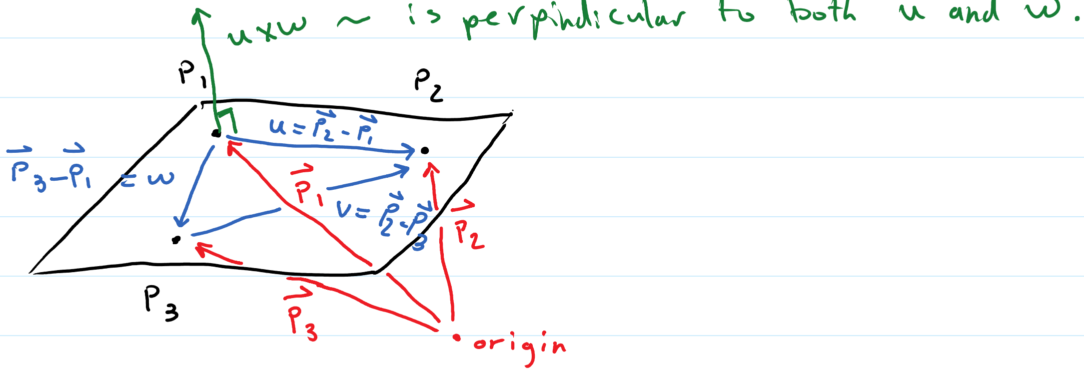
Problem 1

Sunday, April 5, 2020 10:43 PM

11.2/11.3

① Show that the points $P_1 = (1,-1,1)$, $P_2 = (0,1,-2)$, and $P_3 = (-2,1,0)$ are coplanar, i.e., P_1, P_2, P_3 lie in the same plane.



Solution From the picture, we can see that P_1, P_2, P_3 are coplanar, if and only if the vectors u, v, w are coplanar.

Compute u, v, w:

$u = P_2 - P_1 = (-1, 2, -3)$, $v = P_2 - P_3 = (2, 0, -2)$, $w = (-3, 2, -1)$.

We see that u, v, w are coplanar if $u \times w$ is orthogonal to v.

To show that $u \times w \perp v$ we need to show that $v \cdot (u \times w) = 0$.

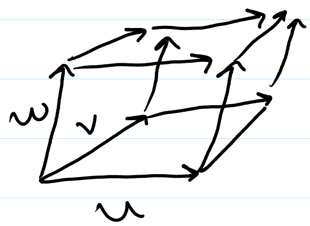
We have
$$v \cdot (u \times w) = (2,0,-2) \begin{vmatrix} i & j & k \\ -1 & 2 & -3 \\ -3 & 2 & -1 \end{vmatrix}$$
$$= (2,0,-2)\left(\begin{vmatrix} 2 & -3 \\ 2 & -1 \end{vmatrix} i - \begin{vmatrix} -1 & -3 \\ -3 & -1 \end{vmatrix} j + \begin{vmatrix} -1 & 2 \\ -3 & 2 \end{vmatrix} k \right)$$
$$= 2\begin{vmatrix} 2 & -3 \\ 2 & -1 \end{vmatrix} - 0\begin{vmatrix} -1 & -3 \\ -3 & -1 \end{vmatrix} - 2\begin{vmatrix} -1 & 2 \\ -3 & 2 \end{vmatrix}$$
$$= 0.$$

Another possible approach: find the plane that contains P_1, P_2, P_3.

Fun fact: Scalar Triple Product $u \cdot (v \times w)$

set $u = (x_1, y_1, z_1)$, $v = (x_2, y_2, z_2)$, $w = (x_3, y_3, z_3)$ $\rightsquigarrow$ $u \cdot (v \times w) = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$

= Volume of the parallelepiped spanned by u, v, w



Parallelepiped

Parallepiped has 0 volume when u, v, w are coplanar!

$\boxed{11.2/11.3}$

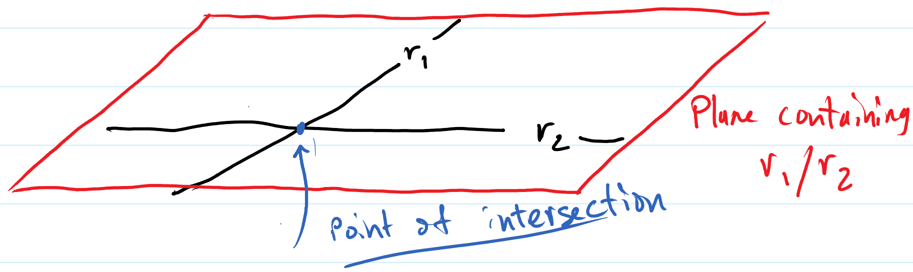
② Consider the lines given by

Direction vectors

$$r_1(t) = \boxed{(1,1,0)} + t\,\boxed{(1,-1,2)} = (1+t,\ 1-t,\ 2t)$$
$$r_2(s) = \boxed{(2,0,2)} + s\,\boxed{(-1,1,0)} = (2-s,\ s,\ 2)$$

a) Show that r_1 and r_2 intersect.

b) Find an equation of the plane that contains both r_1 and r_2.



Solution (a) set $r_1(t) = r_2(s)$. We have

$$(1+t, 1-t, 2t) = (2-s, s, 2)$$

Two vectors are equal if and only if their components are equal.

$$\Rightarrow \begin{cases} 1+t = 2-s \\ 1-t = s \\ 2t = 2 \end{cases} \Rightarrow t=1 \Rightarrow \begin{cases} 2 = 2-s \\ \boxed{\begin{matrix} 0 = s \\ t = 1 \end{matrix}} \end{cases}$$

Set $t=1$: r_1, r_2 intersect at $(2,0,2)$ which occurs when $s=0$, $t=1$.

(b) To find eq. of a plane we need a normal vector and a point in the plane. We already know $r_0 = (2,0,2)$ lies in the plane. To find the normal vector n, we compute

$$n = (1,-1,2) \times (-1,1,0)$$

[since $(1,-1,2)$ and $(-1,1,0)$ point in the direction of r_1, r_2 resp.]

$$= \begin{vmatrix} i & j & k \\ 1 & -1 & 2 \\ -1 & 1 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 2 \\ 1 & 0 \end{vmatrix} i - \begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} j + \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} K$$

$$= \begin{vmatrix} -1 & 2 \\ 1 & 0 \end{vmatrix} i - \begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} j + \underbrace{\begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix}}_{=0} K$$

$$= -2i - 2j$$

$$= (-2, -2, 0).$$

Let $r = (x, y, z)$ be arbitrary. An eq. for the plane is

$$n \cdot (r - r_0) = 0$$

we have

$$(-2, -2, 0) \cdot (x-2, y-0, z-2) = 0$$

$$\Rightarrow \quad \boxed{-2x + 4 - 2y = 0} \quad \text{Plane that contains } r_1 \text{ and } r_2 \text{!}$$

Problem 3

Tuesday, April 14, 2020 7:46 PM

$\boxed{11.7}$

3

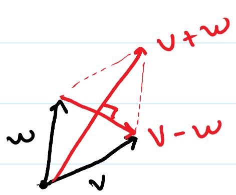
a) Let $v, w \in \mathbb{R}^n$. If $\|v\| = \|w\|$, prove that $v+w$ and $v-w$ are orthogonal.

b) Suppose that three points a, b, c lie on a circle such that a and b are antipodal. Use part (a) to show that Δabc is a right triangle.

<u>Proof</u> (of (a)) Let $v, w \in \mathbb{R}^n$ and assume $\|v\| = \|w\|$. We know that two vectors are orthogonal if and only if their dot product is 0. We have

$$
\begin{aligned}
(v+w)\cdot(v-w) &= v\cdot v - v\cdot w + w\cdot v - w\cdot w \\
&= v\cdot v - \cancel{v\cdot w} + \cancel{v\cdot w} - w\cdot w \\
&= v\cdot v - w\cdot w \\
&= \|v\|^2 - \|w\|^2 \\
&= \|v\|^2 - \|v\|^2 \quad \text{(by assumption)} \\
&= 0.
\end{aligned}
$$

Note: $\|u\| = \sqrt{u\cdot u}$



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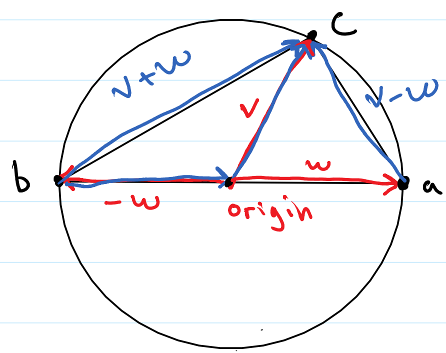
<u>Proof</u> (of (b))



Prove Δabc is a right triangle

Note that $\|v\| = r = \|w\|$ where r is the radius of the circle. Clearly, Δabc is a right triangle if and only if $v+w$ and $v-w$ are orthogonal. But $\|v\| = \|w\|$ so this is true by part (a)!

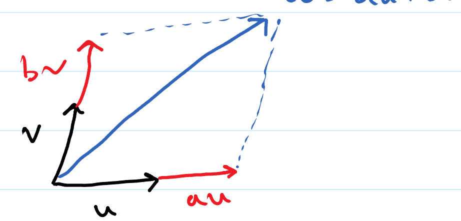
∎

Problem 4

Tuesday, April 14, 2020 8:07 PM

11.3

④ Let $u, v, w \in \mathbb{R}^3$. Suppose that there exists $a, b \in \mathbb{R}$ such that $w = au + bv$. Find the value of $u \cdot (v \times w)$.



Solution We have

$$\begin{aligned}
u \cdot (v \times w) &= u \cdot (v \times (au + bv)) \\
&= u \cdot (v \times au + v \times bv) \\
&= u \cdot (a(v \times u) + b(v \times v)) \\
&= au \cdot (v \times u) + bu \cdot (v \times v) \\
&= 0 + 0 \\
&= 0.
\end{aligned}$$

Notice $v \times u$ is perpendicular to au. So $au \cdot (v \times u) = 0$.

Also, $v \times v = 0$.