① Compute the derivitive $Df(x)$ for each function:

a) $f:\mathbb{R}^2\to\mathbb{R}^2$, $f(x,y)=(x+e^z+y,\ yx^2)$

b) $f:\mathbb{R}^2\to\mathbb{R}^3$, $f(x,y)=(xe^y+\cos y,\ x,\ x+e^y)$

c) $f:\mathbb{R}^3\to\mathbb{R}^3$, $f(\rho,\theta,\phi)=(\rho\sin\phi\cos\theta,\ \rho\sin\phi\sin\theta,\ \rho\cos\theta)$

Def The derivative $Df(x)$ of a function $f:\mathbb{R}^n\to\mathbb{R}^m$ w/ component functions $(x_1,\dots,x_n)\longmapsto(f_1,f_2,\dots,f_m)$ is the $m\times n$ matrix whose ij-entry is $\dfrac{\partial f_i}{\partial x_j}$

$$Df(x)=\begin{bmatrix}\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n}\\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n}\\ \vdots & & & \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n}\end{bmatrix}$$

a) $f:\mathbb{R}^2\to\mathbb{R}^2$, $f(x,y)=(x+e^x+y,\ yx^2)$

Solution Note $Df(x,y)$ is a 2×2 matrix.

$$Df(x,y)=\begin{bmatrix}1+e^x & 1\\ 2xy & x^2\end{bmatrix}$$

$\square$

b) $f:\mathbb{R}^2\to\mathbb{R}^3$, $f(x,y)=(xe^y+\cos y,\ x,\ x+e^y)$

Solution Note $Df(x,y)$ is a 3×2 matrix.

$$Df(x,y)=\begin{bmatrix}e^y & xe^y-\sin y\\ 1 & 0\\ 1 & e^y\end{bmatrix}$$

(Sometimes $Df(x)$ is called the Jacobian)

$\square$

c) $f: \mathbb{R}^3 \to \mathbb{R}^3$, $f(\rho, \theta, \phi) = (\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho\cos\theta)$

Solution Df is a 3×3 matrix.

$$Df(\rho, \theta, \phi) = \begin{bmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\theta & -\rho\sin\theta & 0 \end{bmatrix}$$

$\blacksquare$

Problem 2

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(2) Let $f(x,y) = xe^{y^2} - ye^{x^2}$.

a) Find an equation for the plane tangent to the graph of f at $(1,2)$

b) Which point on the surface $z + y^2 - x^2 = 0$ has a tangent plane parallel to the plane in part (a)?

Def If $f: \mathbb{R}^2 \to \mathbb{R}$ is differentiable at (x_0, y_0), then the tangent plane is given by the equation

$$z - f(x_0,y_0) = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

Solution (a) Compute $f_x(1,2)$, $f_y(1,2)$, $f(1,2)$:

$$f_x(x,y) = e^{y^2} - 2yxe^{x^2} \Rightarrow f_x(1,2) = e^4 - 4e$$

$$f_y(x,y) = 2xye^{y^2} - e^{x^2} \Rightarrow f_y(1,2) = 4e^4 - e$$

$$f(1,2) = e^4 - 2e$$

So equation of the tangent plane at $(1,2)$ is

$$P_1: \quad z - (e^4 - 2e) = (e^4 - 4e)(x-1) + (4e^4 - e)(y-2)$$

(b) Two planes are parallel if and only if their normal vectors are parallel. The plane P_1 has normal vector

$$n_1 = (e^4 - 4e, 4e^4 - e, -1)$$

The surface defined by the eq. $z + y^2 - x^2 = 0$ is the graph of the function

$$\boxed{g(x,y) = x^2 - y^2}$$

and the normal vector to the tangent plane at a point (x,y)

$\left(\begin{array}{l} n_2 = \nabla G \text{ where} \\ G(x,y,z) = g(x,y) - z \end{array} \right)$

$$n_2 = (g_x(x,y), g_y(x,y), -1)$$
$$= (2x, -2y, -1)$$

Then n_1 is parallel to n_2 iff $n_1 = c n_2$ for $c \in \mathbb{R}$.

This yields a system of eq.'s :

$$(e^4-4e,\ 4e^4-e,\ -1) = c(2x,\ -2y,\ -1)$$

$$\Rightarrow c=1 \Rightarrow x = \frac{e^4-4e}{2}$$

$$y = \frac{e-4e^4}{2}$$

Problem 3

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③ Let $f(x,y) = \begin{cases} \dfrac{x^2 y^4}{x^4+6y^8}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$

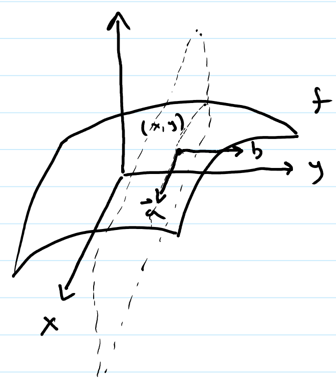
a) Show that $\frac{\partial f}{\partial x}(0,0)$ and $\frac{\partial f}{\partial y}(0,0)$ exist.

b) Show that f is not differentiable at $(0,0)$ by showing that f is not continuous.

<u>Definition</u> The partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for a function $f: \mathbb{R}^2 \to \mathbb{R}$ are defined by:

$$\frac{\partial f}{\partial x}(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}, \quad \frac{\partial f}{\partial y}(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

provided that these limits exist.



$\frac{\partial f}{\partial x}(x,y)$ = slope of line containing $\vec{a}$

$\frac{\partial f}{\partial y}(x,y)$ = slope of line containing b.

Let $f(x,y) = \begin{cases} \dfrac{x^2 y^4}{x^4+6y^8}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$

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a) Show that $\frac{\partial f}{\partial x}(0,0)$ and $\frac{\partial f}{\partial y}(0,0)$ exist.

<u>Proof</u>

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = \lim_{h\to 0} 0 = 0.$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{f(0,0+h) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0.$$

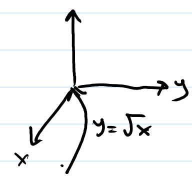
$$= \lim_{h \to 0} \frac{0 - 0}{h} = 0 .$$

So the partial derivatives exists.

b) Show that f is not differentiable at $(0,0)$ by showing that f is not continuous.

Proof If f is not continuous, then f is not differentiable.

Note $f(0,0) = 0$ by definition. Lets check the limit along the path in the xy-plane where $y = \sqrt{x}$:



$$\lim_{(x,y) \to (0,0)} \frac{x^2 y^4}{x^4 + 6y^8} = \lim_{(x,\sqrt{x}) \to (0,0)} \frac{x^2 (\sqrt{x})^4}{x^4 + 6(\sqrt{x})^8}$$

$$= \lim_{x \to 0} \frac{x^4}{7x^4}$$

$$= \lim_{x \to 0} \frac{1}{7} = \frac{1}{7} \neq f(0,0) .$$

Thus, f is not continuous at $(0,0)$. Moral of the story: the existence of all partial derivatives is not sufficient to guarantee differentiability.

Problem 4

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④ Compute the gradient ∇f for the following functions

a) $f(x,y,z) = \dfrac{xyz}{x^2+y^2+z^2}$

b) $f(x,y,z) = \ln(x^2+y^2+z^2)$

(∇ = "Del")

Solution The gradient of $f:\mathbb{R}^n \to \mathbb{R}$ is the vector

$$\nabla f(x_1,\dots,x_n) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{x_n}\right)$$

a) $$\nabla f(x,y,z) = \left(\frac{yz(x^2+y^2+z^2) - 2x^2yz}{(x^2+y^2+z^2)^2}, \frac{xz(x^2+y^2+z^2) - 2y^2xz}{(x^2+y^2+z^2)^2}, \frac{xy(x^2+y^2+z^2) - 2z^2xy}{(x^2+y^2+z^2)^2}\right)$$

b) $$\nabla f(x,y,z) = \left(\frac{2x}{x^2+y^2+z^2}, \frac{2y}{x^2+y^2+z^2}, \frac{2z}{x^2+y^2+z^2}\right)$$

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