Problem 1

Tuesday, May 12, 2020 8:08 PM

① Let $u = u(x,y)$ and let (r,θ) be polar coordinates. Show that

$$\|\nabla u\|^2 = u_r^2 + \frac{1}{r^2} u_\theta^2 .$$

<u>Thm</u> (Chain Rule) Let $g : \mathbb{R}^n \to \mathbb{R}^m$ and $f : \mathbb{R}^m \to \mathbb{R}^p$ be function $f \circ g$ is defined and such that f and g are differentiable. Then $f \circ g$ is differentiable and $Df\circ g$ is given by

$$\underbrace{D(f \circ g)(x_0)}_{p \times n} = \underbrace{Df(g(x_0))}_{p \times m} \cdot \underbrace{Dg(x_0)}_{m \times n}$$

<u>Solution</u> Express u_r and u_θ in terms of u_x and u_y using chainrule.

When set $x = r\cos\theta$ and $y = r\sin\theta$, we can of this as a map $h : \mathbb{R}^2 \to \mathbb{R}^2$, $(r,\theta) \longmapsto (r\cos\theta, r\sin\theta)$.

Apply chain rule: as a function of (r,θ) — as a function of x,y

$$[u_r \ u_\theta] = D(u \circ h) = Du \cdot Dh$$

$$= [u_x \ u_y] \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix}$$

$$= [u_x \ u_y] \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

$$= [u_x\cos\theta + u_y\sin\theta \quad -u_x r\sin\theta + u_y r\cos\theta]$$

Comparing entries:

$$u_r = u_x\cos\theta + u_y\sin\theta$$
$$u_\theta = -u_x r\sin\theta + u_y r\cos\theta$$

Then,

$$u_r^2 + \frac{1}{r^2}u_\theta^2 = u_x^2\cos^2\theta + 2u_x \underline{u_y\sin\theta}\cos\theta + u_y^2\sin^2\theta +$$

$$\frac{1}{r^2}\left(u_x^2 r^2 \sin\theta - r^2 u_x u_y \sin\theta\cos\theta + r^2 u_y^2 \cos^2\theta\right)$$

$$= u_x^2\cos^2\theta + u_y^2\sin^2\theta + u_x^2\sin^2\theta + u_y^2\cos^2\theta$$

$$= u_x^2 + u_y^2$$

$$= \|\nabla u\|^2 \qquad \blacksquare$$

Problem 2

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② Let $f : \mathbb{R}^3 \to \mathbb{R}$ be differentiable. Find formulas for $\frac{\partial f}{\partial \rho}$, $\frac{\partial f}{\partial \theta}$, $\frac{\partial f}{\partial \phi}$ in terms of $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial z}$ where (ρ, θ, ϕ) are spherical coordinates.

Solution Apply chain rule as in Problem 1!

$$\begin{bmatrix} \frac{\partial f}{\partial \rho} & \frac{\partial f}{\partial \theta} & \frac{\partial f}{\partial \phi} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{bmatrix}$$

By comparing entries, $(x = \rho \sin\phi \cos\theta \quad y = \rho \sin\phi \sin\theta \quad z = \rho \cos\phi)$

$$\frac{\partial f}{\partial \rho} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \rho} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial \rho} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \rho}$$

$$= \sin\phi\cos\theta \frac{\partial f}{\partial x} + \sin\phi\sin\theta \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\cos\phi$$

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial \theta} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \theta}$$

$$= -\rho\sin\phi\sin\theta \frac{\partial f}{\partial x} + \rho\sin\phi\cos\theta \frac{\partial f}{\partial y}$$

$$\frac{\partial f}{\partial \phi} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \phi} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial \phi} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \phi}$$

$$= \rho\cos\phi\cos\theta \frac{\partial f}{\partial x} + \rho\cos\phi\sin\theta \frac{\partial f}{\partial y} - \rho\sin\phi \frac{\partial f}{\partial z}.$$

Problem 3

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e.g. $x^2+y^2=4$

③ Define $y(x)$ implicitly via $G(x, y(x)) = K$ where $G: \mathbb{R}^2 \to \mathbb{R}$.
Prove the implicit differentiation formula: if $y(x)$ and G are differentiable and $\frac{\partial G}{\partial y} \neq 0$, then

$$\frac{dy}{dx} = -\frac{\partial G/\partial x}{\partial G/\partial y}.$$

Proof Define $H: \mathbb{R} \to \mathbb{R}$ by $H(x) = (G \circ F)(x)$ where $F: \mathbb{R} \to \mathbb{R}^2$ defined by $F(x) = (x, y(x))$. Note $H(x) = K$ so $H'(x) = 0$.

By def $H(x) = (G \circ F)(x)$
$= G(F(x))$
$= G(x, y(x)) = K$

Using Chain Rule we obtain

$$0 = H'(x) = DH = DG \cdot DF$$

$$= \begin{bmatrix} \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{bmatrix} \begin{bmatrix} 1 \\ \frac{dy}{dx} \end{bmatrix}$$

($Y: \mathbb{R} \to \mathbb{R}$ so $\frac{\partial y}{\partial x} = \frac{dy}{dx}$)

$$= \frac{\partial G}{\partial x} + \frac{\partial G}{\partial y} \cdot \frac{dy}{dx}$$

Since $\frac{\partial G}{\partial y} \neq 0$ we obtain

$$\frac{\partial y}{\partial x} = -\frac{\partial G/\partial x}{\partial G/\partial y}.$$

$\blacksquare$

Ex Find $\frac{dy}{dx}$ if $x^2+y^2=4$.

Solution (Calc III) $\frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}$

(Calc I) $2x + 2y\frac{dy}{dx} = 0$

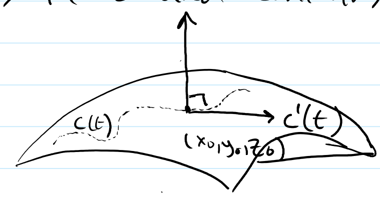
$$\Rightarrow \frac{dy}{dx} = \frac{-2x}{2y} = -\frac{x}{y}$$

Ex Find $\frac{dy}{dx}$ if $x\cos y + y\sin x = 0$

(calc III) $\frac{dy}{dx} = -\frac{\cos y + y\cos x}{-x\sin y + \sin x}$

④ Let $f:\mathbb{R}^3 \to \mathbb{R}$ be a C^1 map and suppose (x_0, y_0, z_0) lies on the level surface S defined by $f(x,y,z) = K$. Show that $\nabla f(x_0,y_0,z_0)$ is normal to S.

Proof Idea To show ∇f is normal to S, I will show that $\nabla f \cdot c' = 0$ for any curve c that lies in S and contains (x_0, y_0, z_0).



Let C be a curve that lies in S parameterized by

$$c(t) = (x(t), y(t), z(t))$$

and such that $c(0) = (x_0, y_0, z_0)$. We have

$$\begin{aligned}\nabla f(x,y,z) \cdot c'(t) &= (f_x, f_y, f_z) \cdot (x'(t), y'(t), z'(t)) \\ &= f_x x'(t) + f_y y'(t) + f_z z'(t) \\ &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \\ &= \frac{\partial}{\partial t}(f \circ c)(t) = \frac{d}{dt}(f \circ c)(t)\end{aligned}$$

Evaluate at $t=0$,

$$\begin{aligned}\nabla f(x_0,y_0,z_0) \cdot c'(0) &= \frac{d}{dt}(f \circ c)(0) \\ &= \frac{d}{dt} K \qquad \text{(since } c \text{ lies in } S \text{, } (f \circ c)(t) = K\text{)} \\ &= 0.\end{aligned}$$

Since $c(t)$ was an arbitrary curve, this shows that ∇f is normal to S. ∎

Definition The tangent plane to the surface $f(x,y,z) = K$ at (x_0, y_0, z_0) is given by

$$(*) \quad \nabla f(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0.$$

<u>Recall</u> The plane tangent to the graph G of $f(x,y)$ is given by

$$f_x(x-x_0)+f_y(y-y_0)=z-f(x,y)$$

Notice that the surface defined by the equation $F(x,y,z)=0$ where $F(x,y,z)=f(x,y)-z$ is precisely the graph G of $f(x,y)$. By $(*)$, the tangent plane is

$$\nabla F(x,y,z)(x-x_0, y-y_0, z-z_0)=0 \quad =$$

$$\Rightarrow \quad (f_x, f_y, -1)\cdot(x-x_0, y-y_0, z-z_0)=0$$