Cylindrical / Spherical coordinates

Let $p=(x,y,z)\in\mathbb{R}^3$.

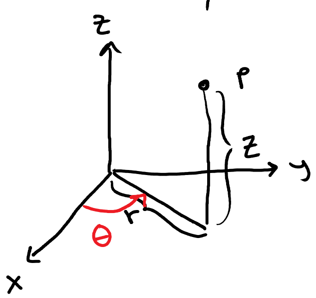
Def (Cylindrical) The cylindrical coordinates (r,θ,z) are given by

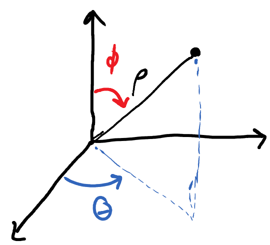
$$x=r\cos\theta \qquad y=r\sin\theta \qquad z=z$$

Def (spherical) The spherical coordinates (ρ,θ,ϕ) are given by

$$x=\rho\sin\phi\cos\theta \qquad y=\rho\sin\phi\sin\theta \qquad z=\rho\cos\phi$$

$\mathbb{R}^3$



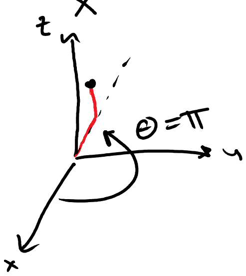


Ex Convert $(-1,0,1)$ from cartesian to spherical coordinates.

Use $\rho^2=x^2+y^2+z^2 = 2 \implies \rho=\sqrt{2}$

For θ, use $\tan\theta=\frac{y}{x} \implies \theta=\tan^{-1}(0) \quad 0\le\theta<2\pi$

Draw the picture



So $\theta=\pi$

For ϕ, use $\phi=\cos^{-1}\left(\frac{z}{\rho}\right) = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$

$= \frac{\pi}{4}$

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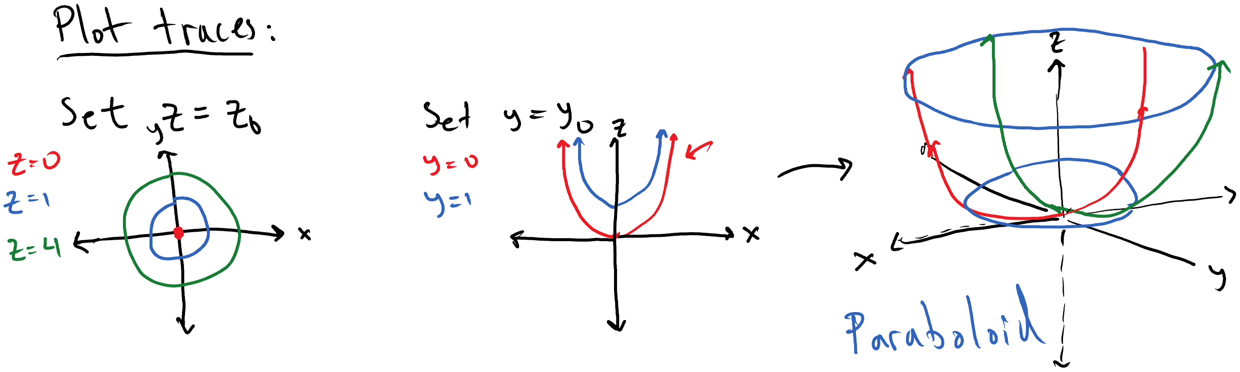
Plot the surfaces defined by the equations

(a) $z = r^2$

One idea: convert to cartesian coordinates:

$$z = r^2 = x^2 + y^2 \qquad \text{(Paraboloid)}$$

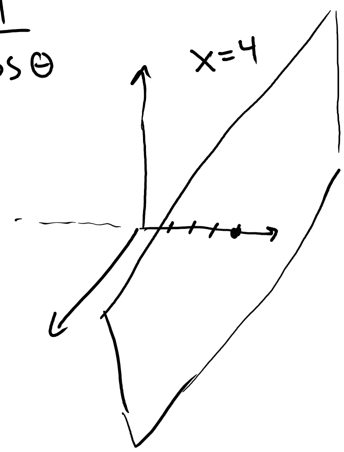
Plot traces:



(b) $\rho = 4 \csc\phi \sec\theta = 4 \cdot \frac{1}{\sin\phi} \cdot \frac{1}{\cos\theta}$

$\Rightarrow \quad 4 = \rho \sin\phi \cos\theta$

$\qquad = x$



(c) $r = 4\sin\theta$

$\Rightarrow \quad x^2 + y^2 = r^2 = r\,(4\sin\theta)$

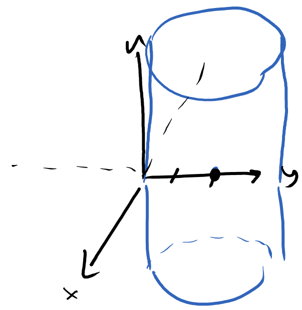
$\qquad\qquad = 4\,r\sin\theta$

$\qquad\qquad = 4\,y$

$\Rightarrow \quad x^2 + \underbrace{y^2 - 4y} = 0 \Rightarrow x^2 + \underbrace{y^2 - 4y + 4} = 4$

Complete the square

$\Rightarrow x^2 + (y-2)^2 = 4$



Cylinder of radius 2

centered at $(0,2)$

Consider the det. of a 4x4 matrix

$$\det\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix}$$

$$+ a_{13}\begin{vmatrix} \cdots \end{vmatrix} - a_{14}\begin{vmatrix} \cdots \end{vmatrix}$$

Some Useful properties: Let A be an $n \times n$ matrix

- Suppose B is obtained from A by adding a multiple of one row to another, then $\det(B) = \det(A)$
- Suppose B is obtained by multiplying a row of A by a constant $c \in \mathbb{R}$, then $\det(B) = c \det A$
- If B is obtained by switching two rows of A, then $\det B = -\det A$

Ex

$$\det\begin{bmatrix} 1 & 0 & 2 & 3 \\ -1 & 0 & 2 & -3 \\ 1 & 4 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} \overset{R_1 \leftarrow R_1 + R_2}{=} \det\begin{bmatrix} 0 & 0 & 4 & 0 \\ -1 & 0 & 2 & -3 \\ 1 & 4 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}$$

$$= +4\begin{vmatrix} -1 & 0 & -3 \\ 1 & 4 & 1 \\ 1 & 2 & 1 \end{vmatrix}$$

$$\overset{R_3 \leftarrow R_3 - R_2}{=} 4\begin{vmatrix} -1 & 0 & -3 \\ 1 & 4 & 1 \\ {}^{+}0 & {}^{-}-2 & {}^{+}0 \end{vmatrix}$$

$$= 4\left(-(-2)\begin{vmatrix} -1 & -3 \\ 1 & 1 \end{vmatrix}\right)$$

$$= 8\begin{vmatrix} -1 & -3 \\ 1 & 1 \end{vmatrix} = 8\left(-1-(-3)\right)$$

$$= 16 \qquad \blacksquare$$

<u>Recall</u> A matrix B is invertible if and only if $\det(B) \neq 0$.

<u>Ex</u> For which values of x will the matrix be invertible:

$$\begin{bmatrix} x-1 & x-2 & x-3 & x-4 \\ x+1 & x+2 & x+3 & x+4 \\ 2x & -x & 2x & -x \\ 0 & x & 0 & x+1 \end{bmatrix}$$

Compute the determinant:

$$\det\begin{bmatrix} x-1 & x-2 & x-3 & x-4 \\ x+1 & x+2 & x+3 & x+4 \\ 2x & -x & 2x & -x \\ 0 & x & 0 & x+1 \end{bmatrix} \overset{R_1 \leftarrow R_1+R_2}{=} \det\begin{bmatrix} 2x & 2x & 2x & 2x \\ x+1 & x+2 & x+3 & x+4 \\ 2x & -x & 2x & -x \\ 0 & x & 0 & x+1 \end{bmatrix}$$

$$\overset{R_1 \leftarrow R_1-R_3}{=} \det\begin{bmatrix} 0 & 3x & 0 & 3x \\ x+1 & x+2 & x+3 & x+4 \\ 2x & -x & 2x & -x \\ 0 & x & 0 & x+1 \end{bmatrix}$$

$$\overset{C_4 \leftarrow C_4-C_2}{=} \det\begin{bmatrix} 0 & 3x & 0 & 0 \\ x+1 & x+2 & x+3 & 2 \\ 2x & -x & 2x & 0 \\ 0 & x & 0 & 1 \end{bmatrix}$$

$$= -3x\begin{vmatrix} x+1 & x+3 & 2 \\ 2x & 2x & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= -3x\begin{vmatrix} x+1 & x+3 \\ 2x & 2x \end{vmatrix}$$

$C_2 \leftarrow C_2 - C_1$

$$= -3x \begin{vmatrix} x+1 & 2 \\ 2x & 0 \end{vmatrix}$$

$$= -3x(0 - 4x) = 12x^2$$

So A is not invertible when $12x^2 = 0$, i.e. when $x = 0$.

$\blacksquare$