Problem 1

Tuesday, April 28, 2020 10:13 PM

① Compute the following limits, if they exist:

a) $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2+2}$

b) $\lim\limits_{(x,y)\to(0,0)} \dfrac{(x-y)^2}{x^2+y^2}$

c) $\lim\limits_{(x,y)\to(0,0)} (3x^2+3y^2)\log(x^2+y^2)$

d) $\lim\limits_{(x,y)\to(0,0)} \dfrac{y}{x^2+y^2}$

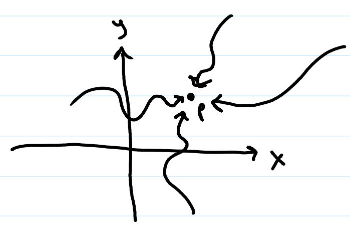
Recall: f is continuous at (a,b) if $\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

Solutions

a) $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2+2} = 0$ since $\dfrac{xy}{x^2+y^2+2}$ is continuous at $(0,0)$.

b) $\lim\limits_{(x,y)\to(0,0)} \dfrac{(x-y)^2}{x^2+y^2}$

Note that $f(x,y) = \dfrac{(x-y)^2}{x^2+y^2}$ is not continuous at $(0,0)$ since $f(0,0)$ is undefined. Level curves suggest the limit doesn't exist.



To show the limit does not exist, we need to find two paths that approach $(0,0)$, but give different values in the limit.

Along the line $y=0$ with $x>0$ the limit is

$$\lim_{(x,0)\to(0,0)} \frac{(x-0)^2}{x^2+0^2} = \lim_{x\to 0} \frac{x^2}{x^2} = \lim_{x\to 0} 1 = 1.$$

But along the line $\boxed{y=x}$ with $x>0$ the limit is

$(x,0)\to(0,0) \quad x^2+0^2 \quad x\to 0 \quad x^2 \quad x \to 0$

But along the line $\boxed{y=x}$ with $x>0$ the limit is

$$\lim_{(x,x)\to(0,0)} \frac{(x-x)^2}{x^2+x^2} = \lim_{x\to 0} 0 = 0.$$

Since $0 \neq 1$ the limit doesn't exist.

c) $\displaystyle\lim_{(x,y)\to(0,0)} (3x^2+3y^2)\ln(x^2+y^2)$

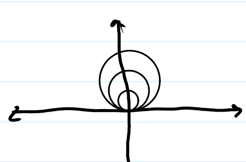
The function is not continuous at $(0,0)$. We convert to polar coordinates via $x = r\cos\theta$, $y = r\sin\theta$, $r^2 = x^2+y^2$. Note $\displaystyle\lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2} = 0$.

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} (3x^2+3y^2)\ln(x^2+y^2) &= \lim_{r\to 0} 3r^2 \ln r^2 \qquad \left(\text{indeterminant form } 0\cdot\infty\right)\\
&= 3\lim_{r\to 0} \frac{\ln r^2}{1/r^2}\\
&\overset{L'H}{=} 3\lim_{r\to 0} \frac{2r\cdot 1/r^2}{-2\, 1/r^3}\\
&= 3\lim_{r\to 0} -r^2 \qquad (\text{continuous at } 0)\\
&= 3\cdot 0\\
&= 0.
\end{aligned}$$

☒

d) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y}{x^2+y^2}$

The function is not continuous at $(0,0)$. The level curves look like



$$\frac{y}{x^2+y^2} = c \quad \text{for } c = 1, 2, 3$$

Since each level curve corresponds to a different function value, and the curves all pass through $(0,0)$, the limit doesn't exist.

☆ Another way to see that limit DNE is to use polar coordinates:

$$\lim_{(x,y)\to(0,0)} \frac{y}{x^2+y^2} = \lim_{r\to 0} \frac{\cos\theta}{r}$$

(note: $(x,y)\to(0,0)$ implies $r\to 0$, independent of θ!)

Fix θ. Then $\cos\theta$ is a constant and

$$\lim_{r\to\infty} \frac{c}{r} \quad \text{DNE}$$

for any constant $c \in \mathbb{R}$.

Problem 2

Tuesday, April 28, 2020 9:19 PM

② Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are functions such that

$$f(x,y) = g(xy) \qquad (\text{for all } (x,y) \in \mathbb{R}^2)$$

If $(a,b) \in \mathbb{R}^2$ and g is continuous at ab, then $\lim_{(x,y)\to(a,b)} f(x,y)$ exists and is equal to $\lim_{t \to ab} g(t) = g(ab)$.

__Proof__ Define $h : \mathbb{R}^2 \to \mathbb{R}$ via $h(x,y) = x \cdot y$. Notice $h(x,y)$ is continuous everywhere since h is a polynomial.

$$\begin{aligned}
\lim_{(x,y)\to(a,b)} f(x,y) &= \lim_{(x,y)\to(a,b)} g(xy) \\
&= \lim_{(x,y)\to(a,b)} g(h(x,y)) && \left[\begin{array}{l}\text{as } (x,y)\to(a,b),\ h(x,y)\to ab, \\ \text{and } g \text{ is continuous at } ab\end{array}\right] \\
&= g\left(\lim_{(x,y)\to(a,b)} h(x,y)\right) && \left[\ h \text{ is continuous at } (a,b)\ \right] \\
&= g(ab).
\end{aligned}$$

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Problem 3

Tuesday, April 28, 2020 9:58 PM

③ Compute the following limits, if they exist:

a) $\lim\limits_{(x,y)\to(0,0)} \dfrac{e^{xy}-1}{y}$

b) $\lim\limits_{(x,y)\to(0,0)} \dfrac{\cos(xy)-1}{x^2y^2}$

c) $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(x^2+y^2)}{x^2+y^2}$

Solutions

a) $\lim\limits_{(x,y)\to(0,0)} \dfrac{e^{xy}-1}{y} = \lim\limits_{(x,y)\to(0,0)} x \cdot \left(\dfrac{e^{xy}-1}{xy}\right)$

Consider the continuous function $g(t) = \begin{cases} \dfrac{e^t-1}{t}, & t \neq 0 \\ 1, & t=0. \end{cases}$

Note that $g(t)$ is continuous at $t=0$ since

$$\lim_{t\to 0} \frac{e^t-1}{t} = \lim_{t\to 0} \frac{e^t}{1} = 1 = g(1)$$

Also $g(xy) = \dfrac{e^{xy}-1}{xy}$. By ②, $\lim\limits_{(x,y)\to(0,0)} \dfrac{e^{xy}-1}{xy} = g(0) = 1.$

Thus, $\lim\limits_{(x,y)\to(0,0)} x \cdot \left(\dfrac{e^{xy}-1}{xy}\right) = \left(\lim\limits_{(x,y)\to(0,0)} x\right)\left(\lim\limits_{(x,y)\to(0,0)} \dfrac{e^{xy}-1}{xy}\right)$

$$= 0 \cdot 1 = 0$$

b) $\lim\limits_{(x,y)\to(0,0)} \dfrac{\cos(xy)-1}{x^2y^2}$ $\rightsquigarrow$ setting $u=xy$ $\rightsquigarrow$ $\lim\limits_{u\to 0} \dfrac{\cos u - 1}{u^2}$

Consider $g(t) = \begin{cases} \dfrac{\cos t - 1}{t^2}, & t \neq 0 \\ -\dfrac{1}{2}, & t = 0. \end{cases}$

Note that $g(t)$ is continuous at $t=0$ since $\lim\limits_{t\to 0} \dfrac{\cos t - 1}{t^2} = -\dfrac{1}{2}$.

Also, $g(xy) = \dfrac{\cos(xy)-1}{x^2y^2}$. Thus by ②

$$\lim_{(x,y)\to(0,0)} \frac{\cos(xy)-1}{x^2y^2} = \lim_{t\to 0} g(t)$$
$$= -\tfrac{1}{2}. \qquad \blacksquare$$