① Compute the derivitive $Df(x)$ for each function:

a) $f: \mathbb{R}^2 \to \mathbb{R}^2$, $f(x,y) = (x + e^z + y, yx^2)$

b) $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = (xe^y + \cos y, x, x + e^y)$

c) $f: \mathbb{R}^3 \to \mathbb{R}^3$, $f(\rho, \theta, \phi) = (\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\theta)$

<u>Definition</u> If $f: \mathbb{R}^n \to \mathbb{R}^m$ is differentiable, the derivative Df is the $m \times n$ matrix whose ij-entry is $\frac{\partial f_i}{\partial x_j}$:

$$Df(x_1, \dots, x_n) = \begin{bmatrix} \underbrace{\frac{\partial f_1}{\partial x_1}}_{f_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & & \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

a) $f: \mathbb{R}^2 \to \mathbb{R}^2$, $f(x,y) = (x + e^y + y, yx^2)$ $\quad \frac{\partial f_1}{\partial y} = 0 + e^y + 1$

<u>Solution</u>:

$$Df(x,y) = \begin{bmatrix} 1 & e^y + 1 \\ 2xy & x^2 \end{bmatrix}$$

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<u>Note</u> $\mathbb{R}^n = \{(x_1, \dots, x_n) : x_i \in \mathbb{R}\}$

when we write $f: \mathbb{R}^n \to \mathbb{R}^m$ means domain $f = \mathbb{R}^n$ and range $f = \mathbb{R}^m$

b) $f: \mathbb{R}^2 \to \mathbb{R}^3$, $f(x,y) = (xe^y + \cos y, x, x + e^y)$

<u>Solution</u>

$$Df(x,y) = \begin{bmatrix} e^y & xe^y - \sin y \\ 1 & 0 \\ 1 & e^y \end{bmatrix}$$

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c) $f: \mathbb{R}^3 \to \mathbb{R}^3$, $f(\rho, \theta, \phi) = (\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\theta)$

<u>Solution</u>

$$\begin{bmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \end{bmatrix}$$

Solution

$$Df(\rho, \theta, \phi) = \begin{bmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\theta & -\rho\sin\theta & 0 \end{bmatrix}$$

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Problem 2

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(2) Let $f(x,y) = xe^{y^2} - ye^{x^2}$.

a) Find an equation for the plane tangent to the graph of f at $(1,2)$

b) Which point on the surface $z + y^2 - x^2 = 0$ has a tangent plane parallel to the plane in part (a)?

Recall If $f: \mathbb{R}^2 \to \mathbb{R}$ is differentiable at (x_0, y_0), then the plane tangent to the graph of f at $(x_0, y_0, f(x_0,y_0))$ is given by

$$\boxed{z - f(x_0,y_0) = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)}$$

Note The normal vector is $n = (f_x(x_0,y_0), f_y(x_0,y_0), -1)$

Solution: First compute $f_x, f_y, f(1,2)$:

$f_x(x,y) = e^{y^2} - 2xy\,e^{x^2}$ $\qquad f(x,y) = xe^{y^2} - ye^{x^2}$.

$f_x(1,2) = e^4 - 4e$

$f_y(x,y) = 2xye^{y^2} - e^{x^2}$

$f_y(1,2) = 4e^4 - e$ $\qquad f(1,2) = e^4 - 2e$

So eq of tangent plane is:

$$P_1: \quad z - (e^4 - 2e) = (e^4 - 4e)(x-1) + (4e^4 - e)(y-2)$$

b) Which point on the surface $z + y^2 - x^2 = 0$ has a tangent plane parallel to the plane in part (a)?

Solution Note that two planes are parallel if and only if their normal vectors are parallel. The normal vector for P_1 is $n_1 = (e^4 - 4e, 4e^4 - e, -1)$. Next notice that the graph of the function $g(x,y) = x^2 - y^2$ is the surface $z + y^2 - x^2 = 0$. By the above definition the normal vector n_2 for the plane tangent to the graph of g at (x,y) is $n_2 = (g_x, g_y, -1)$

$= (2x, -2y, -1)$

$= (2x, -2y, -1)$

If n_1 is parallel to n_2, then $\boxed{n_1 = c\,n_2}$ for some $c \in \mathbb{R}$:

$$(e^4 - 4e, 4e^4 - e, -1) = c(2x, -2y, -1) \implies -1 = -1 \cdot c$$

So $c = 1$, and $e^4 - 4e = 2x$ and $4e^4 - e = -2y$

$$\implies \boxed{x = \frac{e^4 - 4e}{2} \quad y = \frac{e - 4e^4}{2}}$$

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Problem 3

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③ Let $f(x,y) = \begin{cases} \dfrac{x^2y^4}{x^4+6y^8}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$

a) Show that $\frac{\partial f}{\partial x}(0,0)$ and $\frac{\partial f}{\partial y}(0,0)$ exist.

b) Show that f is not differentiable at $(0,0)$ by showing that f is not continuous.

Def If $f: \mathbb{R}^2 \to \mathbb{R}$, then the partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ are given by

$$\frac{\partial f}{\partial x}(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h} \;;\; \frac{\partial f}{\partial y}(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

if these limits exist.

a) Show that $\frac{\partial f}{\partial x}(0,0)$ and $\frac{\partial f}{\partial y}(0,0)$ exist.

Proof we have

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0.$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{f(0,0+h) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0.$$

So both partial derivatives exist at $(0,0)$. But,

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$$f(x,y) = \begin{cases} \dfrac{x^2y^4}{x^4+6y^8}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

is not continuous at the origin $\left(\lim_{(x,y)\to(0,0)} f(x,y) \neq f(0,0) = 0 \right)$

is not continuous at the origin $\left(\lim_{(x,y)\to(0,0)} f(x,y) \neq f(0,0) = 0 \right)$

Take the path $y = \sqrt{x}$, $x > 0$. The limit is

$$\lim_{(x,\sqrt{x})\to(0,0)} f(x,y) = \lim_{x\to 0} \frac{x^2(\sqrt{x})^4}{x^4 + 6(\sqrt{x})^8}$$

$$= \lim_{x\to 0} \frac{x^4}{x^4 + 6x^4} = \lim_{x\to 0} \frac{x^4}{7x^4}$$

$$= \lim_{x\to 0} \frac{1}{7} = \frac{1}{7}$$

So if $\lim_{(x,y)\to(0,0)} f(x,y)$ exists, it must be $=$ to $\frac{1}{7}$. So f is not continuous and hence not differentiable at $(0,0)$. $\blacksquare$

Problem 4

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④ Compute the gradient ∇f for the following functions

a) $f(x,y,z) = \dfrac{xyz}{x^2+y^2+z^2}$

b) $f(x,y,z) = \ln(x^2+y^2+z^2)$

<u>Recall</u> If $f: \mathbb{R}^m \to \mathbb{R}$ then the gradient ∇f is given by

$$\nabla f(x_1, \dots, x_n) = (f_{x_1}, f_{x_2}, \dots, f_n).$$

<u>Solution</u>

a) $$\nabla f(x,y,z) = \left(\frac{yz(x^2+y^2+z^2) - 2x^2yz}{(x^2+y^2+z^2)^2}, \frac{xz(x^2+y^2+z^2) - 2y^2xz}{(x^2+y^2+z^2)^2}, \frac{xy(x^2+y^2+z^2) - 2z^2xy}{(x^2+y^2+z^2)^2} \right)$$

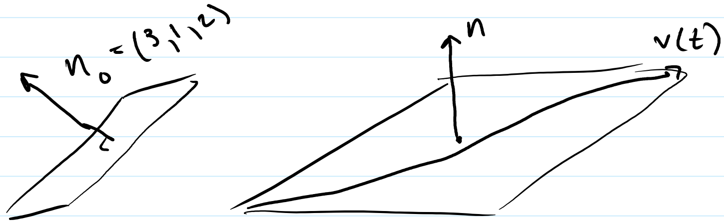
b) $$\nabla f(x,y,z) = \left(\frac{2x}{x^2+y^2+z^2}, \frac{2y}{x^2+y^2+z^2}, \frac{2z}{x^2+y^2+z^2} \right)$$ ▨

11.3.11 Find eq of plane containing the line

$$v(t) = (-3,4,2) + t(2,-3,-4)$$

and perpindicular to $3x + y + 2z + 4 = 0$

Solution We need a vector perpindicular to our plane and a point in the plane. Since the plane contains $v(t)$, we can take the point $v(0) = (-3,4,2)$ as a point in the plane.



We need a vector n that is perpindicular $(3,1,2)$ and also perpindicular to $v(t)$. The vector $(2,-3,-4)$ points in the direction of $v(t)$, so we can take

$$n = (3,1,2) \times (2,-3,-4)$$

$$= \begin{vmatrix} i & \hat{j} & k \\ 3 & 1 & 2 \\ 2 & -3 & -4 \end{vmatrix}$$

$$= i\begin{vmatrix} 1 & 2 \\ -3 & -4 \end{vmatrix} - \hat{j}\begin{vmatrix} 3 & 2 \\ 2 & -4 \end{vmatrix} + \hat{k}\begin{vmatrix} 3 & 1 \\ 2 & -3 \end{vmatrix}$$

$$= (-4+6)i - (-12-4)j + (-9-2)k$$

$$= (2, 16, -11).$$

So the eq of plane is

$$\boxed{2(x-(-3)) + 16(y-4) - 11(z-2) = 0}$$