NOLCOS 1992 conf proc.
Bordeaux France.

The structure of optimal controls for a steering problem

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1 Introduction

We study some facts about the structure of optimal controls in a steering problem for a nonlinear control system. This problem is of considerable current interest in the context of so-called *nonholonomic motion planning* for robots. In ([7],[8]), we showed that the problem of motion planning for such mobile robots under non-integrable Pfaffian constraints was equivalent to a steering problem for nonlinear control systems. The control laws that we proposed in that work was motivated to a large extent by work of Brockett on sinusoidal optimal controls for a class of nonlinear systems ([4]). In parallel, there has been a tremendous amount of recent work on nonholonomy calculations in mechanics and of the connections between gauge theory and problems of reorienting deformable bodies, such as falling cats and space robots([6]). The primary function of this paper is to bring together results from these two areas in the context of optimal control; which is, of course, generalized classical mechanics to control theorists. The results on derivatives of the optimal controls are known to many practitioners (for example, Sussmann uses similar calculations in ([10], pg. 90-91) for studying singular optimal controls and Baillieul derived sinusoidal optimal controls for $SO(3)$ in ([2])). Our symplectic point of view is new and of some interest. The original problem which we set out to solve: namely, to obtain a Newton-Puiseux like expansion for the optimal control costs associated with the optimal steering problem (more precisely, the sub Riemannian geodesic problem), alluded to by Brockett ([3]) remains unsolved, when more than one level of Lie brackets is needed to achieve controllability.

*Research supported in part by NSF IRI90-14490
†Research supported in part by a Faculty Research Grant

2 First derivative of the optimal control

2.1 The Drift Free Case

Consider the problem of steering a system without drift on $\Re^n$ of the form

$$\dot{x} = g_1 u_1 + \cdots + g_m u_m \tag{1}$$

from an initial state $x(0) = x_i$ to a final state $x(1) = x_f$ in one second. Moreover, we will do so with a minimum L_2 norm of the control, namely, we minimize

$$\frac{1}{2}\int_0^1 |u(t)|^2 dt \tag{2}$$

We take a Hamiltonian point of view; thus, we define the Hamiltonian

$$H(x,u,p) = p^T(\sum_{i=1}^m g_i(x)u_i) + \frac{1}{2}\sum_{i=1}^m |u_i|^2$$

The normal optimal controls are then obtained by minimizing the Hamiltonian as

$$u_i^* = -p^T g_i(x) \tag{3}$$

and the optimal Hamiltonian is given by

$$H^*(x,p) = -\frac{1}{2}\sum_{i=1}^m (p^T g_i(x))^2 \tag{4}$$

The Hamiltonian system for the case of abnormal controls will be discussed in a companion paper in these proceedings by the second author. The Hamiltonian equations for the optimal control are

$$\begin{array}{rcl} \dot{x} = & \frac{\partial H^*}{\partial p}^T = & -\sum_{i=1}^m g_i(x)(p^T g_i(x)) \\ \dot{p} = & -\frac{\partial H^*}{\partial x}^T = & \sum_{i=1}^m \frac{\partial g_i}{\partial x}^T p(p^T g_i(x)) \end{array} \tag{5}$$

with the boundary conditions $x(0) = x_i, x(1) = x_f$. We will assume that smooth solutions to the equations (5) exist. In particular, this will require that it is, in fact, possible to find trajectories steering the system from x_i to x_f. In turn, this requires that the control Lie algebra generated by the g_i vector fields spans $\Re^n$ at every point (the controllability rank condition). Under these assumptions, some very interesting results are obtained by differentiating the optimal controls of (3). We will use the convention that the Lie bracket of two vector fields is given (in coordinates) by

$$[g_1, g_2] = \frac{\partial g_2}{\partial x} g_1 - \frac{\partial g_1}{\partial x} g_2$$

Proposition 1 Optimal Controls are Unitary
Consider the optimal control problem of (2) for the system of (1). Then the optimal controls of (3) satisfy the following differential equation:

$$\begin{bmatrix} \dot{u}_1 \\ \vdots \\ \dot{u}_i \\ \vdots \\ \dot{u}_m \end{bmatrix} = \Omega(p,x) \begin{bmatrix} u_1 \\ \vdots \\ u_i \\ \vdots \\ u_m \end{bmatrix} \qquad (6)$$

where $\Omega(p,x) =$

$$\begin{bmatrix} 0 & \cdots & p^T[g_1,g_i] & \cdots & p^T[g_1,g_m] \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ p^T[g_i,g_1] & \cdots & 0 & & p^T[g_i,g_m] \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ p^T[g_m,g_1] & \cdots & p^T[g_m,g_i] & \cdots & 0 \end{bmatrix} \qquad (7)$$

In particular, this implies that the controls are unitary, that is

$$\sum_{i=1}^{m} |u_i(t)|^2 = \sum_{i=1}^{m} |u_i(0)|^2 \qquad (8)$$

for all t.

Proof: The proof of equation (6) follows by direct differentiation of (3) using the Hamiltonian system (5). For the unitarity of the control, we see that the form of equation (6) is a linear equation with a skew symmetric right hand side. (The skew symmetry of Ω follows from the skew symmetry of the Lie bracket.) The unitarity of the controls now follows directly from (6).

Remarks:

1. By virtue of the fact that the magnitude of the optimal controls which solve the problem of (2) are constant, it follows that the same optimal controls also solve an other optimization problem with a different cost, namely,

$$\int_0^1 \sqrt{\sum_{i=1}^{m} (u_i(t))^2 dt} \qquad (9)$$

   The problem solved by the optimization of the criterion (9) is a geodesic problem, associated with the sub-Riemannian metric induced by the drift free control system (1). Indeed, the optimal controls also solve the problem

$$\int_0^1 \psi\left(\sum_{i=1}^{m} (u_i(t))^2\right) dt$$

   for any monotone function $\psi(.) : \Re \rightarrow \Re$.

2. Another consequence of the constant magnitude of the optimal control is that the optimal control normalized by $|u(0)|$, i.e. $\frac{u(t)}{|u(0)|}$ solves the minimum time problem: minimize T subject to

$$x(0) = x_i, x(T) = x_f \quad \sum_{i=1}^{m} |u_i(t)|^2 \leq 1 \ \ \forall t$$

   See the companion paper by Montgomery for a proof of this claim.

3. For the case of $m = 2$, the preceding result is particularly pleasing:

$$\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} = \begin{bmatrix} 0 & \omega(t) \\ -\omega(t) & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

   for $\omega(t) = p^T[g_1(x), g_2(x)]$ and the optimal controls have the form

$$\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} \cos(\psi) & \sin(\psi) \\ -\sin(\psi) & \cos(\psi) \end{bmatrix} \begin{bmatrix} u_1(0) \\ u_2(0) \end{bmatrix}$$

   where $\psi(t) = \int_0^t \omega(t) dt$.

4. Let us set $\mathcal{D}_x = Span\{g_i(x) : i = 1, \ldots m\}$. In the case where $\mathcal{D}$ is the horizontal distribution for some connection (on a principal G bundle, say) then the Lie brackets $[g_i, g_j]$ are essentially the curvature of this connection. $\Omega_{ij}(x,p)$ is the component of the curvature in the 'p' direction. The equations we derived for u^* are called the (first) Wong equations in this case. See Montgomery ([6], [5]).

A symplectic geometric version of these calculations is obtained by introducing the following notation: ([1], pg. 242). If f is a vector field on $\Re^n$ then its "momentum function"

$$P(f) : T^*\Re^n = \Re^n \oplus \Re^n \rightarrow \Re$$

is the function

$$P(f)(x,p) = p^T f(x)$$

We have the following basic relation

$$\{P(f), P(g)\} = -P([f,g])$$

where $\{\cdot,\cdot\}$ denotes the Poisson bracket acting on functions of x, p, defined by

$$\{\phi(x,p), \psi(x,p)\} := \sum_{i=1}^{n} \frac{\partial \phi}{\partial x_i} \frac{\partial \psi}{\partial p_i} - \frac{\partial \phi}{\partial p_i} \frac{\partial \psi}{\partial x_i}$$

Let us set $P_i = P(g_i)$. Then

$$H = -\frac{1}{2}\sum_{i=1}^{m} P_i^2$$

and $u_i^* = -P_i$. Recall that if F is any function of (x,p) then

$$\dot{F} = \{F, H\}$$

describes its time rate of change along trajectory $(x(t), p(t))$ of Hamilton's equations. It follows that

$$\dot{u}_i^* = -\{P_i, H\} = \sum_{j=1}^{m}\{P_i, P_j\}P_j$$

or

$$\dot{u}^* = \Omega u^* \qquad (10)$$

where Ω is the time-dependent skew symmetric matrix with entries

$$\Omega_{ij}(t) = -\{P_i, P_j\} = P([g_i, g_j]) = p^T[g_i, g_j]$$

evaluated at the point $(x(t), p(t))$ (cf. equation (7)).

It is of interest to apply the calculations of this theorem to some simple model systems: first, the example of the so-called Heisenberg Lie algebra for the control system:

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= x_1 u_2 - x_2 u_1 \end{aligned} \qquad (11)$$

A simple calculation yields that the optimal controls satisfy

$$\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} = \begin{bmatrix} 0 & -2p_3(t) \\ 2p_3(t) & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \qquad (12)$$

and the Hamiltonian equations yield that $\dot{p}_3 = 0$ (indeed, by inspection the optimal Hamiltonian is not a function of x_3, since g_1, g_2 are not), so that $p_3(t) \equiv p_3(0)$ and thus the optimal controls are sinusoids at frequency $2p_3(0)$.

2.2 Systems with Drift

The preceding proposition can be generalized to the case of a control system with drift:

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i \qquad (13)$$

We will be interested in steering this system from x_i to x_f minimizing the cost criterion of (2). The Hamiltonian is modified to

$$H(x, u, p) = p^T f(x) + p^T\left(\sum_{i=1}^{m} g_i(x) u_i\right) + \frac{1}{2}\sum_{i=1}^{m}|u_i|^2$$

and the optimal control is still given by the formula of (3), namely

$$u_i = -p^T g_i(x) = -P_i$$

where, P_i is the momentum function defined earlier, but the optimal Hamiltonian is given by

$$\begin{aligned} H^*(x,p) &= -\tfrac{1}{2}\sum_{i=1}^{m}(p^T g_i(x))^2 + p^T f(x) \\ &= -\tfrac{1}{2}\sum_{i=1}^{m} P_i^2 + P(f) \end{aligned} \qquad (14)$$

with $P(f)$ standing for the momentum function associated by f. The Hamiltonian equations for optimal control are given by

$$\begin{aligned} \dot{x} &= f(x) - \sum_{i=1}^{m} g_i(x)(p^T g_i(x)) \\ \dot{p} &= -\frac{\partial f}{\partial x}^T p + \sum_{i=1}^{m} \frac{\partial g_i}{\partial x}^T p(p^T g_i(x)) \end{aligned} \qquad (15)$$

The analog of Proposition 1 is now

Proposition 2 First derivative of the optimal controls for a system with drift

Consider the optimal control problem of (2) for the system of (13). Then the optimal controls of (3) satisfy the following differential equation:

$$\begin{bmatrix} \dot{u}_1 \\ \vdots \\ \dot{u}_i \\ \vdots \\ \dot{u}_m \end{bmatrix} = \Omega(x,p) \begin{bmatrix} u_1 \\ \vdots \\ u_i \\ \vdots \\ u_m \end{bmatrix} + \begin{bmatrix} -p^T[f, g_1] \\ \vdots \\ -p^T[f, g_i] \\ \vdots \\ -p^T[f, g_m] \end{bmatrix} \qquad (16)$$

where $\Omega(x,p)$ is as defined in equation (7) above.

Remarks:

1. It is easy to see that the controls in the case of systems with drift are not unitary. Roughly speaking, the optimal controls have to overcome drift to the extent given by $-p^T[f, g_i]$ for the ith component.

2. In the symplectic geometry notation the constant term in the equation for $\dot{u}$ has as its i th entry $\{P(f), P_i\}$.

3. For a linear system, namely

$$\dot{x} = Ax + \sum_{i=1}^{m} b_i u_i \qquad (17)$$

it follows that the optimal controls satisfy:

$$\begin{bmatrix} \dot{u}_1 \\ \vdots \\ \dot{u}_i \\ \vdots \\ \dot{u}_m \end{bmatrix} = \begin{bmatrix} p^T A b_1 \\ \vdots \\ p^T A b_i \\ \vdots \\ p^T A b_m \end{bmatrix} \qquad (18)$$

3 Higher Order Derivatives of the Optimal Controls

The calculations leading to Proposition 1 can be iterated to get formulas for all order derivatives of the optimal inputs u_i. Recall that the optimal inputs satisfy the differential equation

$$\dot{u} = \Omega u \tag{19}$$

with $\Omega_{ij} = p^T[g_i, g_j] = -\{P_i, P_j\}$, a skew symmetric matrix. A calculation similar to that of Proposition 1, yields that

$$\dot{\Omega}_{ij} = \sum_{k=1}^{m}(p^T[g_k, [g_i, g_j]])u_k \tag{20}$$

Using this calculation we may now verify that

$$\ddot{u} = \Omega_2(u, u) + \Omega^2 u \tag{21}$$

where $\Omega_2(u, u)$ is a bilinear map from $\Re^m \times \Re^m \rightarrow \Re^m$ with

$$(\Omega_2(u, u))_i = \sum_{j=1}^{m}\sum_{k=1}^{m}(p^T[g_k, [g_i, g_j]])u_k u_j$$

and Ω^2 is the (symmetric) square of the skew-symmetric matrix Ω. The ijk th entry of Ω_2 is $-\{\{P_i, P_j\}, P_k\}$. This calculation can be iterated to yield

$$\begin{aligned} u^{(3)} = \; & \Omega_3(u, u, u) + \Omega_2(\Omega u, u) + \Omega_2(u, \Omega u) \\ & + 2\Omega\, \Omega_2(u, u) + \Omega^3 u \end{aligned} \tag{22}$$

with $\Omega_3(u, u, u)$ a multi-linear map from $\Re^m \times \Re^m \times \Re^m \rightarrow \Re^m$

$$(\Omega_3(u, u, u))_i = \sum_{l=1}^{m}\sum_{j=1}^{m}\sum_{k=1}^{m}(p^T[g_l, [g_k, [g_i, g_j]]])u_k u_j u_l$$

Thus, the i th entry has coefficient in j,k,l given by $-\{\{P_i, P_j\}, P_k\}, P_l\}$ For systems with drift, the equation (21) is modified to

$$\ddot{u} = \Omega_2(u, u) + \Omega^2 u + \Psi_1 u + \Psi \tag{23}$$

where the matrix $\Psi_1 \in \Re^{m\times m}$ has as it's ij th entry

$$p^T[[f, g_i], g_j] + p^T[[g_i, g_j], f]$$

and $\Psi \in \Re^m$ has the form

$$\begin{bmatrix} p^T[f, [f, g_1]] \\ \vdots \\ p^T[f, [f, g_m]] \end{bmatrix} + \Omega \begin{bmatrix} -p^T[f, g_1] \\ \vdots \\ -p^T[f, g_m] \end{bmatrix}$$

In the symplectic notation, the ij th entry of Ψ_1 is $\{\{P(f), P_i\}, P_j\} + \{\{P_i, P_j\}, P(f)\}$ and the i th entry of Ψ is $\{P(f), \{P(f), P_i\}\} + \sum_{j=1}^{m}\{P_i, P_j\}\{P_i, P(f)\}$.

4 Optimal Controls on Lie Groups

Consider the case that the state space is the Lie Group $SO(3)$, the rotation group, which we view as the configuration space of a rigid body, a satellite. Assume that the satellite has a momentum wheel on each of two of its orthogonal principal axes and that the satellite and its momentum wheels are isolated and as a system has angular momentum 0. Let v_i be the angular velocities of the wheels. Choose a basis set E_1, E_2, E_3 of the space of skew symmetric matrices, representing infinitesimal rotations about the x, y, z axes respectively. Then, the control vector fields are *left invariant* with $g_i(x) = xE_i$ and the control system modeling the satellite (expressing the statement that angular momentum = 0) is

$$\dot{x} = g_1(x)u_1 + g_2(x)u_2 \tag{24}$$

Here the controls $u_i = -(m_i/I_i)v_i$, where m_i is the moment of inertia of the i th wheel and I_i is the moment of inertia of the satellite about the corresponding principal axis, $i = 1, 2$. We pose the same optimal steering problem as before, namely to steer from x_i to x_f minimizing $\int_0^1(u_1^2 + u_2^2)dt$. Let P_i be the momentum functions corresponding to the g_i, as above. P_3 is defined from $g_3 := xE_3$. Then, we have that $[g_1, g_2] = -g_3$ and cyclic permutations of this relationship. The P_i satisfy the Poisson bracket relationships: $\{P_1, P_2\} = P_3$, $\{P_2, P_3\} = P_1$, $\{P_3, P_1\} = P_2$ and the optimal control Hamiltonian is $H^* = -(1/2)(P_1^2 + P_2^2)$ with the optimal controls given by $u_i^* = -P_i$.

In the instance that the total angular momentum of the satellite is (c_1, c_2, c_3), then the system corresponds to a system with drift and it is easy to verify that the optimal Hamiltonian is

$$H^* = -(1/2)(P_1^2 + P_2^2) + c_1 P_1 + c_2 P_2 + c_3 P_3$$

In the case of zero angular momentum, to verify that the controls are unitary, we compute

$$\dot{u}_i^* = -\{P_i, H^*\}$$

so that we have that

$$\begin{bmatrix} \dot{u}_1^* \\ \dot{u}_2^* \end{bmatrix} = \begin{bmatrix} 0 & -\{P_1, P_2\} \\ -\{P_2, P_1\} & 0 \end{bmatrix} \begin{bmatrix} u_1^* \\ u_2^* \end{bmatrix} \tag{25}$$

This is exactly the same formula as was derived in Section 2. However, in this case a more explicit solution to the state trajectory corresponding to the optimal control may be given. The optimal Hamiltonian may be rewritten as $H^* = H_0 - H_3$, where $H_0 = -(1/2)(P_1^2 + P_2^2 + P_3^2)$ and $H_i = -(1/2)(P_i^2)$. Notice that $\{P_i, H_0\} = 0$ so that in particular, the flows of H_3 and H_0 commute. In fact, since H_0 generates the geodesic flow on $SO(3)$, corresponding to the free motion of a completely symmetric rigid body, any solution of Hamilton's equations using H_0 is given by

$$x(t) = x(0)e^{t\omega\times} \tag{26}$$

for some skew symmetric matrix $\omega\times \in \Re^{3\times 3}$ of the form,

$$\begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix}$$

which has the interpretation of body angular velocity. The vector field associated with H_3 has the interpretation of being the flow associated with rotation about the third body axis. Thus, the flow associated with H_3 is $x(t) = x(0)exp(c_3 t E_3)$. Composing these two flows, we get that the flow corresponding to the optimal Hamiltonian is

$$x(t) = x(0)e^{t\omega\times}e^{c_3 t E_3} \tag{27}$$

Using this formula in the equations of the control system above yields that $c_3 = -\omega_3$. The optimal controls in this case are sinusoids as in the case of the Heisenberg algebra control system (11) as may be verified by noting that in (25) that $\{P_1, P_2\} = P_3$ and further that $\dot{P}_3 = \{P_3, H^*\} = -1/2\{P_3, P_1^2 + P_2^2\} = 0$, so that u_1^*, u_2^* are sinusoids at frequency $P_3(0)$.

This example is a special case of the following general problem: Let X be an n-dimensional Lie group and $G \subset X$ an $n-k$ dimensional Lie subgroup. Let $< .,. >$ be a bi-invariant inner product (Killing form) on the associated Lie Algebra, Lie (X). Let $U = Lie(G)^\perp$ be the space of controls. Then, we have that $LieX = LieG \oplus U$. Now consider the control problem

$$\dot{x} = xu \tag{28}$$

where $x \in X$ and $u \in U$ with cost functional $\frac{1}{2}\int_0^1 < u(t), u(t) > dt$ to be minimized subject to the constraint that u steers x_i to x_f. This problem has the following general solution (see [2], [6]):

Theorem 3 *The normal optimal extremals corresponding to the problem of (28) are*

$$x(t) = x_0 exp(t\omega)exp(-t\omega^\perp) \tag{29}$$

with exp: $Lie(G) \to G$ being the usual exponential map, ω an arbitrary element of $Lie(G)$ and $\omega^\perp$, the orthogonal projection of ω onto $Lie(G) = U^\perp$.

The theorem is proved in the same way as the preceding by writing the optimal Hamiltonian as

$$-\frac{1}{2}\left(\sum_{i=1}^{n} P_i^2 - \sum_{i=k+1}^{n} P_i^2\right)$$

5 Sub-Riemannian Balls

In this section, we discuss the shape of the balls in the sub-Riemannian metric induced by the control. These balls are of interest, in as much as they represent the correct metric for path planning in the instance of non-holonomic motion planning. We define $S(x_0, x_1)$ to be the value function corresponding to minimizing (2) for steering from x_0 to x_1. We define $S(x_0, x_1) = 1/2d^2(x_0, x_1)$, where d is called the *sub-Riemannian* distance function. The *sub-Riemannian ball* centered at x_0 is the set of points $\{x : d(x_0, x) \leq \delta\}$. Detailed calculations of the sub-Riemannian balls are seldom explicit. Here, we discuss in detail the case that we have a three dimensional state space with two controls in $\Re^3$ and $SO(3)$. in the instance of $\Re^3$ the following is a re-expression of work in [4].

Thus, we consider the case where

$$g_1 = (1, 0, -a_1(x_1, x_2))^T, g_2 = (0, 1, -a_2(x_1, x_2))^T$$

Then $H = -\frac{1}{2}P_1^2 - \frac{1}{2}P_2^2$ with $P_i = p_i - p_3 a_i(x_1, x_2)$. Since H is independent of the third coordinate, x_3, we have p_3 is a constant. The optimal control equations are the Lorentz equations for a charge p_3 traveling through a magnetic field:

$$\begin{aligned} \ddot{u}_1 = &\ p_3 b(x_1, x_2) u_2 \\ \ddot{u}_2 = &\ -p_3 b(x_1, x_2) u_1 \end{aligned} \tag{30}$$

with $b(x_1, x_2) = \frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2}$. If $a_1 = -1/2x_2$ and $a_2 = 1/2x_1$ as in the Heisenberg example of Section 2.1, then $b(x_1, x_2) \equiv$ constant $\neq 0$ and these equations can be immediately integrated. Set $u = u_i + iu_2 \in$ C Then the "Lorentz equation" becomes

$$\ddot{u} = -i\lambda u \quad ; \lambda = p_3 \;\; b = \text{ constant}$$

so that $u(t) = e^{-i\lambda t}u(0)$ and, if we write $w = x_1 + ix_2$ then

$$w(t) = w(0) - \frac{u(0)}{i\lambda}(e^{-it\lambda} - 1)$$

which is the parametric form of a circle C with center $w(0) + \frac{u(0)}{i\lambda}$ and radius $R = \frac{|u(0)|}{\lambda}$. Further, assume that $x_1(0) = x_2(0) = x_3(0) = 0$. Then, $x_3(t) = \int \frac{1}{2}(x_1 dx_2 - x_2 dx_1)$ which is the area bounded by the moving secant $[o, w(t)]$ and the arc of the circle C:

$$x_3(t) = \frac{1}{2}R^2(\lambda t - sin(\lambda t))$$

Following Riemannian geometry, we write

$$exp(v, \lambda) = (w(1), z(1))$$

where $v = v(0) = p_1(0) + ip_2(0), \lambda = p_3$ and $(x_1(0), x_2(0), x_3(0)) = (0, 0, 0)$. We have that the map $exp : \Re^3 \to \Re^3$.

The *sub-Riemannian ball* of radius δ is the image under the map exp of the solid cylinder $|v| \leq \delta$. The sub-Riemannian sphere is the boundary of this set. These are shown in Figure (1) for radii of 1,2. The *sub-Riemannian wave front* is defined to be the image of the set $|v| = \delta$. It properly contains the sub-Riemannian sphere. Using the above solutions, we have:

$$exp(v, \lambda) = \left(\frac{iv}{\lambda}(e^{-it\lambda} - 1), \frac{1}{2}\left(\frac{|v|}{\lambda}\right)^2(\lambda - sin\lambda)\right)$$

where we think of $\Re^3 as C \oplus \Re$ One calculates

$$\left|\frac{iv}{\lambda}(e^{-it\lambda} - 1)\right| = 2\frac{|v|}{\lambda}\left(\frac{sin\lambda}{2}\right)$$

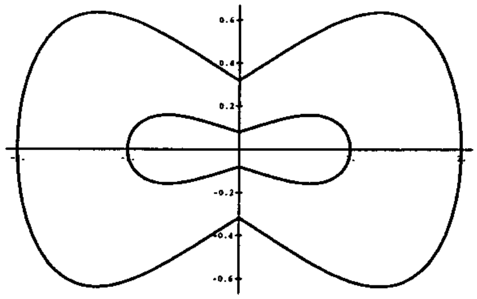

Figure 1: Showing the sub-Riemannian balls of radius 1 and 2

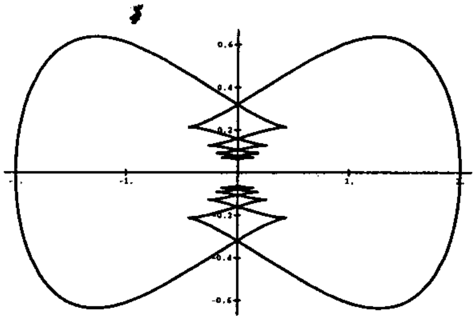

Figure 2: Sub Riemannian wave fronts

Setting $|v| = \delta$, we see that we have described the sub-Riemannian wavefront, and sphere as surface of rotations. The sub-Riemannian wave fronts are visualized in Figure (2). This wavefront figure can also be found in an appendix to the little known, but excellent paper of Rayner ([9]).

For the case of the steering problem for the satellite in $SO(3)$, let us compute the sub-Riemannian metric distance required to steer from $x(0) = I$ to $x(1) = e^{\theta E_3}$. A simple calculation using (27) yields that $\omega_3 = 2\pi - \theta$ (to be interpreted mod 2π if $\theta > \pi$) and that $|\omega| = 2\pi$. One may, thus, choose the optimal inputs to be of the form $u_1^*(t) = (2\pi - \theta)\cos(\omega_1 t)$ and $u_2^* = (2\pi - \theta)\sin(\omega_1 t)$ with $\omega_1^2 + (2\pi - \theta)^2 = (2\pi)^2$. The metric distance from $x(0)$ to $x(1)$ is, thus,

$$2\pi\sqrt{1 - (1 - \frac{\theta}{2\pi})^2}$$

so that one may conjecture that the shape of the ball at least for small θ is like that of the Heisenberg ball. Some important differences do exist: one of the more interesting ones is that the frequency ω_1 of the optimal input sinusoids for motion along E_3, a conjugate direction, is less than 2π.

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