Chain Rule Let $f: \mathbb{R}^n \to \mathbb{R}^m$ and $g: \mathbb{R}^m \to \mathbb{R}^p$ be differentiable at $x_0 \in \mathbb{R}^n$ and $y_0 = f(x_0)$, respectively. Then $g \circ f$ is differentiable at x_0 and the Jacobian is

matrix multiplication

$$\underbrace{D(g\circ f)(x_0)}_{p \times n} = \underbrace{Dg(y_0)}_{p \times m} \cdot \underbrace{Df(x_0)}_{m \times n}$$

Special Cases

(1) (Derivative of a map along a path) Let $f: \mathbb{R}^n \to \mathbb{R}$ and $c: \mathbb{R} \to \mathbb{R}^n$ and form $h: \mathbb{R} \to \mathbb{R}$, $h = f \circ c$. By Chain Rule,

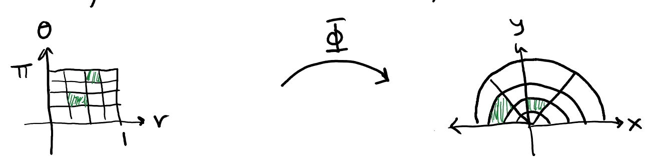
$c(t) = (x_1(t), \dots, x_n(t))$

$x_i: \mathbb{R} \to \mathbb{R}$

$$\frac{dh}{dt} = Dh = Df \cdot Dc$$
$$= \left[\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{x_n} \right] \begin{bmatrix} \frac{dx_1}{dt} \\ \vdots \\ \frac{dx_n}{dt} \end{bmatrix}$$
$$= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{dx_i}{dt}$$
$$= \boxed{\nabla f \cdot c'(t).}$$

(2) (Change of coordinates) Let $f: \mathbb{R}^2 \to \mathbb{R}$ and $g: \mathbb{R}^2 \to \mathbb{R}^2$ with component functions $g(x,y) = (u(x,y), v(x,y))$. The map g "changes the coordinates" from (u,v) to (x,y) (when we compose f w/ g).

(i) (Polar Coordinates) $\Phi(r,\theta) = (r\cos\theta, r\sin\theta)$, $\Phi: \mathbb{R}^2 \to \mathbb{R}^2$



(ii) (Spherical Coordinates) $\Psi(\rho,\theta,\phi) = (\rho\sin\phi\sin\theta, \rho\sin\phi\cos\theta, \rho\cos\phi)$

Form $h(x,y) = f \circ g(x,y)$, $g(x,y) = (u(x,y), v(x,y))$. By chain rule,

$$\left[\frac{\partial h}{\partial x} \ \frac{\partial h}{\partial y} \right] = Dh = Df \cdot Dg$$
$$= \left[\frac{\partial f}{\partial u} \ \frac{\partial f}{\partial v} \right] \left[\frac{\partial u}{\partial x} \ \frac{\partial u}{\partial y} \right]$$

$$\left[\frac{\partial h}{\partial x} \ \ \frac{\partial h}{\partial y}\right] = Dh = Df \cdot Dg$$
$$= \left[\frac{\partial f}{\partial u} \ \ \frac{\partial f}{\partial v}\right]\begin{bmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{bmatrix}$$

Comparing entries gives formulas for $\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}$ in terms of $\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}$.

$$\frac{\partial h}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} \qquad \frac{\partial h}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y}$$

Back to polar coordinates: $f:\mathbb{R}^2 \to \mathbb{R}$, $\Phi(r,\theta) = (r\cos\theta, r\sin\theta)$. By Chain Rule:

$$\left[\frac{\partial f}{\partial r} \ \ \frac{\partial f}{\partial \theta}\right] = D f\circ\Phi = Df\, D\Phi$$
$$= \left[\frac{\partial f}{\partial x} \ \ \frac{\partial f}{\partial y}\right]\begin{bmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{bmatrix}$$

Comparing entries:

$$\frac{\partial f}{\partial r} = \cos\theta\frac{\partial f}{\partial x} + \sin\theta\frac{\partial f}{\partial y} \qquad \frac{\partial f}{\partial \theta} = -r\sin\theta\frac{\partial f}{\partial x} + r\cos\theta\frac{\partial f}{\partial y}$$

$\blacksquare$

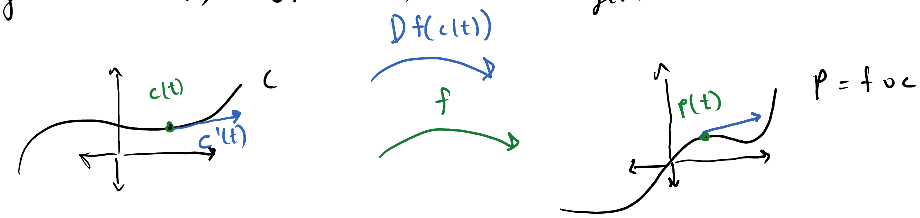
<u>Ex</u> (Geometric interpretation of the Jacobian) Let $f:\mathbb{R}^2 \to \mathbb{R}^2$ and $c:\mathbb{R} \to \mathbb{R}^2$. Composing with f gives a new path $p(t) = f\circ c(t)$. By chain rule,

$$p'(t) = Dp(t) = Df(c(t))\cdot Dc(t)$$
$$= Df(c(t))\ c'(t) \qquad \left(c'(t) = \begin{bmatrix}x'(t)\\ y'(t)\end{bmatrix}\right)$$

a vector tangent to $p(t)$

a vector tangent to $c(t)$

This equation says the linear map $L_{Df(c(t))}:\mathbb{R}^2 \to \mathbb{R}^2$, $L\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = Df(c(t))\begin{bmatrix}x\\y\end{bmatrix}$, maps tangent vectors of $c(t)$ to tangent vectors of $p(t)$.



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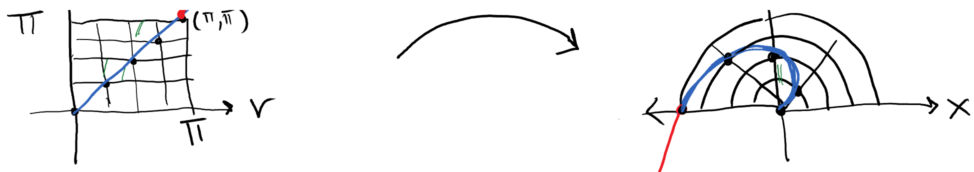
<u>Ex</u> Consider $c(t) = (t,t)$ and $\Phi:\mathbb{R}^2 \to \mathbb{R}^2$, $\Phi(r,\theta) = (r\cos\theta, r\sin\theta)$.

Compose to get a new path $p(t) = \Phi\circ c(t) = (t\cos\theta, t\sin t)$ (spiral)



By chain rule, $\quad p'(t) = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Then $p'(\pi) = \begin{bmatrix} -1 & 0 \\ 0 & -\pi \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$= \begin{bmatrix} -1 \\ \pi \end{bmatrix}$$

∎

<u>Gradient</u> Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable. The gradient of f is the map $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \ \ldots\ldots, \ \frac{\partial f}{\partial x_n} \right)$$

The gradient assigns to each point $x_0 \in \mathbb{R}^n$ a vector $\nabla f(x_0) \in \mathbb{R}^n$.

∎

<u>Directional Derivative</u> Let $f : \mathbb{R}^3 \to \mathbb{R}$ be differentiable and fix $x, v \in \mathbb{R}^n$. Form the line $c(t) = x + tv$. Then $f \circ c$ maps $\mathbb{R} \to \mathbb{R}$. The directional deriv. of f at x in the direction of v is

$c : \mathbb{R} \to \mathbb{R}^3$

$$\frac{d}{dt}(f \circ c)\Big|_{t=0} = \frac{d}{dt} f(x+tv)\Big|_{t=0}.$$

By chain rule Special Case #1,

$$\boxed{\begin{aligned} \frac{d}{dt}(f \circ c)\Big|_{t=0} &= \nabla f(c(0)) \cdot c'(0) \\ &= \nabla f(x) \cdot v \end{aligned}}$$

∎

Usually, v is taken to be a unit vector.

<u>Thm</u> Assume $\nabla f(x) \neq 0$. Then $\nabla f(x)$ points in the direction of steepest ascent of f.

<u>Proof</u> Let n be a unit vector. The rate of change of f in the direction of n is given by

$$\begin{aligned} \nabla f(x) \cdot n &= \|\nabla f(x)\| \, \|n\| \cos\theta \\ &= \|\nabla f(x)\| \cos\theta \end{aligned}$$

This is maximized when $\theta = 0$. If $\theta = 0$, $\nabla f(x)$ and n are parallel.

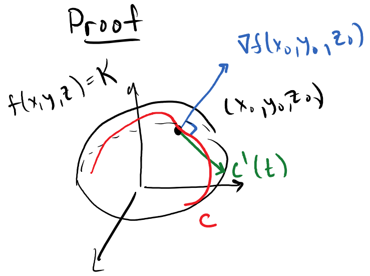
∎

<u>Tangent Plane to Level Curves</u>

<u>Thm</u> Let $f : \mathbb{R}^3 \to \mathbb{R}$ be differentiable and consider the level surface S defined by $f(x,y,z) = K$, $K \in \mathbb{R}$. If $(x_0, y_0, z_0) \in S$, then $\nabla f(x_0, y_0, z_0)$

is perpendicular to S.

Proof



Let $c(t)$ be any path in S such that $c(0)=(x_0,y_0,z_0)$.
We need to show: $\nabla f(x_0,y_0,z_0)\cdot c'(0)=0$.
We have

$$\begin{aligned}\nabla f(x_0,y_0,z_0)\cdot c'(0) &= \nabla f(c(0))\cdot c'(0)\\ &= \frac{d}{dt}(f\circ c)\Big|_{t=0} \qquad (\text{Special Case } \#1)\\ &= \frac{d}{dt}(K)\Big|_{t=0}\\ &= 0.\end{aligned}$$

■

Def By the Theorem, the tangent plane to a level surface $f(x,y,z)$ is given by the eq.

$$\boxed{\nabla f(x_0,y_0,z_0)\cdot(x-x_0,\,y-y_0,\,z-z_0)=0}$$

This generalizes to maps $f:\mathbb{R}^n\to\mathbb{R}$. In this case, $f(x)=K$ defines an n-dimensional hypersurface. The same equation

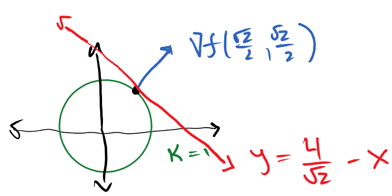
$$\nabla f(x_0)\cdot(x-x_0)=0$$

defines an $(n-1)$-dimensional tangent space to the level set. ■

Ex $(n=2)$ Consider $f(x,y)=x^2+y^2$ and form the level curves

$$x^2+y^2=f(x,y)=K^2.$$

These level curves are circles of radius K.



The gradient is $\nabla f(x,y)=(2x,2y)$ so a vector normal to the circle at $(\frac{\sqrt2}{2},\frac{\sqrt2}{2})$ is $\nabla f(\frac{\sqrt2}{2},\frac{\sqrt2}{2})=(\sqrt2,\sqrt2)$. The tangent space at this point is

$$(\sqrt2,\sqrt2)\cdot(x-\tfrac{\sqrt2}{2},\,y-\tfrac{\sqrt2}{2})=0$$

$$\Rightarrow\quad \sqrt2x+\sqrt2y=4 \quad\Rightarrow\quad y=\frac{4}{\sqrt2}-x$$

■