

Weekly Assignment 3

WRITE YOUR NAME HERE
MATH 117: Advanced Linear Algebra

August 7, 2022

Some hints for this assignment are written in the footnotes. See the [weekly assignment webpage](#) for due dates, templates, and assignment description. Make sure to justify any claims you make. You may not appeal to any results that we have not discussed in class.

Definition 1. Let V be a vector space and let S be a subset of V . The annihilator S^0 of S is the set of linear functionals whose kernel contains S , that is,

$$S^0 := \{f \in V^* : f(v) = 0 \text{ for all } v \in S\} \subset V^*.$$

1. Let V be a vector space.

- (a) Let S be a subset of V . Prove that S^0 is a subspace of V^* .
- (b) Let W be a subspace of V . Prove that W^0 is isomorphic to $(V/W)^*$.¹
- (c) Suppose that V is finite-dimensional and let W be a subspace of V . By part (b), $\dim(W^0) = \dim V - \dim W$. Provide another proof of this equation using dual bases.²

Proof. Write your proof here.

□

Definition 2. Let V be a vector space. A bilinear form $B : V \times V \rightarrow F$ is called reflexive if $B(v, v') = 0$ implies $B(v', v) = 0$ for all $v, v' \in V$. The radical of a reflexive bilinear form is the set

$$\text{rad}(V) := \{v \in V : B(v, v') = 0 \text{ for all } v' \in V\}.$$

A reflexive bilinear form is called nondegenerate if $\text{rad}(V) = \{0\}$.

2. Let V be a vector space. Let $B : V \times V \rightarrow F$ be a bilinear form on V .

- (a) For any $v \in V$, define a function $\Phi_B(v) : V \rightarrow F$ by the rule $(\Phi_B(v))(w) = B(v, w)$. Show that $\Phi_B(v)$ is a linear functional and show that the assignment $v \mapsto \Phi_B(v)$ defines a linear map $\Phi_B : V \rightarrow V^*$.
- (b) Suppose that V is finite-dimensional and that B is reflexive and nondegenerate.
 - (i) Prove that Φ_B is an isomorphism.
 - (ii) Let W be a subspace of V . Describe the the preimage $W^\perp := \Phi_B^{-1}(W^0)$ of W^0 under Φ_B . In particular, $W^0 \cong W^\perp$.

¹Hint: Universal Property of the Quotient.

²Hint: Start with a basis for W and extend to a basis for V . Can you construct a basis for W^0 using these vectors?

- (iii) Suppose that B is nondegenerate when restricted to W , i.e., $\text{rad}(W) = \{0\}$. Prove that $V = W \oplus W^\perp$.³

Proof. Write your proof here.

□

3. (i) Suppose that $L_1 : V_1 \rightarrow W_1$ and $L_2 : V_2 \rightarrow W_2$ are linear maps. Prove that there is a unique linear map

$$L_1 \otimes L_2 : V_1 \otimes V_2 \rightarrow W_1 \otimes W_2$$

with the property that $(L_1 \otimes L_2)(v_1 \otimes v_2) = L_1(v_1) \otimes L_2(v_2)$ for all $v_1 \in V_1$ and $v_2 \in V_2$.⁴

- (ii) Let $F = \mathbb{Z}_5$ and let $V = F^2$. Let $L : V \rightarrow V$ be the linear map defined by left multiplication with the matrix $A = \begin{pmatrix} 0 & 1 \\ 4 & 2 \end{pmatrix}$. Let $E = (e_1, e_2)$ denote the standard basis for V . Compute the matrix

$$[L \otimes L]_B$$

for linear map $L \otimes L : V \otimes V \rightarrow V \otimes V$, where B is the basis $(e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2)$ for $V \otimes V$.

Proof. Write your proof here.

□

4. (i) Suppose that V is a finite-dimensional vector space. Use the Universal Property of the Tensor Product to construct an isomorphism

$$\Psi : V \otimes V^* \rightarrow \text{End}_F(V)$$

satisfying $(\Psi(v \otimes f))(w) = f(v)w$ for all $v, w \in V$ and $f \in V^*$.

- (ii) Let $F = \mathbb{Z}_5$ and let $V = F^3$. Define a linear functional $\varphi \in V^*$ via $f(x, y, z) = x + z$. Let $E = (e_1, e_2, e_3)$ denote the standard basis for V . Compute the following matrix

$$[\Psi((1, 2, 4) \otimes \varphi)]_E.$$

Proof. Write your proof here.

□

³Hint: Use 1(c) and 2(b)(ii).

⁴Hint: Universal Property of the Tensor Product.