① Compute the following limits, if they exist:

a) $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2+2}$

b) $\lim\limits_{(x,y)\to(0,0)} \dfrac{(x-y)^2}{x^2+y^2}$

c) $\lim\limits_{(x,y)\to(0,0)} (3x^2+3y^2)\log(x^2+y^2)$

d) $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$

Solutions:

a) $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2+2}$

The function $f(x) = \dfrac{xy}{x^2+y^2+2}$ is continuous since xy and x^2+y^2+2 are continuous everywhere (polynomials) and $x^2+y^2+2 \neq 0$ at $(0,0)$. Thus,

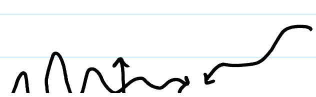
$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2+2} = 0$$

$\boxtimes$

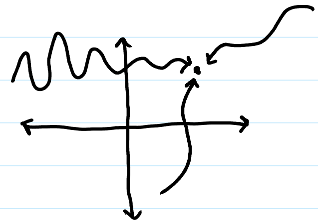
b) $\lim\limits_{(x,y)\to(0,0)} \dfrac{(x-y)^2}{x^2+y^2}$

The function $f(x) = \dfrac{(x-y)^2}{x^2+y^2}$ is not continuous since $x^2+y^2=0$ at $(0,0)$, i.e, $f(0,0)$ is not defined! In fact, this limit does not exist. To show that the limit doesn't exist, we need to find two paths that approach $(0,0)$, but give different values in the limit.

e.g.



e.g.



For instance, along the line $y=x$ we have

$$\lim_{(x,x)\to(0,0)} \frac{(x-x)^2}{x^2+x^2} = 0.$$

Along the line $y=0$, w/ $x>0$, we obtain

$$\lim_{(x,0)\to(0,0)} \frac{(x-0)^2}{x^2+0^2} = \lim_{x\to 0} \frac{x^2}{x^2} = 1 .$$

Since the values do not agree, the limit cannot exists. ∎

c) $\lim\limits_{(x,y)\to(0,0)} (3x^2+3y^2)\log(x^2+y^2)$

The function $f(x) = (3x^2+3y^2)\log(x^2+y^2)$ is not continuous at $(0,0)$ since $\log(0)$ is undefined. Let's convert to polar coordinates via the identity $r^2 = x^2+y^2$. Note that $\lim\limits_{(x,y)\to(0,0)} r = 0$ since $\sqrt{x^2+y^2}$ is continuous. The limit in question becomes:

$$\lim_{(x,y)\to(0,0)} (3x^2+3y^2)\log(x^2+y^2) = \lim_{r\to 0} 3r^2 \log r^2 \quad \begin{pmatrix}\text{indeterminate} \\ \text{form } 0\cdot\infty\end{pmatrix}$$

$$= 3\lim_{r\to 0} \frac{\log r^2}{1/r^2}$$

$$\overset{L'H}{=} 3\lim_{r\to 0} \frac{\frac{1}{r^2}\cdot 2r}{\frac{-2}{r^3}}$$

$$= 3\lim_{r\to 0} -r^2$$

$$= 3\cdot 0 = 0$$ ∎

d) $\lim\limits_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$

The function is not continuous at $(0,0)$. If we approach $(0,0)$ along the path where $y=x$, $x>0$, we obtain

$$\lim_{(x,x)\to(0,0)} \frac{x^2}{2x^2} = \frac{1}{2}$$

But along the line $y=0$, $x>0$ we obtain

$$\lim_{(x,0)\to(0,0)} \frac{x\cdot 0}{x^2+0^2} = 0.$$

Thus, the limit doesn't exist. $\blacksquare$

Problem 2

Tuesday, April 28, 2020 9:19 PM

(2) Suppose $f: \mathbb{R}^2 \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ are functions such that

$$f(x,y) = g(xy) \quad (\text{for all } (x,y) \in \mathbb{R}^2)$$

If $(a,b) \in \mathbb{R}^2$ and g is continuous at ab, then $\lim_{(x,y)\to(a,b)} f(x,y)$ exists and is equal to $\lim_{t \to ab} g(t) = g(ab)$.

<u>Proof</u> Define a function $h: \mathbb{R}^2 \to \mathbb{R}$ via $h(x,y) = xy$. Notice that h is continuous everywhere since it is a polynomial. Now, we have

$$\boxed{\lim_{(x,y)\to(a,b)} f(x,y)} = \lim_{(x,y)\to(a,b)} g(xy) \quad (\text{by assumption})$$

$$= \lim_{(x,y)\to(a,b)} g(h(x,y)) \quad (\text{by def. of } h)$$

$$= g\left(\lim_{(x,y)\to(a,b)} h(x,y)\right) \quad (\text{since } g \text{ is continuous at } ab)$$

$$= g(ab) \quad (\text{since } h \text{ is continuous at } (a,b))$$

$$\boxed{= \lim_{t\to ab} g(t)}$$

∎

Problem 3

Tuesday, April 28, 2020 9:58 PM

③ Compute the following limits, if they exist:

a) $$\lim_{(x,y)\to(0,0)} \frac{e^{xy}-1}{y}$$

b) $$\lim_{(x,y)\to(0,0)} \frac{\cos(xy)-1}{x^2y^2} \overset{\substack{\text{set}\\ u=xy}}{=} \lim_{u\to 0} \frac{\cos u - 1}{u^2}$$

c) $$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$$

Solutions

a) $$\lim_{(x,y)\to(0,0)} \frac{e^{xy}-1}{y} = \lim_{(x,y)\to(0,0)} x \cdot \frac{e^{xy}-1}{xy}$$

Now, if $\displaystyle\lim_{(x,y)\to(0,0)} \frac{e^{xy}-1}{xy}$ exists, then we can write:

$$\lim_{(x,y)\to(0,0)} x \cdot \frac{e^{xy}-1}{xy} = \left(\lim_{(x,y)\to(0,0)} x\right)\left(\lim_{(x,y)\to(0,0)} \frac{e^{xy}-1}{xy}\right)$$
$$= 0 \cdot 1$$
$$= 0 \checkmark$$

To compute $\displaystyle\lim_{(x,y)\to(0,0)} \frac{e^{xy}-1}{xy}$ consider the function $g(t) = \begin{cases} \frac{e^t-1}{t}, & t\neq 0 \\ 1, & t=0. \end{cases}$

Then $g(t)$ is continous at $t=0$ since

$$\lim_{t\to 0} \frac{e^t-1}{t} \overset{L'H}{=} \lim_{t\to 0} \frac{e^t}{1}$$
$$= e^0$$
$$= 1$$
$$= g(0). \checkmark$$

Also, $g(xy) = \dfrac{e^{xy}-1}{xy}$ for all points $(x,y)\in\mathbb{R}^2$. Thus, by ②

$$\lim \frac{e^{xy}-1}{} = \lim g(t)$$

$$\lim_{(x,y)\to(0,0)} \frac{e^{xy}-1}{xy} = \lim_{t\to 0} g(t)$$
$$= g(0)$$
$$= 1.$$

b) $$\lim_{(x,y)\to(0,0)} \frac{\cos(xy)-1}{x^2y^2}$$

Consider the function $g(t) = \begin{cases} \dfrac{\cos t - 1}{t^2}, & t \neq 0 \\ -\frac{1}{2}, & t = 0. \end{cases}$

Then $g(t)$ is continuous at $t=0$ since

$$\lim_{t\to 0} \frac{\cos t - 1}{t^2} \overset{L'H}{=} \lim_{t\to 0} \frac{-\sin t}{2t}$$
$$\overset{L'H}{=} \lim_{t\to 0} \frac{-\cos t}{2}$$
$$= -1/2 = g(0) \checkmark$$

And also, $g(xy) = \dfrac{\cos xy - 1}{xy}$. By ②,

$$\lim_{(x,y)\to(0,0)} \frac{\cos(xy)-1}{x^2y^2} = g(0) = -1/2.$$