Problem 1

Tuesday, May 19, 2020 10:40 PM

$\boxed{13.5}$

① You are walking on the graph of $f(x,y) = y\cos\pi x - x\cos\pi y + 10$ starting at the point $(2,1,13)$. Which direction should you walk to maintain a constant elevation?

Solution The directional derivative gives the rate of change (slope) in the direction of a unit vector. So we need to find a unit vector $v = (x, y)$ such that

$$\underbrace{\nabla f(2,1) \cdot v}_{\text{Directional Derivative}} = 0.$$

Since $\nabla f = (-\pi y \sin\pi x - \cos\pi y, \cos\pi x + \pi x \sin\pi y)$

$\nabla f(1,2) = (1,1)$

So we get $x + y = \nabla f(2,1) \cdot (x,y) = 0$. Since v is a unit vector, we also know $x^2 + y^2 = \|v\|^2 = 1$. By substitution

$$2x^2 = x^2 + (-x)^2 = 1$$

So $x = \pm\frac{\sqrt{2}}{2}$ and $y = \pm\frac{\sqrt{2}}{2}$.

So we can walk on any of the four directions determined by

$$v = \left(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}\right) \blacksquare$$

$\boxed{14.1}$

② (a) Show that $f(x,t) = \sin(x - ct)$ satisfies the one-dimensional wave equation

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}.$$

(b) Let $w = f(x,y)$ be a function of two variables and let $x = u+v$, $y = u-v$. Show that

$$\frac{\partial^2 w}{\partial u \partial v} = \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2}.$$

Proof of (a) We have

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\sin(x-ct)\right)$$

$$= \frac{\partial}{\partial x}\cos(x-ct)$$

$$= -\sin(x-ct)$$

$$= -\frac{c^2}{c^2}\sin(x-ct)$$

$$= -\frac{c}{c^2}\frac{\partial}{\partial t}\cos(x-ct)$$

$$= \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\sin(x-ct)$$

$$= \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}$$

$\blacksquare$

(b) Let $w = f(x,y)$ be a function of two variables and let $x = u+v$, $y = u-v$. Show that

$$\frac{\partial^2 w}{\partial u \partial v} = \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2}.$$

Proof of (b)

For any function $g(x,y)$, if we make the change of variable $x = u+v$, $y = u-v$, then the chain rule gives

$$(1) \quad \frac{\partial g}{\partial v} = \frac{\partial g}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial v} = \frac{\partial g}{\partial x} - \frac{\partial g}{\partial y}$$

$$(2) \quad \frac{\partial g}{\partial u} = \frac{\partial g}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial g}{\partial y}\cdot\frac{\partial y}{\partial u} = \frac{\partial g}{\partial x} + \frac{\partial g}{\partial y}$$

Now with $w = f(x,y)$ we have

$$\frac{\partial^2 w}{\partial u \partial v} = \frac{\partial}{\partial u}\left(\frac{\partial w}{\partial v}\right)$$

$$\overset{(1)}{=} \frac{\partial}{\partial u}\left(\frac{\partial w}{\partial x} - \frac{\partial w}{\partial y}\right)$$

$$= \frac{\partial}{\partial u}\left(\frac{dw}{\partial x}\right) - \frac{\partial}{\partial u}\left(\frac{\partial w}{\partial y}\right)$$ $\left(\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}\right.$ are functions of x and $\left.y\right)$

$$\overset{(2)}{=} \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y \partial x} - \left(\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 w}{\partial y^2}\right)$$

equal by clairaut

$$\overset{\text{Clairaut's}}{=} \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2}$$

$\blacksquare$

Problem 3

Monday, May 18, 2020 7:23 PM

14.1

③ a) Does there exist a C^2 function $f(x,y)$ such that:

$$f_x = 2x - 5y \quad \text{and} \quad f_y = 4x + y \; ?$$

b) Does there exist a C^2 function $g(x,y)$ such that:

$$g_x = 5x - 2y \quad \text{and} \quad g_y = -2x \; ?$$

Solution a) There is no such function. Suppose $f(x,y)$ was C^2 and satisfies $f_x = 2x - 5y$ and $f_y = 4x + y$. By Clairaut's Thm, $f_{xy} = f_{yx}$. But $f_{xy} = -5$ and $f_{yx} = 4$ and even a fool knows $-5 \neq 4$.

□

(b) Note that $g_{xy} = -2 = g_{yx}$, so Clairaut's theorem tells us nothing. In this case, we can use integration to find f.

To find g:
$$\begin{aligned} g(x,y) &= \int g_x \, dx \\ &= \int 5x - 2y \, dx \\ &= \underbrace{\tfrac{5}{2}x^2 - 2xy + h(y)} \end{aligned}$$
($h(y)$ is any function of y)

To find $h(y)$: compute g_y from $g(x,y) = \frac{5}{2}x^2 - 2xy + h(y)$.

$\Rightarrow$
$$g_y = -2x + h'(y)$$

We also know $g_y = -2x$ so $-2x = -2x + h'(y)$. So $h'(y) = 0$

So $h(y) = \int h'(y)\,dy = \int 0 \, dy = 0$. So $g(x,y) = \frac{5}{2}x^2 - 2xy$.

■

$\boxed{14.2}$

Second-Order Taylor Formula: Let $f:\mathbb{R}^n \to \mathbb{R}$ be a C^3 function. Then

$$f(x_0+h) = f(x_0) + \sum_{i=1}^{n} h_i \frac{\partial f}{x_i}(x_0) + \frac{1}{2}\sum_{i,j=}^{n} h_i h_j \frac{\partial f}{\partial x_i \partial x_j}(x_0) + R_2(x_0,h)$$

where $\frac{R_2(x_0,h)}{\|h\|} \to 0$ as $h \to 0$.

(4) Compute the second order Taylor approximation to $f(x,y) = e^{(x-1)^2}\cos y$ at $(1,0)$.

Solution If $f:\mathbb{R}^2 \to \mathbb{R}$, the Taylor polynomial centered at (x_0,y_0)

$$T(h_1,h_2) = f(x_0,y_0) + \sum_{i=1}^{2} h_i \frac{\partial f}{x_i}(x_0,y_0) + \frac{1}{2}\sum_{i,j=1}^{2} h_i h_j \frac{\partial f}{\partial x_i \partial x_j}(x_0,y_0)$$

(set $x_1 = x$, $x_2 = y$)

$$= f(x_0,y_0) + h_1 f_x(x_0,y_0) + h_2 f_y(x_0,y_0) + \frac{1}{2}\Big(h_1^2 f_{xx} + h_1h_2 f_{xy} + h_2h_1 f_{yx} + h_2^2 f_{yy}\Big)$$

So we compute all partial derivatives at $(1,0)$ $\quad \boxed{f(x,y) = e^{(x-1)^2}\cos y}$

$f(1,0) = 1$

$f_x = (2x-2)e^{(x-1)^2}\cos y \xrightarrow{\text{at}(1,0)} 0$

$f_y = -e^{(x-1)^2}\sin y \longrightarrow 0$

$f_{xx} = 2^2(x-1)^2 e^{(x-1)^2}\cdot\cos y + 2e^{(x-1)^2}\cos y \longrightarrow 2$

$f_{xy} = -(2x-2)e^{(x-1)^2}\sin y \longrightarrow 0$

$f_{yx} = \ldots \longrightarrow 0$

$f_{yy} = -e^{(x-1)^2}\cos y \longrightarrow -1$

So the Taylor polynomial of degree 2 at $(1,0)$

$$\boxed{T(x,y) = 1 + x^2 - \frac{1}{2}y^2}$$

■