$, $C = (2, x^2, 1+x)$, $L(a+bx+cx^2) = b+cx+ax^2$. Note: I am using the notation 0,1,2 for the elements of $F = \mathbb{Z}_3$. So for instance $1+2=0$ in F.

(b) $F = \mathbb{Z}_5$, $V = F^2$, $B = ((1,4),(2,4))$, $C = ((1,1),(2,1))$, $L((a,b)) = (2a-b, 3a)$. Note: I am using the notation 0,1,2,3,4 for the elements of $F = \mathbb{Z}_5$. So for instance, $3 \cdot 4 = 2$ in F.

(c) $F = \mathbb{Z}_2$, $V = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in F^{2\times 2} : a+d=0 \right\}$, $B = (\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right))$,
$C = (\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right))$, $L\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) = \left(\begin{smallmatrix} a+b+c & c \\ b & a+b+c \end{smallmatrix}\right)$.

Solution. (a) Since L permutes B, it is easy to calculate $[L]_B = \left(\begin{smallmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix}\right)$. Then using Daily 4.1(a), we have $[L]_B^C = T_B^C [L]_B = \left(\begin{smallmatrix} 0 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{smallmatrix}\right)$.

(b) Let E be the standard basis. Then $[L]_B^E = \left(\begin{smallmatrix} 3 & 0 \\ 3 & 1 \end{smallmatrix}\right)$ and $T_E^C = \left(\begin{smallmatrix} 1 & 2 \\ 1 & 1 \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} 4 & 2 \\ 1 & 4 \end{smallmatrix}\right)$. Thus

$$[L]_B^C = T_E^C [L]_B^E = \left(\begin{smallmatrix} 4 & 2 \\ 1 & 4 \end{smallmatrix}\right)\left(\begin{smallmatrix} 3 & 0 \\ 3 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 3 & 2 \\ 0 & 4 \end{smallmatrix}\right).$$

(c) In this case, there is no standard basis to take advantage of. Just compute $[L]_B^C$ using the definition directly, or compute the product $T_B^C [L]_B$ using the transition matrix from Daily 4.1(c). The former is actually easier in this problem. In the end, I got $[L]_B^C = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{smallmatrix}\right)$.

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[1] The transition matrices from Daily 4 might be useful....