Problem 1

Tuesday, May 12, 2020 8:08 PM

① Let $u = u(x,y)$ and let (r,θ) be polar coordinates. Show that

$$\|\nabla u\|^2 = u_r^{\,2} + \frac{1}{r^2} u_\theta^{\,2}.$$

<u>Theorem</u> Let $g: \mathbb{R}^n \to \mathbb{R}^m$ and $f: \mathbb{R}^m \to \mathbb{R}^p$ be such that $f \circ g$ is defined and g is differentiable at $x_0 \in \mathbb{R}^n$ and f differentiable and $g(x_0)$. Then $f \circ g$ is differentiable at x_0 and

$$\underbrace{D(f\circ g)(x_0)}_{p \times n} = \underbrace{Df(g(x_0))}_{p\times m} \cdot \underbrace{Dg(x_0)}_{m \times n}$$

matrix multiplication

<u>Proof</u> We can find $u_r = \frac{\partial u}{\partial r}$ and $u_\theta = \frac{\partial u}{\partial \theta}$ interms of u_x, u_y using the Chain rule. Let $g: \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$(r,\theta) \mapsto (r\cos\theta, r\sin\theta)$$

$$\begin{aligned}
\begin{bmatrix} u_r & u_\theta \end{bmatrix} &= D(u \circ g) = Du \cdot Dg \\
&= \begin{bmatrix} u_x & u_y \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} \\
&= \begin{bmatrix} u_x & u_y \end{bmatrix} \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta\, u_x + \sin\theta\, u_y & -r\sin\theta\, u_x + r\cos\theta\, u_y \end{bmatrix}
\end{aligned}$$

$\Rightarrow$

$$u_r = \cos\theta\, u_x + \sin\theta\, u_y$$
$$u_\theta = -r\sin\theta\, u_x + r\cos\theta\, u_y$$

Then,

$$u_r^{\,2} + \frac{1}{r^2} u_\theta^{\,2} = \cos^2\theta\, u_x^2 + \underline{2\sin\theta\cos\theta\, u_x u_x} + \sin^2\theta\, u_y^2 + \frac{1}{r^2}\left(r^2\sin^2\theta\, u_x^2 \underline{- 2r^2\sin\theta\cos\theta\, u_x u_y} + r^2\cos^2\theta\, u_y^2\right)$$

$$= \cos^2\theta\, u_x^2 + \sin^2\theta\, u_y^2 + \sin^2\theta\, u_x^2 + \cos^2\theta\, u_y^2$$

$$= u_x^2 + u_y^2$$

$$= \|\nabla u\|^2 . \qquad \blacksquare$$

Problem 2

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② Let $f: \mathbb{R}^3 \to \mathbb{R}$ be differentiable. Find formulas for $\frac{\partial f}{\partial \rho}, \frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial \phi}$ in terms of $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ where (ρ, θ, ϕ) are spherical coordinates.

Solution Apply the chain rule (set $x = \rho \sin\phi \cos\theta$ $y = \rho \sin\phi \sin\theta$ $z = \rho\cos\phi$)

$$\begin{bmatrix} \frac{\partial f}{\partial \rho} & \frac{\partial f}{\partial \theta} & \frac{\partial f}{\partial \phi} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{bmatrix}$$

Comparing entries we get:

$$\frac{\partial f}{\partial \rho} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \rho} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \rho} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \rho} \quad (\text{set } x = \rho \sin\phi \cos\theta \ \ y = \rho \sin\phi \sin\theta \ \ z = \rho\cos\phi)$$

$$= \sin\phi\cos\theta \frac{\partial f}{\partial x} + \sin\phi \sin\theta \frac{\partial f}{\partial y} + \cos\phi \frac{\partial f}{\partial z}.$$

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial \theta} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \theta}$$

$$= -\rho \sin\phi \sin\theta \frac{\partial f}{\partial x} + \rho \sin\phi \cos\theta \frac{\partial f}{\partial y}.$$

$$\frac{\partial f}{\partial \phi} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \phi} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial \phi} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \phi}$$

$$= \rho\cos\phi\cos\theta \frac{\partial f}{\partial x} + \rho\cos\phi \sin\theta \frac{\partial f}{\partial y} - \rho\sin\phi \frac{\partial f}{\partial z}$$

$\blacksquare$

e.g. $x^2+y^2=4$

③ Define $y(x)$ implicitly via $G(x, y(x)) = K$ where $G: \mathbb{R}^2 \to \mathbb{R}$. Prove the implicit differentiation formula: if $y(x)$ and G are differentiable and $\frac{\partial G}{\partial y} \neq 0$, then

$$\frac{dy}{dx} = -\frac{\partial G/\partial x}{\partial G/\partial y}.$$

Proof Define $H: \mathbb{R} \to \mathbb{R}$ via $H(x) = G(x, y(x))$ and $F: \mathbb{R} \to \mathbb{R}^2$ via $F(x) = (x, y(x))$. Then $H(x) = G \circ F(x)$. Since $H(x) = K$, $H'(x) = 0$.

Then by chain rule:

$$\begin{aligned} 0 = H'(x) &= DH \\ &= DG\, DF \\ &= \begin{bmatrix} \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{bmatrix} \begin{bmatrix} 1 \\ \frac{dy}{dx} \end{bmatrix} \end{aligned}$$

(since $y: \mathbb{R} \to \mathbb{R} \Rightarrow \frac{\partial y}{\partial x} = \frac{dy}{dx}$)

So we get $0 = \frac{\partial G}{\partial x} + \frac{\partial G}{\partial y} \cdot \frac{dy}{dx}$. Since $\frac{\partial G}{\partial y} \neq 0$ we can solve for

$$\frac{dy}{dx} = -\frac{\partial G/\partial x}{\partial G/\partial y}.$$

$\blacksquare$

Ex Find $\frac{dy}{\partial x}$ if $x^2 + y^2 = 4$.

Two ways: By Calc I, $2x + 2y\frac{dy}{dx} = 0$

$$\Rightarrow \frac{dy}{dx} = -\frac{2x}{2y} = \frac{-x}{y}$$

By Calc III, $\frac{dy}{dx} = \frac{-2x}{2y} = -\frac{x}{y}$.

Ex Find $\frac{dy}{\partial x}$ if $\frac{x^2 y}{e^y} = 4$

By Calc III, $\frac{dy}{} = \frac{2xy}{}$

By Calc $\underline{\text{III}}$,

$$\frac{dy}{dx} = -\frac{\frac{2xy}{e^y}}{\frac{x^2e^y - e^y x^2 y}{e^{2y}}}.$$

$\blacksquare$

Problem 4

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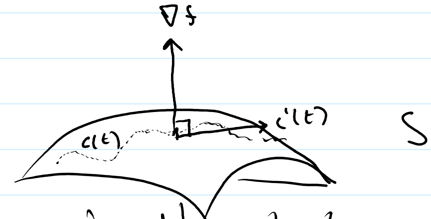
④ Let $f:\mathbb{R}^3 \to \mathbb{R}$ be a C^1 map and suppose (x_0, y_0, z_0) lies on the level surface S defined by $f(x,y,z)=K$. Show that $\nabla f(x_0,y_0,z_0)$ is normal to S.



__Proof__ __Idea__ Show that $\nabla f(x_0,y_0,z_0)$ is perpendicular to the tangent vector of an arbitrary curve contained in S and passing through (x_0,y_0,z_0). So let $c:\mathbb{R}\to\mathbb{R}^3$ be parameterized by $c(t)=(x(t), y(t), z(t))$ such that $c(t)$ lies in S for all $t\in\mathbb{R}$ and such that $c(0)=(x_0,y_0,z_0)$.

Then we have

$$\nabla f(x,y,z)\cdot c'(t) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)\cdot\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$$

$$= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial f}{\partial z}\cdot\frac{dz}{dt}$$

$$= \frac{\partial(f\circ c)}{\partial t} \qquad \text{(by chain rule!)}$$

$$= \frac{d}{dt}(f\circ c)$$

Evaluate at $t=0$,

$$\nabla f(x_0,y_0,z_0)\cdot c'(0) = \frac{d}{dt}(f\circ c)\Big|_{t=0} = \frac{d}{dt}K\Big|_{t=0}$$

$$= 0.$$

■

__Definition__ Suppose $f:\mathbb{R}^3\to\mathbb{R}$ is a C^1 map. The plane tangent to the surface S defined by $f(x,y,z)=K$ at (x_0,y_0,z_0) is

$$\nabla f(x_0,y_0,z_0)\cdot(x-x_0, y-y_0, z-z_0)=0 \quad (*)$$

Now suppose $f:\mathbb{R}^2\to\mathbb{R}$ is C^1. Define $h:\mathbb{R}^3\to\mathbb{R}$ defined by $h(x,y,z)=f(x,y)-z$. Then the graph of f is given by the level surface $h(x,y,z)=0$. By $(*)$, the tangent plane to

$h(x,y,z) = f(x,y) - z$. Then the graph of f is given by the level surface $h(x,y,z)=0$. By (*), the tangent plane to the graph of f is given by

$$(f_x(x_0,y_0), f_y(x_0,y_0), -1) \cdot (x-x_0, y-y_0, z-z_0) = 0.$$

<u>Cool Fact</u> If $f:\mathbb{R}^n \to \mathbb{R}$, we can define a tangent hyperplane to the surface $f(\vec{x}) = k$ at $\vec{x}_0 \in \mathbb{R}^n$

$$\nabla f(\vec{x}_0)(\vec{x} - \vec{x}_0) = 0.$$

This is an $(n-1)$ dimensional <u>affine</u> subspace of $\mathbb{R}^n$ that is tangent to the surface.