Problem 1

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$\boxed{14.3}$

① For each function, find all critical points and determine whether they are local maxima/minima or saddle points

a) $f(x,y) = x^2 + 2xy + y^2$
b) $g(x,y) = x \sin y$

Thm If $f:\mathbb{R}^n \to \mathbb{R}$ has a local max/min at $x_0 \in \mathbb{R}^n$, then $Df(x_0) = 0$. Equivalently $\frac{\partial f}{\partial x_i}(x_0) = 0$ for all $i = 1, \dots, n$.

Def A point $x_0 \in \mathbb{R}^n$ is a critical point for f if $Df(x_0) = 0$.

Critical points are potential maxima/minima.

Solution (a) To find the critical points for f, we solve the system of equations:

$$\begin{cases} 0 = f_x = 2x + 2y \\ 0 = f_y = 2x + 2y \end{cases}$$

Any point on the line $y = -x$ is a solution so the critical points are

$$\{(x,-x) : x \in \mathbb{R}\}$$

Notice that $f(x,y) = x^2 + 2xy + y^2 = (x+y)^2$ so if $y = -x$, then

$$f(x,y) = (x-x)^2 = 0$$

Since $(x+y)^2 \geq 0$ for all (x,y), all points on the line $y = -x$ are local minima.

(b) We solve the system of eq's:

$$\begin{cases} 0 = f_x = \sin y \\ 0 = f_y = x \cos y \end{cases}$$

The eq. $\sin y = 0$ implies $y = k\pi$, $k \in \mathbb{Z}$. If $y = k\pi$ then $x \cos k\pi = 0$ implies $x = 0$. So the critial points are $\{(0, k\pi) : k \in \mathbb{Z}\}$.

Apply the second derivative test: $f_{xx} = 0$ so all critical points are saddle points. $\blacksquare$

$\boxed{14.4}$

② Find the shortest distance between the point $(1,0,-1)$ and the plane $2x-2y+2z=6$.

<u>Solution</u> The distance between a point (x,y,z) and $(1,0,-1)$ is given by

$$d(x,y,z) = \sqrt{(x-1)^2 + y^2 + (z+1)^2}$$

Notice that if (x,y,z) is a minimum for $d^2 = (x-1)^2 + y^2 + (z+1)^2$ then it is also a minimum for d. By Lagrange Multiplier Thm, if d^2 attains a max/min subject to the constraint $g(x,y,z) = 2x-2y+2z$, then there exists $\lambda \in \mathbb{R}$ such that

$$\nabla d^2 = \lambda \nabla G$$

This gives the system of eq's:

$$\begin{cases} 2x-2 = 2\lambda & \Rightarrow \quad x = \lambda+1 \\ 2y = -2\lambda & \Rightarrow \quad y = -\lambda \\ 2z+2 = 2\lambda & \Rightarrow \quad z = \lambda-1 \\ 2x-2y+2z=6 & \Rightarrow \quad 2(\lambda+1) - 2(-\lambda) + 2(\lambda-1) = 6 \\ & \Rightarrow \quad 6\lambda = 6 \Rightarrow \lambda = 1 \end{cases}$$

So $\lambda = 1$ which implies that $x=2$, $y=-1$, $z=0$. So the only critical point is $(2,-1,0)$.

Theorem 10 Let $f\colon U \subset \mathbb{R}^2 \to \mathbb{R}$ and $g\colon U \subset \mathbb{R}^2 \to \mathbb{R}$ be smooth (at least C^2) functions. Let $\mathbf{v}_0 \in U$, $g(\mathbf{v}_0) = c$, and S be the level curve for g with value c. Assume that $\nabla g(\mathbf{v}_0) \neq \mathbf{0}$ and that there is a real number λ such that $\nabla f(\mathbf{v}_0) = \lambda \nabla g(\mathbf{v}_0)$. Form the auxiliary function $h = f - \lambda g$ and the ***bordered Hessian*** determinant

$$|\overline{H}| = \begin{vmatrix} 0 & -\frac{\partial g}{\partial x} & -\frac{\partial g}{\partial y} \\ -\frac{\partial g}{\partial x} & \frac{\partial^2 h}{\partial x^2} & \frac{\partial^2 h}{\partial x \partial y} \\ -\frac{\partial g}{\partial y} & \frac{\partial^2 h}{\partial x \partial y} & \frac{\partial^2 h}{\partial y^2} \end{vmatrix} \quad \text{evaluated at } \mathbf{v}_0.$$

(i) If $|\overline{H}| > 0$, then $\mathbf{v}_0$ is a local maximum point for $f|S$.
(ii) If $|\overline{H}| < 0$, then $\mathbf{v}_0$ is a local minimum point for $f|S$.
(iii) If $|\overline{H}| = 0$, the test is inconclusive and $\mathbf{v}_0$ may be a minimum, a maximum, or neither.

This theorem is proved in the Internet supplement for this section.

for $f,g\colon \mathbb{R}^3 \to \mathbb{R}$

✓

Using theorem 10, define $h = d^2 - \lambda g = (x-1)^2 + y^2 + (z+1)^2 - (2x-2y+2z)$

So
$$|\overline{H}| = \begin{vmatrix} 0 & -\frac{\partial g}{\partial x} & -\frac{\partial g}{\partial y} & -\frac{\partial g}{\partial z} \\ -\frac{\partial g}{\partial x} & \frac{\partial^2 h}{\partial x^2} & \frac{\partial^2 h}{\partial y \partial x} & \frac{\partial^2 h}{\partial z \partial x} \end{vmatrix}$$

So $|H| = \begin{vmatrix} 0 & \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & -\frac{\partial g}{\partial z} \\ -\frac{\partial g}{\partial x} & \frac{\partial^2 h}{\partial x^2} & \frac{\partial^2 h}{\partial x \partial y} & \frac{\partial^2 h}{\partial x \partial z} \\ \frac{\partial g}{\partial y} & \frac{\partial^2 h}{\partial x \partial y} & \frac{\partial^2 h}{\partial y^2} & \frac{\partial^2 h}{\partial y \partial z} \\ -\frac{\partial g}{\partial z} & \frac{\partial^2 h}{\partial x \partial z} & \frac{\partial^2 h}{\partial y \partial z} & \frac{\partial^2 h}{\partial z^2} \end{vmatrix}$

$$= \begin{vmatrix} 0 & -2 & 2 & -2 \\ -2 & 2 & 0 & 0 \\ 2 & 0 & 2 & 0 \\ -2 & 0 & 0 & 2 \end{vmatrix}$$

$$= -(-2)\begin{vmatrix} -2 & 2 & -2 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{vmatrix} + 2\begin{vmatrix} 0 & -2 & 2 \\ -2 & 2 & 0 \\ 2 & 0 & 2 \end{vmatrix}$$

$$= 2(-2)\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix}$$

$$+ 2\left(2\begin{vmatrix} -2 & 0 \\ 2 & 2 \end{vmatrix} + 2\begin{vmatrix} -2 & 2 \\ 2 & 0 \end{vmatrix}\right)$$

$$= -16 + 2\left(2(-4) + 2(-4)\right)$$

$$= -16 + -16 = \boxed{-32}$$

So $|H| < 0$ which means $(2,-1,0)$ is a local minimum. ∎

$\boxed{14.4}$

③ Let P be a point on the surface S in $\mathbb{R}^3$ defined by the equation $f(x,y,z)=1$, where f is continuously differentiable. Suppose the distance between S and $(0,0,0)$ is maximized at P. Show that the vector emanating from $(0,0,0)$ and ending at P is orthogonal to S.

<u>Proof</u> Let $\vec{P}=(x,y,z)$. Since ∇f is orthogonal to S so we need to show that $\vec{P}$ is parallel to $\nabla f(x,y,z)$, i.e., $(x,y,z)=\alpha\nabla f(x,y,z)$. The distance between a point and the origin is given

$$d(a,b,c)=\sqrt{a^2+b^2+c^2}$$

As in problem (2), $\vec{P}$ maximizes d if and only if $\vec{P}$ maximizes d^2. By Lagrange Multiplier thm, there exists $\lambda\in\mathbb{R}$ such that

$$\nabla d^2(x,y,z)=\lambda\nabla f(x,y,z)$$

this yields the system of eq's,

$$\begin{cases} 2x = \lambda f_x \\ 2y = \lambda f_y \\ 2z = \lambda f_z \end{cases}$$

So, $P=(x,y,z)=\left(\frac{\lambda}{2}f_x,\frac{\lambda}{2}f_y,\frac{\lambda}{2}f_z\right)=\frac{\lambda}{2}\nabla f(x,y,z)$. So

$P=\alpha\nabla f$ where $\alpha=\frac{\lambda}{2}$ which is what we needed to show.

∎

$\boxed{14.4}$

(4) Let A be a non-zero symmetric 3×3 matrix. Define $f:\mathbb{R}^3 \to \mathbb{R}$ via

$$f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \frac{1}{2}\Bigg(\underbrace{A\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{3\times 1}\Bigg) \underset{\text{dot product.}}{\cdot} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{3\times 1}$$

(a) Find ∇f.

(b) Restrict f to the unit sphere S. Does f achieve a global max/min?

(c) Show that there exists a point $x \in S$ and $\lambda \neq 0$ such that

$$Ax = \lambda x.$$

<u>Proof</u> Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$. Then we have

$$f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \frac{1}{2}\left(a_{11}x^2 + a_{12}xy + a_{13}xz + a_{12}xy + a_{22}y^2 + a_{23}yz + a_{13}xz + a_{23}yz + a_{33}z^2\right)$$

So $f_x = \frac{1}{2}\left(2a_{11}x + 2a_{12}y + 2a_{13}z\right) = a_{11}x + a_{12}y + a_{13}z$

$f_y = \frac{1}{2}\left(2a_{12}x + 2a_{22}y + 2a_{23}z\right) = a_{12}x + a_{22}y + a_{23}z$

$f_z = a_{13}x + a_{23}y + a_{33}z$

So $\nabla f = A\begin{bmatrix} x \\ y \\ z \end{bmatrix}$. The unit sphere is the set

$$S = \{(x,y,z) : x^2+y^2+z^2 = 1\}.$$

S is clearly bounded and it's closed since it is a level surface. Since f is continuous on a closed and bounded set, it must attain a max/min.

For (c), let $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$ be a point where f achieves a max. By Lagrange, there is $\alpha \in \mathbb{R}$ such that (w/ $g(x,y,z) = x^2+y^2+z^2$)

$$A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \nabla f = \alpha \nabla g = \alpha \begin{bmatrix} 2x \\ 2y \\ 2y \end{bmatrix}$$

So $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ where $\lambda = 2\alpha$. And $\alpha = 0$ since A is

So $A\begin{bmatrix}x\\y\\z\end{bmatrix} = \lambda\begin{bmatrix}x\\y\\z\end{bmatrix}$ where $\lambda = 2\alpha$. And $\alpha = 0$ since A is non-zero and $\begin{bmatrix}x\\y\\z\end{bmatrix} \neq 0$. We just proved that every real symmetric matrix has at least one non-zero real eigenvalue. $\boxtimes$