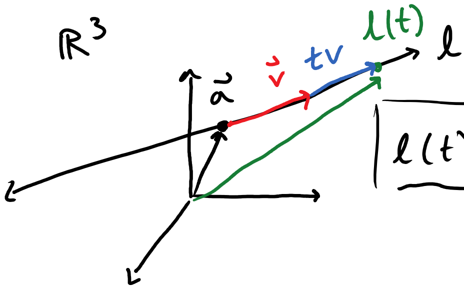
Equation of a line

Picture:



$$\boxed{l(t) = \vec{a} + t\vec{v}}$$

Want to find an equation for l.

(1) Let $\vec{a}$ be any point on the line
(2) Let $\vec{v}$ be a vector parallel to the line

Then the equation of a line is

$$l(t) = \vec{a} + t\vec{v}$$

Problem 1 **Chapter 11.1**

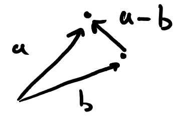
Are the points $(2,1,3)$, $(2,3,4)$, and $(2,-3,1)$ collinear?

Do the points lie on the same line?

Solution

Find eq of line between 2 points and check to see if the third point satisfies the equation

Step 1 Eq of line between $(2,1,3)$, $(2,3,4)$



(1) A point on the line is $(2,1,3)$.

(2) A vector parallel to the line is $(2,1,3)-(2,3,4)=(0,-2,-1)$

then the eq of the line is

$$\ell(t) = (2,1,3) + t(0,-2,-1)$$
$$= (2,\, 1-2t,\, 3-t)$$

Step 2 Does $(2,-3,1)$ lie on ℓ?

Set $(2,-3,1) = (2,\, 1-2t,\, 3-t)$

$$-3 = 1-2t \implies t = 2 \quad \checkmark$$

So $\ell(2) = (2,-3,1)$ so the points are collinear.

Note There are infinitely many ways to parameterize the same line.

Two lines
$$r(t) = a + tv$$
$$s(t) = b + tw$$

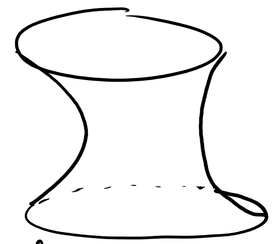
are the same if a lies on s and v and w are parallel: $v = cw$ for some $c \in \mathbb{R}$.

Problem 2

Find a line that lies inside the surface defined by the equation:

$$x^2 + y^2 - z^2 = 1$$

The surface defined by $x^2+y^2-z^2=1$ is a hyperboloid of one-sheet



Solution

(1) Choose a nice point that lies on the surface

Say $a=(1,0,0)$

(2) Let $\ell(t) = a + tv \qquad (v=(a,b,c))$

$$= (1,0,0) + t(a,b,c)$$
$$= (\underbrace{1+at}_{x(t)}, \underbrace{bt}_{y(t)}, \underbrace{ct}_{z(t)})$$

If $\ell(t)$ lies in the surface, then for all $t \in \mathbb{R}$

$$x(t)^2 + y(t)^2 - z(t)^2 = 1$$

$$\Rightarrow \quad (1+at)^2 + \underbrace{(bt)^2 - (ct)^2}_{\text{set } b=c} = 1$$

$$\Rightarrow \quad (1+at)^2 = 1 \qquad \text{set } a=0.$$

So $v = (0,b,b)$ for any $b \in \mathbb{R}$

So a line lying in the surface is

$$\boxed{\ell(t) = (1,0,0) + t(0,1,1)}$$

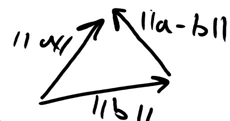
(say $b=1$ but any $b \in \mathbb{R}$ will work)

Chapter 11.2

The Dot Product

Def The dot product of $a = (a_1, a_2, a_3)$ $b = (b_1, b_2, b_3)$ is

$$a \cdot b = a_1 b_1 + a_2 b_2 + a_3 b_3 \in \mathbb{R}$$



Note $\cdot \|a\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

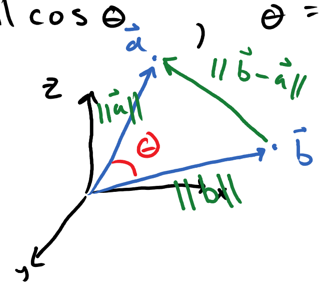
$= \sqrt{a \cdot a}$

- The distance between a and b is $\|b-a\| = \|a-b\|$.

Q: What does the dot product measure?
A: The angle between the vectors a and b.

Claim: $a \cdot b = \|a\| \|b\| \cos\theta$, θ = angle between them

Proof: Picture



By the Law of Cosines:

$$\|b-a\|^2 = \|a\|^2 + \|b\|^2 - 2\|a\|\|b\| \cos\theta$$

But $\|b-a\|^2 = (b-a)\cdot(b-a) = b \cdot b - 2 a \cdot b + a \cdot a$

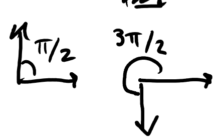
$\|a\|^2 = a \cdot a$ $\quad \|b\|^2 = b \cdot b$

$\Rightarrow \cancel{b \cdot b} - 2a \cdot b + \cancel{a \cdot a} = \cancel{a \cdot a} + \cancel{b \cdot b} - 2\|a\|\|b\| \cos\theta$

$\Rightarrow -2a \cdot b = -2\|a\|\|b\| \cos\theta$

$\Rightarrow a \cdot b = \|a\|\|b\| \cos\theta$

Thm $a \cdot b = 0$ if and only if $a \perp b$.



Proof If $a \cdot b = 0$, then $\cos\theta = 0 \Rightarrow \theta = \pi/2$ or $3\pi/2$
If $a \perp b$, then $\theta = 3\pi/2$ or $\pi/2 \Rightarrow \cos\theta = 0$
$\Rightarrow a \cdot b = 0$

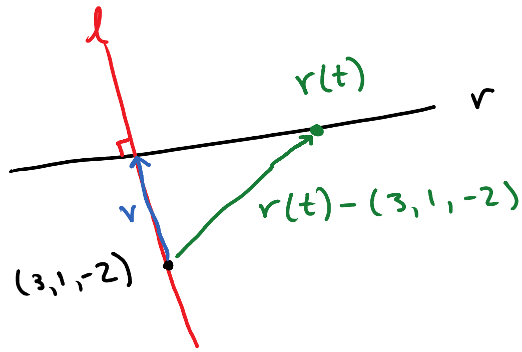
If $a \perp b$, then $\theta = 3\pi/2$ or $\pi/2$ $\Rightarrow$ $\cos\theta = 0$

$\Rightarrow a \cdot b = 0$ $\square$

Problem 3 Chapter 11.2

Find a line through $(3,1,-2)$ that intersects and is perpendicular to the line $r(t) = \langle t-1, t-2, t-1\rangle$.

Solution



Want: to find ℓ. A point that lies on ℓ is $a = (3,1,-2)$

Need to find v, a vector perpendicular to r. We can find $t \in \mathbb{R}$ so that $r(t) - (3,1,-2)$ is perpendicular to r. The direction of $r(t)$ is $(1,1,1)$. The vectors are perpendicular if:

$$\begin{aligned} 0 &= (1,1,1)\cdot\left(r(t) - (3,1,-2)\right) \quad (*) \\ &= (1,1,1)\,(t-4,\ t-3,\ t+1) \\ &= t-4 + t-3 + t+1 \\ &= 3t-6 \end{aligned}$$

So $t=2$. So $r(2) - (3,1,-2) = (-2,-1,3)$ is perpendicular to r, but parallel to ℓ. So the eq of ℓ is

$$\boxed{\ell(t) = (3,1,-2) + t(-2,-1,3)}$$

$\blacksquare$

$(*)$ $$\begin{aligned} r(t) - (3,1,-2) &= (t-1, t-2, t-1) - (3,1,-2) \\ &= (t-4,\ t-3,\ t+1) \end{aligned}$$

Problem 4

Chapter 11.2

Find the shortest distance from $v = (1,1,1)$ to the line $r(t) = (t, 2t, -t)$