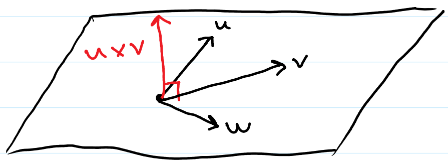
Problem 1

Sunday, April 5, 2020   10:43 PM

① Determine whether the vectors $u = (-1, 2, -3)$, $v = (-2, 0, 2)$, and $w = (-3, 2, -1)$ are coplanar, i.e., u, v, w lie in the same plane.



we need to find a test to determine if u, v, w are in the same plane.

Solution The vector $u \times v$ is perpendicular to both of u and v, hence it is perpendicular to the plane that contains u and v. Notice that w is coplanar to u and v if and only if $w \perp u \times v$.
But $w \perp u \times v$ if and only if $w \cdot (u \times v) = 0$.
We have

$$w \cdot (u \times v) = (-3, 2, -1) \cdot \begin{vmatrix} i & j & k \\ -1 & 2 & -3 \\ -2 & 0 & 2 \end{vmatrix}$$

$$= (-3, 2, -1) \cdot \left(\begin{vmatrix} 2 & -3 \\ 0 & 2 \end{vmatrix} i - \begin{vmatrix} -1 & -3 \\ -2 & 2 \end{vmatrix} j + \begin{vmatrix} -1 & 2 \\ -2 & 0 \end{vmatrix} k \right)$$

$$= -3 \begin{vmatrix} 2 & -3 \\ 0 & 2 \end{vmatrix} - 2 \begin{vmatrix} -1 & -3 \\ -2 & 2 \end{vmatrix} - 1 \begin{vmatrix} -1 & 2 \\ -2 & 0 \end{vmatrix}$$

$$= -3(4-0) - 2(-2-6) - 1(0-(-4))$$

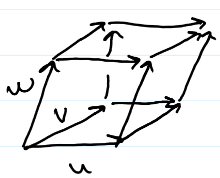
$$= -12 + 16 - 4 = 0$$

So, $w \perp (u \times v)$. So u, v, w are coplanar! ▨

Another viewpoint: Scalar triple product

Fact: $w \cdot (u \times v)$ = volume of the parallelepiped spanned by u, v, w.

Picture



Parallelepiped
If u, v, w are coplanar, then the volume is zero!

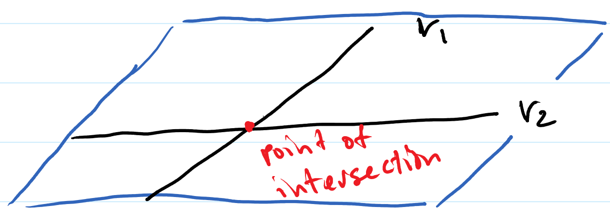
$(a,b,c)+(x,y,z)=(a+x,\ b+y,\ c+z)$

② Consider the lines given by

$$r_1(t) = (1,1,0) + t(1,-1,2) = (1+t,\ 1-t,\ 2t)$$
$$r_2(s) = (2,0,2) + s(-1,1,0) = (2-s,\ s,\ 2)$$

a) Show that r_1 and r_2 intersect.

b) Find an equation of the plane that contains both r_1 and r_2.



r_1 and r_2 determine a unique plane if and only if r_1 and r_2 intersect at a unique point.

Solution (a) We need to find $s,t \in \mathbb{R}$ such that

$$(1+t,\ 1-t,\ 2t) = (2-s,\ s,\ 2)$$

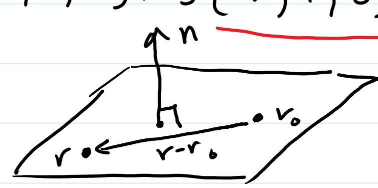
Recall that two vectors are equal if and only if their components are equal. This gives us a system of equations:

$$\boxed{\begin{cases} 1+t = 2-s \\ 1-t = s \\ 2t = 2 \end{cases}} \rightsquigarrow \boxed{t=1}.$$

Setting $t=1$ we obtain $\begin{cases} 2 = 2-s \\ \boxed{0 = s} \end{cases}$

So r_1 and r_2 intersect precisely when $s=0$ and $t=1$. So the point of intersection is $(2,0,2)$. ☒

(b) Find eq. of a plane containing $r_1(t) = (1,1,0) + t\underline{(1,-1,2)}$
$r_2(s) = (2,0,2) + s\underline{(-1,1,0)}$



Recall: Eq. of a plane is given by

$$(*) \quad n\cdot(r-r_0) = 0$$

where $n=(a,b,c)$ is orthogonal to the plane, $r_0=(x_0,y_0,z_0)$ is a point in the plane, and $r=(x,y,z)$ is arbitrary. Expanding $(*)$ we get

$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

We already know $r_0 = (2,0,2)$ is contained in the plane. The vectors $(1,-1,2)$ and $(-1,1,0)$ point in the direction of r_1 and r_2 respectively. So $n = (1,-1,2) \times (-1,1,0)$ will be orthogonal to the plane.

$$n = (1,-1,2) \times (-1,1,0) = \begin{vmatrix} i & j & k \\ 1 & -1 & 2 \\ -1 & 1 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 2 \\ 1 & 0 \end{vmatrix} i - \begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} j + \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} k$$

$$= -2i - 2j + 0k$$

$$= (-2,-2,0)$$

So an eq. of the plane will be

$$\boxed{-2(x-2) - 2(y-0) = 0}$$

∎

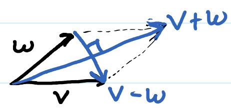
Problem 3

Tuesday, April 14, 2020 7:46 PM

(3)

a) Let $v, w \in \mathbb{R}^n$. If $\|v\| = \|w\|$, prove that $v+w$ and $v-w$ are orthogonal.

b) Suppose that three points a, b, c lie on a circle such that a and b are antipodal. Use part (a) to show that $\triangle abc$ is a right triangle.



Proof (of (a))

Let $v, w \in \mathbb{R}^n$ and assume $\|v\| = \|w\|$. We know that $v+w \perp v-w$ if and only if $(v+w)\cdot(v-w) = 0$. We have

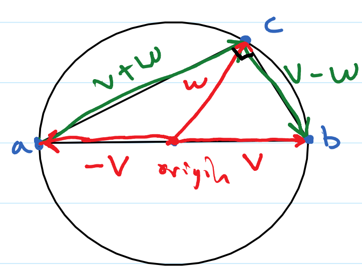
$$\begin{aligned}(v+w)\cdot(v-w) &= v\cdot v - w\cdot v + v\cdot w - w\cdot w \\ &= v\cdot v - w\cdot v + w\cdot v - w\cdot w \\ &= v\cdot v - w\cdot w \\ &= \|v\|^2 - \|w\|^2 \\ &\overset{(*)}{=} \|v\|^2 - \|v\|^2 = 0\end{aligned}$$

Note $\|u\| = \sqrt{u\cdot u}$

$(*)$ by hypothesis $\|v\| = \|w\|$

∎

Proof (of (b))



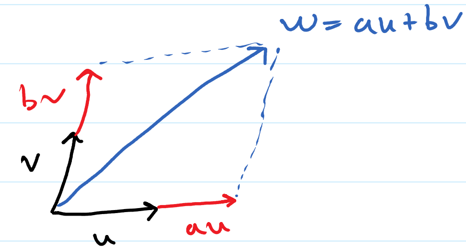
We need to show that $v+w \perp v-w$. By construction, both v and w have magnitude equal to r, where r is the radius of the circle. So $\|v\| = r = \|w\|$. So by part (a), $v+w$ and $v-w$ are orthogonal. So $\triangle abc$ is a right triangle.

∎

Problem 4

Tuesday, April 14, 2020 8:07 PM

(4) Let $u, v, w \in \mathbb{R}^3$. Suppose that there exists $a, b \in \mathbb{R}$ such that $w = au + bv$. Find the value of $u \cdot (v \times w)$.



Solution We have

$$
\begin{aligned}
u \cdot (v \times w) &= u \cdot (v \times \underline{(au + bv)}) \\
&= u \cdot (v \times au + v \times bv) \\
&= u \cdot (a(v \times u) + b(v \times v)) \\
&= au \cdot (v \times u) + bu \cdot (v \times v) \\
&= 0 + 0 \\
&= 0.
\end{aligned}
$$

Notice $v \times u$ is perpendicular to au. So $au \cdot (v \times u) = 0$.

Also, $v \times v = 0$.

$\square$