M23A_Fin_S18v1

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Math 23A, Spring 2018 **Final EXAM** **UCSC, Bäuerle & Tromba, 06/13/18**

Notes: No calculators are allowed. **Please put away your cell phones and other electronic devices, including smart watches, turned off or in airplane mode.** Put your answers in the provided boxes.

Warning: No cheating will be tolerated. **Students who communicate with other students during the exam must immediately turn in their exam.**

Your Name: Jadyn Breland **Your Score:** 0 /100 **Secret No.:** 1.618

1. (5 points) The equation of the tangent plane to the graph of $f(x,y) = 2 - 5y^2 + x^2y$ at the point $(1,1)$.
(A) $z = 2x - 9y$ (B) $z = -2 + 2(x-1) - 9(y-1)$ (C) $z = 2 + 2(x-1) - 9(y-1)$
(D) $z = 2 + 2xy(x-1) + (-10y^2 + x^2)(y-1)$ (E) $z = -2 - 2(x-1) + 9(y-1)$

Answer (Letter): B

2. (5 points) For each of the questions below, indicate if the statement is true (T) or false (F).

(a) Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be a function of class C^2. Then curl $(\nabla f) = \mathbf{0}$. Answer (T/F): T

(b) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a function where all second order partial derivatives exist and are continuous for all points $(x,y) \in \mathbb{R}^2$. Then $\frac{\partial^2 f}{\partial y \partial x}(x,y) = \frac{\partial^2 f}{\partial x \partial y}(x,y)$ for all points $(x,y) \in \mathbb{R}^2$.

Answer (T/F): T

(c) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a function where all second order partial derivatives exist but are not continuous for all points $(x,y) \in \mathbb{R}^2$. Then $\frac{\partial^2 f}{\partial y \partial x}(x,y) = \frac{\partial^2 f}{\partial x \partial y}(x,y)$ for all points $(x,y) \in \mathbb{R}^2$.

Answer (T/F): F

(d) Let $f(x,y)$ be a C^2 function which has a local minimum at $(0,0)$. Then the Hessian matrix of f at $(0,0)$ is necessarily negative definite. Answer (T/F): F

(e) Let $D \subset \mathbb{R}^2$ be a closed and bounded set. Every continuous function $f : D \longrightarrow \mathbb{R}$ has a global maximum and a global minimum value on D. Answer (T/F): T

3. (9 points) Let S be the quadratic surface given by $x^2 + y^2 + 2z^2 = 10$.
(a) Classify S:
(A) S is an ellipsoid (B) S is a hyperboloid of one sheet (C) S is a hyperboloid of two sheets
(D) S is an elliptic cone (E) S is an elliptic paraboloid Answer (Letter): A

(b) Find the equation of the tangent plane to S at the point $(2,2,1)$

(c) Find point(s) $P(x_0, y_0, z_0)$ on S so that the tangent plane to S at the point P is normal to the vector $\vec{k} = (0,0,1)$. Write DNE if no such point exists.

$P(x_0, y_0, z_0)=$

4. (8 points) Consider the function

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases} \qquad (1)$$

(a) Compute $\frac{\partial f}{\partial x}(0,0)$ $\qquad \frac{\partial f}{\partial x}(0,0) =$ 0

(b) Compute $\frac{\partial f}{\partial y}(0,0)$ $\qquad \frac{\partial f}{\partial y}(0,0) =$ 0

For (c) and (d), state whether the statement is true (T) or false (F)

(c) The function $f(x,y)$ is continuous at $(0,0)$. Answer (T/F): T

(d) The function $f(x,y)$ is differentiable at $(0,0)$. Answer (T/F): F

5. (5 points) Suppose $f : \mathbb{R}^3 \to \mathbb{R}$ is a differentiable function which has an absolute maximum value M and an absolute minimum m. Suppose further that $m = M$. What can be said about its derivative $\mathbf{D}f$?

$\mathbf{D}f =$ $(0,0,0)$

6. (6 points) Find the (x,y,z) coordinates of the point P where the line $\mathbf{l}(t) = (x,y,z) = (3-t, 4+t, -1+t)$ intersects the plane $2z = 2x + y$.

$P =$

7. (12 points) Consider $f(x,y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$
(a) Find the four critical points of $f(x,y)$

(i) $(0,0)$ (ii) $(0,2)$ (iii) $(1,1)$ (iv) $(-1,1)$

(b) State whether (i), (ii), (iii) and (iv) are a local maximum (MAX), local minimum (MIN) or saddle point (SAD).

(i) Max (ii) Min (iii) Sad (iv) Sad

① A point in the plane is given by $f(1,1) = -2$. A normal vector is given by $\nabla g(1,1,-2)$ where $g(x,y,z) := f(x,y) - z$, that is $\nabla g(1,1,-2) = (2,-9,-1)$. So eq. is

$$\boxed{2(x-1) - 9(y-1) - (z+2) = 0.}$$

② a) $$\text{Curl}(\nabla f) = \nabla \times \nabla f = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \partial/\partial y & \partial/\partial z \\ f_x & f_y & f_z \end{vmatrix} = (f_{zy} - f_{yz}, f_{zx} - f_{xz}, f_{yx} - f_{xy})$$
$$= (0,0,0) \quad (\text{by Clairaut})$$

b) True. This is Clairaut's theorem.

c) False. I gave a counter-example in week 6 (problem 3).

d) False. See second derivative test.

e) True. Multivariable analog of "a continuous fnc on a closed interval $[a,b]$ achieves a max/min"

③ a) Ellipsoid. See classification of quadratic surfaces in book.

b) A point in the plane is $(2,2,1)$ and a normal vector is $\nabla(x^2+y^2+2z^2)\big|_{(2,2,1)} = (4,4,4)$. A simpler normal vector is then $n = (1,1,1)$. So eq of plane is

$$\boxed{x - 2 + y - 2 + z - 1 = 0}$$

c) Let (x_0, y_0, z_0) live on S. Then $x_0^2 + y_0^2 + 2z_0^2 = 10$. The tangent plane at (x_0,y_0,z_0) has normal vector $n = (2x_0, 2y_0, 4z_0)$. We require that n is parallel to $(0,0,1)$, i.e.

$$(2x_0, 2y_0, 4z_0) = c(0,0,1)$$

for some $c \in \mathbb{R}$. This shows that $x_0 = y_0 = 0$. Thus, $z_0 = \pm\sqrt{5}$ since the point lies on the plane. So the two solutions are $(0,0,\sqrt{5}), (0,0,-\sqrt{5})$.

④ a) By definition $\frac{\partial f}{\partial x}(0,0) = \lim_{h \to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0.$

b) Same argument as 4(a), $\frac{\partial f}{\partial y}(0,0) = 0$

c) Set $x = r\cos\theta$ and $y = r\sin\theta$. Then $r \to 0$ as $(x,y) \to (0,0)$.
Thus, $$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r \to 0} \frac{r^2 \sin\theta\cos\theta}{r} = \lim_{r\to 0} r\sin\theta\cos\theta$$
$$= 0 \quad (\text{since } r\sin\theta\cos\theta \text{ is continuous})$$

d) If f were differentiable at $(0,0)$, we would have
$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - f(0,0) - \left[\frac{\partial f}{\partial x}(0,0)\ \frac{\partial f}{\partial y}(0,0)\right]\begin{bmatrix} x-0 \\ y-0 \end{bmatrix}}{\|(x,y)\|} = 0$$
by definition of the derivative (see Ch. 13.3!).
Since $f(0,0) = \frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0$, this becomes

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y)}{\sqrt{x^2+y^2}} = \frac{xy}{x^2+y^2}.$$ Along the path $y = 0, x > 0$, the limit is 0 but along the path $y = x, x > 0$ the limit is $1/2$. So the limit does not exist, a contradiction.

⑤ By assumption $M = m \le f(x,y,z) \le M$ for all $(x,y,z) \in \mathbb{R}^3$.
Thus f is constant. So $Df = [0 \ 0 \ 0]$.

⑥ Since $x(t) = 3-t$, $y(t) = 4+t$, $z(t) = t-1$, just solve $2z(t) = 2x(t) + y(t)$ for t. We have
$$2t - 2 = 6 - 2t + 4 + t$$
$$\rightarrow t = 4 \quad \Rightarrow P = (x(4), y(4), z(4)) = (-1, 8, 3)$$

⑦ a) The critical points are the solutions to the system
(1) $0 = f_x = 6xy - 6x = 6x(y-1)$
(2) $0 = f_y = 3x^2 + 3y^2 - 6y$

Eq (1) $\Rightarrow$ $x = 0$ or $y = 1$. If $x = 0$, eq (2) becomes $3y^2 - 6y = 0$ $\Rightarrow$ $y = 0$ or $y = 2$. So this case gives $(0,0), (0,2)$. If $y = 1$, then eq (2) gives $3x^2 = 3$ $\Rightarrow$ $x = \pm 1$. So in this case we have $(-1,1), (1,1)$.

b) Apply 2nd Der. Test: the Hessian at (x,y) is

$$H(x,y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} 6y-6 & 6x \\ 6x & 6y-6 \end{vmatrix} = (6y-6)^2 - 36x^2$$

Critical pt	f_{xx}	H	Max/Min/Saddle

8. (10 points) Use the method of Lagrange Multipliers to find the maximum and minimum values of $f(x, y, z) = x + y + 2z$ subject to the constraint $x^2 + y^2 + z^2 = 3$.

(a) Give the three Lagrange multiplier equations:

Equation 1: $\boxed{1 = \lambda 2x}$ Equation 2. $\boxed{1 = \lambda 2y}$ Equation 3. $\boxed{2 = \lambda 2z}$

(b) Give a comma-separated list of all critical points of f subject to the given constraint:

$\boxed{\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \sqrt{2}\right), \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, -\sqrt{2}\right)}$

(c) Give the global extreme values of f subject to the given constraint.

Maximum value of $f = \boxed{3\sqrt{2}}$ Minimum value of $f = \boxed{-3\sqrt{2}}$

9. (12 points) Let $f(u, v) = (u^2 v, u - v, u + v^2)$ and $g(x, y) = (y, x)$. Find $\mathbf{D}f(u, v)$, $\mathbf{D}g(x, y)$ and use the chain rule to compute the derivative $\mathbf{D}(f \circ g)(0, 1)$.

$\mathbf{D}f(u, v) =$ $\mathbf{D}g(x, y) =$ $\mathbf{D}(f \circ g)(0, 1) =$

10. (12 points) Let $f(x, y) = \dfrac{x^2}{x + y}$. Compute the following:

(a) $\nabla f(x, y)$

$\nabla f(x, y) =$

(b) The direction of the fastest rate of increase of f at the point $P(-1, 0)$. Give your answer as a unit vector.

(c) The directional derivative (i.e. rate of change) of f at the point $P(-1, 0)$ in the direction $\mathbf{v} = (2, 1)$.

(d) Give a direction $\mathbf{u} = (u_1, u_2)$ so that the directional derivative of f at the point $P(-1, 0)$ in the direction $\mathbf{u}$ has value 0. Give your answer as a unit vector.

$\mathbf{u} =$

11. (8 points) A vector field $\mathbf{F} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is defined by $\mathbf{F}(x, y, z) = (-z^2 + y, y^2 + x, 2z(x - y))$. Compute the following:

(a) (3 points) $\mathrm{div}(\mathbf{F}) = \nabla \cdot \mathbf{F} =$

(b) (3 points) $\mathrm{curl}(\mathbf{F}) = \nabla \times \mathbf{F} =$

(c) (2 points) True (T) or False (F): $\mathbf{F}$ is a gradient vector field.

12. (8 points)

(a) Find the first order Taylor polynomial (or Taylor approximation) of $f(x, y) = x \sin(2y) + 1$ at $(0, \pi)$.

$T_1(x, y) =$

(b) Find the second order Taylor polynomial (or Taylor approximation) of $f(x, y) = x \sin(2y) + 1$ at $(0, \pi)$.

$T_2(x, y) =$

$$H(x,y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} 6y-6 & 6x \\ 6x & 6y-6 \end{vmatrix} = (6y-6)^2 - 36x^2$$

Critical pt	f_{xx}	H	Max/Min/Saddle
$(0,0)$	$-6<0$	$36>0$	Max
$(0,2)$	$6>0$	$36>0$	Min
$(-1,1)$	0	$-36<0$	Saddle
$(1,1)$	0	$-36<0$	saddle

8 a) By Lagrange, there exist $\lambda \in \mathbb{R}$ such that $(1,1,2) = \lambda(2x, 2y, 2z)$. The critical points are the solutions to the system of eq.

(1) $1 = \lambda 2x$
(2) $1 = \lambda 2y$
(3) $2 = \lambda 2z$
(4) $x^2 + y^2 + z^2 = 3$

(b) By eq. (1), $\lambda \neq 0$. So we can divide by λ to obtain $x = \frac{1}{2\lambda}$, $y = \frac{1}{2\lambda}$ and $z = \frac{1}{\lambda}$. Substitute these into eq (3) to obtain

$$\frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} + \frac{1}{\lambda^2} = 3.$$

This gives $\lambda^2 = \frac{1}{3}\left(\frac{1}{4} + \frac{1}{4} + 1\right) = \frac{1}{2}$. So $\lambda = \pm\sqrt{\frac{1}{2}}$. When $\lambda = \frac{1}{\sqrt{2}}$ we obtain $(x,y,z) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \sqrt{2}\right)$. We $\lambda = -\frac{1}{\sqrt{2}}$, we obtain $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, -\sqrt{2}\right)$

(c) Since $D = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 3\}$ is a closed and bounded set and f is continuous, f attains a max/min on D (see Problem 2e above!). The critical points are the only possible points where extrema occur. Compute:

$$f\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \sqrt{2}\right) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + 2\sqrt{2} = 3\sqrt{2},$$
global max (on D)

$$f\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, -\sqrt{2}\right) = -3\sqrt{2}$$
global min (on D)

9 $f: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ so Df is the 3×2 matrix

$$Df(u,v) = \begin{bmatrix} 2uv & u^2 \\ 1 & -1 \\ 1 & 2v \end{bmatrix}.$$

Similarly, $g: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ so Dg is the 2×2 matrix $Dg(x,y) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. We have $g(0,1) = (1,0)$ so by the chain rule

$$\begin{aligned} D(f \circ g)(0,1) &= Df(g(0,1)) \cdot Dg(0,1) \\ &= Df(1,0) \cdot Dg(0,1) \\ &= \begin{bmatrix} 0 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

10 a) $\nabla f = (f_x, f_y) = \left(\frac{2x(x+y) - x^2}{(x+y)^2}, \frac{-x^2}{(x+y)^2}\right)$

b) $\nabla f(-1,0) = (1, -1)$ is the direction of steepest ascent. Normalized this is $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$.

c) This is $\nabla f(-1,0) \cdot \frac{(2,1)}{\sqrt{5}} = (1,-1) \cdot \frac{(2,1)}{\sqrt{5}} = \frac{1}{\sqrt{5}}$.

d) We seek a vector u s.t. $(1,-1) \cdot u = 0$ and $\|u\| = 1$. If $u = (x,y)$ this become $x - y = 0$ where $x^2 + y^2 = 1$. So $x = y = \pm\frac{\sqrt{2}}{2}$. So take $u = \left(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}\right)$.

11 a) This is straightforward.
b) This is straightforward.
c) False. Suppose F is a gradient vector field. Then there is a C^1 function f such that $F = \nabla f$. Hence, according to Problem 2(b), we must have

$$\nabla \times F = \nabla \times (\nabla f) = 0.$$

This is impossible since $\nabla \times F \neq 0$ by 11 (b).

12 Use the formula

$$\underbrace{f(0,\pi) + f_x(0,\pi)h_1 + f_y(0,\pi)h_2} + \frac{1}{2}f_{xx}(0,\pi)h_1^2 + f_{xy}(0,\pi)h_1h_2 + f_{yy}(0,\pi)h_2^2$$

Answers:

1. B
2. T T F F T
3. (a) A
   (b) $4(x-2)+4(y-2)+4(z-1)=0$ OR $x+y+z=5$
   (c) $(0,0,\pm\sqrt{5})$
4. (a) 0
   (b) 0
   (c) T
   (d) F
5. $Df=(0,0,0)=\vec{0}$
6. (-1,8,3)
7. (a) (0,0) (0,2) (1,1) (-1,1)
   (b) MAX MIN SAD SAD
8. (a) $\begin{cases}1=2\lambda x\\1=2\lambda y\\2=2\lambda z\end{cases}$ (b) $\lambda=\pm\frac{\sqrt{2}}{2}$ (c) MAX: $f=3\sqrt{2}$; MIN $F=-3\sqrt{2}$
9. (a) $\begin{bmatrix}2uv & u^2\\1 & -1\\1 & 2v\end{bmatrix}$ (b) $\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$ (c) $\begin{bmatrix}1 & 0\\-1 & 1\\0 & 1\end{bmatrix}$
10. (a) $\left(\frac{x^2+2xy}{(x+y)^2},\frac{-x^2}{(x+y)^2}\right)$
    (b) $(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}})$
    (c) $\frac{1}{\sqrt{5}}$
    (d) $(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$
11. (a) 0
    (b) $(-2z,-2z,0)$
    (c) F
12. (a) $T_1=1$
    (b) $T_2=1+2x(y-\pi)$

✗ (12) Use the formula

$$\underbrace{\underbrace{f(0,\pi)+f_x(0,\pi)h_1+f_y(0,\pi)h_2}_{\text{First order}}+\frac{1}{2}f_{xx}(0,\pi)h_1^2+f_{xy}(0,\pi)h_1h_2+f_{yy}(0,\pi)h_2^2}_{\text{second order.}}$$